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STRONGLY CONVERGENT ITERATIVE ALGORITHMS FOR SOLVING A CLASS OF VARIATIONAL INEQUALITIES

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Abstract. The paper is concerned with the problem of finding a common solution of a variational inequality problem governed by inverse strongly monotone operators and of a fixed point problem of nonexpansive mappings. Two new iterative algorithms are introduced to solve the problem. Moreover, it is proved the sequence generated by each of the algorithms converges in norm to a solution closest to the anchor from the common solution set. Applications to strict pseudocontractions, the split feasibility problem, and the convexly constrained linear inverse problem are included.

1. Introduction

A variational inequality problem (VIP) is formulated as a problem of finding a point x^* with the property:

$$x^* \in C, \quad \langle Ax^*, z - x^* \rangle \geq 0, \quad z \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $A : C \to \mathcal{H}$ is an operator. We will denote the solution set of VIP (1.1) by $\Omega(A; C)$.

VIP (1.1) is said to be a monotone VIP if the governing operator A is a monotone operator. In this paper we will consider a special case of a monotone VIP where the governing operator $A : C \to \mathcal{H}$ is inverse strongly monotone (ism) (i.e., there exists a constant $\nu > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \nu \|Ax - Ay\|^2$ for all $x, y \in C$).

A fixed point problem (FPP) is to find a point $\hat{x}$ with the property:

$$\hat{x} \in C, \quad S\hat{x} = \hat{x}, \tag{1.2}$$

where $S : C \to C$ is a nonlinear mapping and C is, as above, a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. The set of fixed points of S is denoted as $Fix(S)$.

The problem under consideration in this article is to find a common solution of VIP (1.1) and of FPP (1.2). Namely, we seek a point x^* such that

$$x^* \in Fix(S) \cap \Omega(A; C). \tag{1.3}$$

To solve Problem (1.3), Takahashi and Toyoda [22] introduced an algorithm which generates a sequence (x_n) by the iterative procedure:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \tag{1.4}$$

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where P_C is the projection of C onto $\mathcal{H}$.

Iiduka and Takahashi [14] introduced another algorithm which generates a sequence (x_n) by the iterative procedure:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \quad n \geq 0. \tag{1.5}$$

In both algorithms (1.4) and (1.5), the sequence (α_n) is chosen from the interval $[0,1]$. Under certain assumptions, the sequence (x_n) generated by algorithm (1.4) (resp., (1.5)) can be weakly (resp., strongly) convergent to a solution of problem (1.3) (see [22, 14]).

Recently, some other algorithms, which are connected with Korpelevich's extragradient method [15], were studied (see e.g. [18, 28]). We mention that the main tool for the convergence analysis used in these articles is the maximal monotone operator T defined by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{1.6}$$

where $N_C v = \{w \in \mathcal{H} : \langle v-u, w\rangle \geq 0,\ u \in C\}$ is the normal cone to C at $v \in C$. In this way, VIP (1.1) is equivalent to finding a zero of the maximal monotone T defined by (1.6) (see [21]).

It is the aim of this paper to introduce two new iterative algorithms to solve problem (1.3). Our approach to the the convergence analysis uses averaged operators which is different from the existing methods of using maximal monotone operators. The paper is organized as follows. In the next section, some useful lemmas are given. In Section 3, we prove strong convergence of our new algorithms. In Section 4, we include applications of our algorithms in solving fixed point problems of strict pseudocontractions, the split feasibility problem, the convexly constrained linear problem, as well as the convexly constrained minimization problem.

2. Preliminary and notation

Let $\mathcal{H}$ be a real Hilbert space and C a nonempty closed convex subset of $\mathcal{H}$. We use P_C to denote the projection from $\mathcal{H}$ onto C; namely, for $x \in \mathcal{H}$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is well-known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad z \in C. \tag{2.1}$$

We will use the following notions on nonlinear operators $T : C \to \mathcal{H}$.

(i) T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

(ii) T is firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \quad x, y \in C$$

(Projections are firmly nonexpansive; hence nonexpansive.)

(iii) T is α-averaged if there exist a constant $\alpha \in (0,1)$ and a nonexpansive mapping S such that $T = (1-\alpha)I + \alpha S$, where I is the identity operator on $\mathcal{H}$. (Projections are $\frac{1}{2}$-averaged.)

(iv) T is ν-inverse strongly monotone (ν-ism) if there is a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \nu \|Tx - Ty\|^2, \quad x, y \in C.$$

(Projections are 1-ism.)

Some of the basic properties of the above-stated operators are collected below.

Lemma 2.1 ([5])**.** *The following assertions hold.*

(a) *T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism;*
(b) *T is averaged if and only if $I - T$ is ν-ism for some $\nu > 1/2$;*
(c) *T is firmly nonexpansive if and only if $I - T$ is 1-ism;*
(d) *T is firmly nonexpansive if and only if T is $\frac{1}{2}$-averaged;*
(e) *If T is ν-ism for $\nu > 0$ and if $\gamma > 0$, then γT is (ν/γ)-ism.*

We shall use the following notation:

- $x_n \to x$: strong convergence of (x_n) to x;
- $x_n \rightharpoonup x$: weak convergence of (x_n) to x;
- $\omega_w(x_n)$: the set of the cluster points of (x_n) in the weak topology (i.e., the set $\{x : \exists x_{n_j} \rightharpoonup x\}$, where (x_{n_j}) means a subsequence of (x_n)).

The lemma below is referred to as the demiclosedness principle for nonexpansive mappings (see [12]).

Lemma 2.2 (Demiclosedness principle)**.** *Let C be a nonempty closed convex subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If (x_n) is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Averaged operators will play an important role in the convergence analysis of our algorithms (to be introduced in Section 3). We therefore collect some useful properties of averaged mappings.

Lemma 2.3 ([5, 7])**.** *Let ν_A and ν_B be constants in $(0,1)$. Assume that A is ν_A-averaged and B is ν_B-averaged. Then the following hold.*

(i) *If $Fix(A) \cap Fix(B) \neq \emptyset$, then $Fix(AB) = Fix(A) \cap Fix(B)$;*
(ii) *The composition AB is ν_{AB}-averaged with*

$$\nu_{AB} = \frac{2}{1 + 1/(\nu_A \vee \nu_B)};$$

(iii) *For any $z \in Fix(A)$, there holds the following inequality:*

$$\|Ax - z\|^2 \leq \|x - z\|^2 - \frac{1-\nu_A}{\nu_A}\|Ax - x\|^2.$$

The assumption below will be used in the subsequent section.

Assumption 2.4. Let $\nu > 0$ and $b \geq a > 0$.

(a) The sequence (λ_n) is chosen so that
    (i) $0 < a \leq \lambda_n \leq b < 2\nu$;

(i) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

(b) The sequence (α_n) is chosen in $(0, 1)$ so that

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Finally, we present a sufficient condition for a real sequence to converge to zero.

Lemma 2.5 ([24]). *Let (a_n) be a nonnegative real sequence satisfying*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\mu_n,$$

where the sequences $(\gamma_n) \subset (0, 1)$ and (μ_n) satisfy the conditions:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\lim_{n\to\infty} \gamma_n = 0$;

(iii) *either* $\sum_{n=0}^{\infty} |\gamma_n\mu_n| < \infty$ *or* $\overline{\lim}_{n\to\infty} \mu_n \le 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Algorithms and Their Convergence

In this section, we will introduce two iterative algorithms and prove their strong convergence to a solution of Problem (1.3). We begin with the following lemma.

Lemma 3.1. *Assume that $A : C \to \mathcal{H}$ is ν-ism for some $\nu > 0$. Given a real number λ such that $0 < \lambda < 2\nu$. Set $V = I - 2\nu A$ and $V_\beta = (1 - \beta)I + \beta V$ with $\beta = \lambda/2\nu$. Then the following assertions hold.*

(a) *$Fix(P_C V) = Fix(P_C V_\beta) = \Omega(A; C)$;*

(b) *V_β is β-averaged and also $V_\beta = I - \lambda A$;*

(c) *For $z \in \Omega(A; C)$, there holds the following estimate:*

$$\|P_C V_\beta x - z\|^2 \le \|x - z\|^2 - \frac{1-\gamma}{\gamma}\|P_C V_\beta x - x\|^2, \tag{3.1}$$

where $\gamma := 2/[1 + 1/(1/2 \vee \beta)] < 1$;

(d) *For any given $x \in \mathcal{H}$, there holds the inequality:*

$$\|P_C V_\beta x - x\| \le \|P_C V_{\beta'} x - x\|,$$

where $0 < \beta < \beta' < 1$.

Proof. (a) Observing that, for any $z \in C$,

$$\begin{aligned} \langle x - V_\beta x, x - z\rangle \le 0 \quad &\Leftrightarrow \quad \langle x - Vx, x - z\rangle \le 0 \\ &\Leftrightarrow \quad \langle Ax, x - z\rangle \le 0, \end{aligned}$$

one can easily check that (a) holds.

(b) Since A is ν-ism, $2\nu A$ is $\frac{1}{2}$-ism; thus $V = I - 2\nu A$ is nonexpansive in view of Lemma 2.1. Consequently V_β is β-averaged. The relation $V_\beta = I - \lambda A$ follows from simple computations.

(c) Since P_C is $\frac{1}{2}$-averaged, it follows from Lemma 2.3(ii) that $P_C V_\beta$ is γ-averaged, where

$$\gamma := \frac{2}{1 + 1/(\max\{1/2, \beta\})}.$$

This together with Lemma 2.3(iii) immediately yields

$$\|P_C V_\beta x - z\|^2 \le \|x - z\|^2 - \frac{1-\gamma}{\gamma}\|P_C V_\beta x - x\|^2,$$

where $z \in \Omega(A; C) = Fix(P_C V_\beta)$.

(d) By definition of projections, it follows that

$$\|V_\beta x - P_C V_\beta x\|^2 \le \|V_\beta x - P_C V_{\beta'} x\|^2,$$

which implies that

$$\|(x - P_C V_\beta x) + \beta(Vx - x)\|^2 \le \|(x - P_C V_{\beta'} x) + \beta(Vx - x)\|^2.$$

Expanding the above square-norms and by monotonicity of P_C, we arrive at

$$\begin{aligned}\|x - P_C V_{\beta'} x\|^2 &\ge \|x - P_C V_\beta x\|^2 + 2\beta\langle P_C V_{\beta'} x - P_C V_\beta x, Vx - x\rangle \\ &= \|x - P_C V_\beta x\|^2 + \frac{2\beta}{\beta' - \beta}\langle P_C V_{\beta'} x - P_C V_\beta x, V_{\beta'} x - V_\beta x\rangle \\ &\ge \|x - P_C V_\beta x\|^2,\end{aligned}$$

which completes the proof. □

We now introduce our first iterative algorithm. Take an initial guess $x_0 \in C$ and another point $x \in C$ called an anchor; choose a sequence (α_n) in the interval $[0, 1]$; and define a sequence (x_n) by the iterative procedure:

$$x_{n+1} = S[\alpha_n x + (1 - \alpha_n)P_C(x_n - \lambda_n A x_n)], \quad n \ge 0. \tag{3.2}$$

Below is the convergence of this algorithm.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be ν-ism for some $\nu > 0$ and $S : C \to C$ nonexpansive. Suppose $Fix(S) \cap \Omega(A; C) \ne \emptyset$ and Assumption 2.4 holds. Then the sequence (x_n) generated by the iterative algorithm (3.2) converges strongly to the solution x^* of Problem (1.3) closest to x from the solution set; namely, $x^* = P_{Fix(S)\cap\Omega} x$.*

Proof. Set $y_n = P_C(x_n - \lambda_n A x_n)$. By Lemma 3.1, we can write

$$y_n = P_C V_{\beta_n} x_n = P_C[(1 - \beta_n)x_n + \beta_n V x_n],$$

where $\beta_n = \lambda_n/(2\nu) \in (0, 1)$ satisfying $a/(2\nu) \le \beta_n \le b/(2\nu)$ and

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty. \tag{3.3}$$

We divide our proof into several steps.

Step 1. The sequence (x_n) is bounded.

Taking any $z \in Fix(S) \cap \Omega(A; C)$ (thus $z = P_C V_{\beta_n} z$ by Lemma 3.1), we deduce that

$$\begin{aligned}\|x_{n+1} - z\| &\le \|\alpha_n x + (1 - \alpha_n)y_n - z\| \\ &\le \alpha_n\|x - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\le \max\{\|x - z\|, \|x_n - z\|\}.\end{aligned}$$

By induction, we can easily show that

$$\|x_n - z\| \le \max\{\|x_0 - z\|, \|x - z\|\}$$

for all $n \ge 0$. In particular, (x_n) is bounded.

STEP 2. $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

By Algorithm (3.2), we estimate that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \|[\alpha_n x + (1-\alpha_n)y_n] - [\alpha_{n-1}x + (1-\alpha_{n-1})y_{n-1}]\| \\ &= \|(\alpha_n - \alpha_{n-1})(x - y_{n-1}) + (1-\alpha_n)(y_n - y_{n-1})\| \\ &\le |\alpha_n - \alpha_{n-1}|\|x - y_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \end{aligned} \tag{3.4}$$

and also that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|P_C V_{\beta_n} x_n - P_C V_{n-1} x_{n-1}\| \\ &\le \|V_{\beta_n} x_n - V_{\beta_{n-1}} x_{n-1}\| \\ &\le \|V_{\beta_n} x_n - V_{\beta_n} x_{n-1}\| + \|V_{\beta_n} x_{n-1} - V_{\beta_{n-1}} x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|V x_{n-1} - x_{n-1}\|. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), we arrive at

$$\|x_{n+1} - x_n\| \le (1-\alpha_n)\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|), \tag{3.6}$$

where M is a suitable positive constant. By virtue of Assumption 2.4 and the condition (3.3), we can apply Lemma 2.5 to (3.6) to obtain $x_{n+1} - x_n \to 0$. Consequently we also have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.7}$$

In fact, it follows from Lemma 3.1 that $P_C V_{\beta_n}$ is γ_n-averaged, where

$$\gamma_n := \frac{2}{1 + 1/(\max\{1/2, \beta_n\})}.$$

We therefore derive from (3.1) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \|\alpha_n x + (1-\alpha_n)y_n - z\|^2 \\ &\le \alpha_n\|x - z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\ &\le \alpha_n\|x - z\|^2 + \|P_C V_{\beta_n} x_n - z\|^2 \\ &\le \alpha_n\|x - z\|^2 + \|x_n - z\|^2 - \frac{1-\gamma_n}{\gamma_n}\|P_C V_{\beta_n} x_n - x_n\|^2 \\ &= \alpha_n\|x - z\|^2 + \|x_n - z\|^2 - \frac{1-\gamma_n}{\gamma_n}\|y_n - x_n\|^2. \end{aligned}$$

It turns out that

$$\begin{aligned} \frac{1-\gamma_n}{\gamma_n}\|y_n - x_n\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|x - z\|^2 \\ &\le L(\|x_n - x_{n+1}\| + \alpha_n), \end{aligned} \tag{3.8}$$

where L is a suitable positive constant. Since it is not hard to check that

$$\inf_{n\geq 0} \frac{1-\gamma_n}{\gamma_n} > 0,$$

we see that (3.7) follows from (3.8) by sending $n \to \infty$.

STEP 3. If $x' \in \omega_w(x_n)$, then $x' \in Fix(S) \cap \Omega(A;C)$.

To see this, we set $z_n = \alpha_n x + (1-\alpha_n)y_n$. Then we conclude that

$$\|y_n - z_n\| = \alpha_n \|x - y_n\| \to 0,$$

and also that

$$\begin{aligned} \|Sz_n - z_n\| &= \|x_{n+1} - z_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

Take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x'$; hence $z_{n_k} \rightharpoonup x'$ as well. By the demiclosedness principle (Lemma 2.2), we conclude that $x' \in Fix(S)$.

To show $x' \in \Omega(A;C)$, set $V_\beta = (1-\beta)I + \beta V$ with $\beta = a/(2\nu)$. In view of Lemma 3.1, we have

$$\|P_C V_\beta x_n - x_n\| \leq \|P_C V_{\beta_n} x_n - x_n\| = \|y_n - x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Applying the demiclosedness principle again, we get $x' \in Fix(P_C V_\beta) = \Omega(A;C)$, where the equality follows from Lemma 3.1(a).

STEP 4. $x_n \to x^* := P_{Fix(S)\cap\Omega(A;C)}x$.

To see this we first claim that the following estimate holds:

$$\varlimsup_{n\to\infty} \langle y_n - x^*, x - x^* \rangle \leq 0. \tag{3.9}$$

In fact, by Steps 2 and 3, we get

$$\begin{aligned} \varlimsup_{n\to\infty} \langle y_n - x^*, x - x^* \rangle &= \varlimsup_{n\to\infty} \left(\langle x_n - x^*, x - x^* \rangle + \langle y_n - x_n, x - x^* \rangle \right) \\ &= \varlimsup_{n\to\infty} \langle x_n - x^*, x - x^* \rangle = \lim_{n'\to\infty} \langle x_{n'} - x^*, x - x^* \rangle \\ &= \langle x' - x^*, x - x^* \rangle \leq 0, \end{aligned} \tag{3.10}$$

where $(x_{n'})$ is an appropriately chosen subsequence of (x_n) converging weakly to x' and where the last inequality follows because x^* is the projection of the anchor x onto $Fix(S) \cap \Omega(A;C)$.

Finally we prove $x_n \to x^*$. As a matter of fact, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|S(\alpha_n x + (1-\alpha_n)y_n) - Sx^*\|^2 \\ &\leq \|\alpha_n x + (1-\alpha_n)y_n - x^*\|^2 \\ &= (1-\alpha_n)^2 \|y_n - x^*\|^2 + \alpha_n^2 \|x - x^*\|^2 \\ &\qquad + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^* \rangle \\ &= (1-\alpha_n)^2 \|P_C V_{\beta_n} x_n - x^*\|^2 + \alpha_n^2 \|x - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^* \rangle \end{aligned}$$

$$\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n^2\|x - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle$$

$$\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\delta_n, \tag{3.11}$$

where

$$\delta_n := 2(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle + \alpha_n\|x - x^*\|^2$$

satisfies the property (due to (3.9)):

$$\overline{\lim_{n\to\infty}}\, \delta_n \le 0.$$

Therefore, applying Lemma 2.5 to (3.11) gives that $\|x_n - x^*\| \to 0$, which is the required result. □

Before moving on to our second algorithm, we present a lemma below.

Lemma 3.3. *Let the conditions in Lemma 3.1 hold. If* $A^{-1}(0) \neq \emptyset$, *then* $\Omega(A; C) = Fix(V)$.

Proof. By definitions of V and V_β, it is obvious that

$$A^{-1}(0) = Fix(V) = Fix(V_\beta) \subseteq C. \tag{3.12}$$

However, since

$$Fix(P_C) \cap Fix(V_\beta) = C \cap A^{-1}(0) = A^{-1}(0) \neq \emptyset,$$

it follows from Lemmas 3.1 and 2.3 that

$$\Omega(A; C) = Fix(P_C V_\beta) = C \cap Fix(V_\beta).$$

Now $\Omega(A; C) = Fix(V)$ follows from (3.12). □

Remark 3.4. If the domain of A is the whole space $\mathcal{H}$, then the assumption $A^{-1}(0) \neq \emptyset$ should be replaced by $C \cap A^{-1}(0) \neq \emptyset$.

Our second iterative algorithm generates a sequence (x_n) according to the recursion:

$$x_{n+1} = SP_C[\alpha_n x + (1-\alpha_n)(x_n - \lambda_n A x_n)], \quad n \ge 0, \tag{3.13}$$

where the initial guess $x_0 \in C$ and the anchor $x \in C$ are chosen in C arbitrarily, and the sequence (α_n) is selected in the interval $[0, 1]$. The convergence of this algorithm is given below.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be ν-ism for some $\nu > 0$ and $S : C \to C$ nonexpansive. Suppose $Fix(S) \cap A^{-1}(0) \neq \emptyset$ and Assumption 2.4 holds. Then the sequence (x_n) generated by the algorithm (3.13) converges strongly to the solution x^* of Problem (1.3) closest to x from the solution set; namely, $x^* = P_{Fix(S)\cap\Omega(A;C)}x$.*

Proof. Set $y_n = x_n - \lambda_n A x_n$. Then by Lemma 3.1, we can write

$$y_n = V_{\beta_n} x_n = (1-\beta_n)x_n + \beta_n V x_n,$$

where $\beta_n = \lambda_n/(2\nu) \in (0, 1)$ satisfying condition (3.3). Again we divide our proof into several steps.

Step 1. The sequence (x_n) is bounded.

Take $z \in Fix(S) \cap \Omega(A;C)$. It follows from Lemma 3.3 that

$$z \in Fix(S) \cap \Omega(A;C) = Fix(S) \cap Fix(V) \subseteq C. \tag{3.14}$$

Repeating the argument of Step 1 in the proof of Theorem 3.2, we can obtain that the sequence (x_n) is bounded.

Step 2. $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

By Algorithm (3.13), we estimate that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \|\alpha_n x + (1-\alpha_n)y_n - [\alpha_{n-1}x + (1-\alpha_{n-1})y_{n-1}]\| \\ &= \|(\alpha_n - \alpha_{n-1})(x - y_{n-1}) + (1-\alpha_n)(y_n - y_{n-1})\| \\ &\le |\alpha_n - \alpha_{n-1}|\|x - y_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \end{aligned} \tag{3.15}$$

and also that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|V_{\beta_n}x_n - V_{\beta_{n-1}}x_{n-1}\| \\ &\le \|V_{\beta_n}x_n - V_{\beta_n}x_{n-1}\| + \|V_{\beta_n}x_{n-1} - V_{\beta_{n-1}}x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Vx_{n-1} - x_{n-1}\|. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$\begin{aligned} \|x_{n+1} - x_n\| \le{}& (1-\alpha_n)\|x_n - x_{n-1}\| \\ &+ M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|), \end{aligned} \tag{3.17}$$

where M is a suitable positive constant. Now we can apply Lemma 2.5 to (3.17) to obtain $x_{n+1} - x_n \to 0$.

On the other hand, we deduce by (3.14) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|SP_C[\alpha_n x + (1-\alpha_n)y_n] - z\|^2 \\ &\le \|\alpha_n x + (1-\alpha_n)y_n - z\|^2 \\ &\le (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|x - z\|^2 \\ &\le \|(1-\beta_n)x_n + \beta_n Vx_n - z\|^2 + \alpha_n\|x - z\|^2 \\ &= (1-\beta_n)\|x_n - z\|^2 + \beta_n\|Vx_n - z\|^2 \\ &\quad + \alpha_n\|x - z\|^2 - \beta_n(1-\beta_n)\|x_n - Vx_n\|^2 \\ &\le \|x_n - z\|^2 + \alpha_n\|x - z\|^2 - \beta_n(1-\beta_n)\|x_n - Vx_n\|^2. \end{aligned}$$

It turns out that

$$\begin{aligned} \beta_n(1-\beta_n)\|x_n - Vx_n\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|x - z\|^2 \\ &\le L(\|x_{n+1} - x_n\| + \alpha_n), \end{aligned} \tag{3.18}$$

where L is a suitable positive constant. Now since $0 < a/(2\nu) \le \beta_n \le b/(2\nu) < 1$ for all n and since $\|x_{n+1} - x_n\| \to 0$, it follows from (3.18) that

$$\|x_n - Vx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

However $\|x_n - y_n\| = \beta_n\|Vx_n - x_n\|$. Therefore, $x_n - y_n \to 0$ as $n \to \infty$.

Step 3. If $x' \in \omega_w(x_n)$, then $x' \in Fix(S) \cap \Omega(A;C)$.

Take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x'$. Thus, by (3.19) and the demiclosedness principle (Lemma 2.2), $x' \in Fix(V) = \Omega(A;C)$.

To see $x' \in Fix(S)$, let $z_n = P_C[\alpha_n x + (1-\alpha_n)y_n]$. It is obvious that $x_n \in C$ for every $n \geq 0$ and thus

$$\begin{aligned}\|x_n - z_n\| &= \|P_C x_n - P_C[\alpha_n x + (1-\alpha_n)y_n]\| \\ &\leq \alpha_n\|x_n - x\| + \|x_n - y_n\| \to 0,\end{aligned}$$

which implies that $z_{n_k} \rightharpoonup x'$. On the other hand,

$$\|Sz_n - z_n\| = \|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0.$$

Applying again the demiclosedness principle (Lemma 2.2) gives us that $x' \in Fix(S)$.

STEP 4. $x_n \to x^* := P_{Fix(S)\cap\Omega(A;C)}x$.

To see this, we first repeat the proof of (3.10) to get the following estimate:

$$\overline{\lim_{n\to\infty}}\langle y_n - x^*, x - x^*\rangle \leq 0. \tag{3.20}$$

Next we compute from (3.13) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|SP_C[\alpha_n x + (1-\alpha_n)y_n] - x^*\|^2 \\ &\leq \|\alpha_n x + (1-\alpha_n)y_n - x^*\|^2 \\ &= (1-\alpha_n)^2\|y_n - x^*\|^2 + \alpha_n^2\|x - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle \\ &= (1-\alpha_n)^2\|V_{\beta_n}x_n - x^*\|^2 + \alpha_n^2\|x - x^*\|^2 \\ &\qquad + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n^2\|x - x^*\|^2 \\ &\qquad + 2\alpha_n(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\delta_n, \end{aligned} \tag{3.21}$$

where

$$\delta_n = 2(1-\alpha_n)\langle y_n - x^*, x - x^*\rangle + \alpha_n\|x - x^*\|^2$$

satisfies the property (due to (3.20)):

$$\overline{\lim_{n\to\infty}}\,\delta_n \leq 0.$$

Applying Lemma 2.5 to (3.21) yields that $\|x_n - x^*\| \to 0$. This completes the proof. □

4. APPLICATIONS

We present in this section several applications of the results obtained in Section 3.

4.1. **Strictly pseudocontractive mapping.** We first consider a problem for finding a common fixed point of a nonexpansive mapping and of a strictly pseudocontractive mapping. Recall that an operator $T : C \to C$ is called strictly κ-pseudocontractive if there is constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. It is known that if T is strictly κ-pseudocontractive, then $A = I - T$ is $\frac{1-\kappa}{2}$-ism (see [3]).

Corollary 4.1. *Let $T : C \to C$ be strictly κ-pseudocontractive and $S : C \to C$ nonexpansive. Suppose that Assumption 2.4 holds and $Fix(S) \cap Fix(T) \neq \emptyset$. Then, for any given $x,\ x_0 \in C$, the sequence (x_n) generated by the algorithm*

$$x_{n+1} = S[\alpha_n x + (1 - \alpha_n)((1 - \lambda_n)x_n + \lambda_n T x_n)], \quad n \geq 0$$

converges strongly to the point $P_{Fix(S)\cap Fix(T)}x$.

Proof. Set $A = I - T$. Then A is $\frac{1-\kappa}{2}$-ism. Also $Fix(T) = \Omega(A; C)$ and $P_C(x_n - \lambda_n Ax_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$. Applying Theorem 3.2 yields the result of the lemma. □

4.2. **Convexly constrained minimization problem.** Consider the optimization problem of finding a point x^* with the property:

$$x^* \in \arg\min_{x\in C} f(x), \tag{4.1}$$

where $f : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function. A classical method to solve Problem (4.1) is the well-known gradient projection algorithm, which generates a sequence (x_n) by the iterative procedure:

$$x_{n+1} = P_C(x_n - \gamma\nabla f(x_n)),\ n \geq 0, \tag{4.2}$$

where $x_0 \in \mathcal{H}$, and γ is a positive parameter. If, in addition, $\nabla f(x)$ is $\frac{1}{\nu}$-Lipschitz continuous, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq \frac{1}{\nu}\|x - y\|, \quad x, y \in \mathcal{H},$$

then the sequence (x_n), generated by (4.2) with $0 < \gamma < 2\nu$, converges weakly to a minimizer of f relative to the set C, whenever such minimizers exist (see for example [5, Corollary 4.1]).

Corollary 4.2. *Let $f : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function. Suppose that $\nabla f(x)$ is $\frac{1}{\nu}$-Lipschitz continuous and that Assumption 2.4 holds. Then, for any given x, $x_0 \in \mathcal{H}$, the sequence (x_n) generated by the algorithm*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)P_C(x_n - \lambda_n\nabla f(x_n))$$

converges strongly to a minimizer of f relative to C, whenever such minimizers exist.

Proof. Denote by $\Omega(f; C)$ the solution set of the variational inequality

$$x \in C, \quad \langle\nabla f(x), x - z\rangle \geq 0, \quad z \in C.$$

According to [10, Lemma 5.13], we have $\Omega(f;C) = \arg\min_{x\in C} f(x)$. Further, if ∇f is $(1/\nu)$-Lipschitz continuous, then it is also ν-ism (see [2, Corollary 10]) and therefore, Theorem 3.2 applies by letting $A = \nabla f$. □

Corollary 4.3. *Let the conditions in Corollary 4.2 hold. If, in addition,*

$$C \cap \nabla f^{-1}(0) \neq \emptyset,$$

where $f^{-1}(0) = \{x : \nabla f(x) = 0\}$, *then, for any given* x, $x_0 \in \mathcal{H}$, *the sequence* (x_n) *generated by the algorithm*

$$x_{n+1} = P_C[\alpha_n x + (1-\alpha_n)(x_n - \lambda_n \nabla f(x_n))]$$

converges strongly to a minimizer of f *relative to* C, *whenever such minimizers exist.*

Proof. Applying Theorem 3.5 obtains the result. □

4.3. **Split feasibility problem.** Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. The split feasibility problem (SFP) [6] is formulated as finding a point x satisfying the property:

$$x \in C \text{ and } Ax \in Q, \tag{4.3}$$

where $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator. SFP (4.3) attracts many authors' attention due to its application in signal processing [6]. To solve SFP (4.3), it is very useful to investigate the following convexly constrained minimization problem (CCMP):

$$\min_{x\in C} f(x) := \frac{1}{2}\|(I - P_Q)Ax\|^2. \tag{4.4}$$

Generally speaking, SFP (4.3) and CCMP (4.4) are not fully equivalent: every solution of SFP (4.3) is evidently a minimizer of CCMP (4.4); however a solution to CCMP (4.4) does not necessarily satisfy SFP (4.3). Further, if the solution set of SFP (4.3) is nonempty, i.e.,

$$C \cap A^{-1}(Q) := C \cap \{x : Ax \in Q\} \neq \emptyset,$$

then it follows from [23, Lemma 4.2] that

$$C \cap \nabla f^{-1}(0) = C \cap A^{-1}(Q) \neq \emptyset,$$

where f is defined by (4.4).

Various algorithms have been invented to solve SFP (4.3) (see [4, 5, 20, 25, 26, 27] and reference therein). In particular, Byrne [4] introduced the CQ algorithm: For any initial guess $x_0 \in \mathcal{H}$, define (x_n) recursively as

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \tag{4.5}$$

where $0 < \lambda < 2\nu$ with $\nu = 1/\|A\|^2$. The CQ algorithm is known to have only weak convergence in infinite-dimensional spaces [11], since it is in fact a special case for the Mann iteration [17].

Corollary 4.4. *Suppose that Assumption 2.4 holds. Then, for any given $x \in \mathcal{H}$, the sequence (x_n), generated by the algorithm*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_C[x_n - \lambda_n A^*(I - P_Q)Ax_n]$$

converges strongly to a solution of SFP (4.3)*, whenever its solution set is nonempty.*

Proof. Let f be defined by (4.4). According to [5, p. 113] (see also [1]), we have

$$\nabla f = A^*(I - P_Q)A,$$

which is $(1/\nu)$-Lipschitz continuous with $\nu = 1/\|A\|^2$. Thus Corollary 4.2 applies and the result follows immediately. □

Remark 4.5. Corollary 4.4 recovers the result of [25, Corollary 3.7], which considered the special case where $\lambda_n \equiv \lambda$ for all n.

Corollary 4.6. *Suppose that Assumption 2.4 holds. Then, for any given $x \in \mathcal{H}$, the sequence (x_n), generated by the algorithm*

$$x_{n+1} = P_C[\alpha_n x + (1 - \alpha_n)(x_n - \lambda_n A^*(I - P_Q)Ax_n)]$$

converges strongly to a solution of SFP (4.3)*, whenever its solution set is nonempty.*

Proof. This is a straightforward consequence of Corollary 4.3. □

Remark 4.7. Corollary 4.6 recovers the result of [23, Theorem 4.3], which considered the special case where $x = 0$ and $\lambda_n \equiv \lambda$ for all n.

4.4. **Convexly constrained linear inverse problem.** The convexly constrained linear inverse problem is to solve the constrained linear system (cf. [9, 19])

$$\begin{cases} Ax = b \\ x \in C \end{cases} \tag{4.6}$$

where $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $b \in \mathcal{K}$. A classical way to deal with this problem is the well-known projected Landweber method (see [9]): For any initial guess $x_0 \in \mathcal{H}$, define (x_n) recursively by

$$x_{n+1} = P_C[x_n - \lambda A^*(Ax_n - b)], \tag{4.7}$$

where $0 < \lambda < 2\nu$ with $\nu = 1/\|A\|^2$. An counterexample in [8, Remark 5.12] shows that the projected Landweber iteration converges weakly in infinite-dimensional spaces, in general. To get strong convergence, we have the following result.

Corollary 4.8. *Suppose that Assumption 2.4 holds. Then, for any given $x \in \mathcal{H}$, the sequence (x_n) generated by the algorithm*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_C[x_n - \lambda_n A^*(Ax_n - b)]$$

converges strongly to a solution of Problem (4.6)*, whenever its solution set is nonempty.*

Proof. This is a direct consequence of Corollary 4.4 by taking $Q = \{b\}$. □

Corollary 4.9. *Suppose that Assumption 2.4 holds. Then, for any given* $x \in \mathcal{H}$, *the sequence* (x_n) *generated by the algorithm*

$$x_{n+1} = P_C[\alpha_n x + (1-\alpha_n)(x_n - \lambda_n A^*(Ax_n - b))]$$

converges strongly to a solution of Problem (4.6)*, whenever its solution set is nonempty.*

Proof. This is a direct consequence of Corollary 4.6 by taking $Q = \{b\}$. □

Remark 4.10. Corollary 4.9 recovers the result of [9, Theorem 3.9], which considered the special case where $x = 0$ and $\lambda_n = (1 + n^{-\alpha})^{-1}$ with $0 < \alpha < 1$.

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