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THE 2-KKM PRINCIPLE IN ABSTRACT CONVEX SPACES : EQUIVALENT FORMULATIONS AND APPLICATIONS

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Dedicated to the memory of Ky Fan

Abstract. Since any family of closed balls in a hyperconvex metric space has nonempty intersection whenever each two members of the family intersects. Motivated by this fact, T.-H. Chang et al. [2–4] introduced 2-KKM maps for hyperconvex metric spaces. In this paper, we seek equivalents or consequences of the 2-KKM theorem when it is available.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the non-emptiness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or other solutions of various equilibrium problems. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [12], which concerns with certain types of multimaps called the KKM maps later.

The KKM theory, first named by the author [14], is the study of applications of equivalent formulations or generalizations of the KKM theorem. From 1961, Ky Fan showed that the KKM theorem provides foundations for many of the modern essential results in diverse areas of mathematical sciences. He extended the KKM theorem to arbitrary topological vector spaces and applied it to various problems; see [14, 15]. Fan's works were expanded systematically by Granas [5] to new topological methods in convex analysis mainly on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [13], and to C-spaces (or H-spaces) by Horvath [6, 7], and others. In the last decade, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author; for details, see [15, 16] and references therein.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [17–21]. The partial KKM principle for an abstract convex space is an abstract form of the KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In our recent works [19–21], we studied elements or foundations of

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the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or spaces satisfying the partial KKM principle. Moreover, a number of such results are equivalent to each other.

On the other hand, it is well-known that any family of closed balls in a hyperconvex metric space has a nonempty intersection whenever each two members of the family intersects. Motivated by this fact, recently, T.-H. Chang et al. [2–4] introduced 2-KKM maps and generalized 2-KKM maps on metric spaces and the family 2-KKM(X,Y) of maps defined on a metric space, and then obtained a 2-KKM theorem, a fixed point theorem without compactness condition, some minimax inequalities, and other results for hyperconvex metric spaces.

It is also well-known that any KKM type theorem is equivalent to the Fan type matching theorem, the Fan-Browder type fixed point theorem, the Fan type minimax inequality, and others. Each of such equivalents has many applications in various fields. Similarly, from the 2-KKM theorem, we can establish the basic theory of 2-KKM maps. More precisely, in this paper, we introduce new γ-convex spaces more general than hyperconvex metric ones and obtain equivalents or consequences of the 2-KKM theorem on such new spaces.

2. γ-convex spaces

Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [17–21]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. We gave lots of examples of abstract convex spaces in [18–21].

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

The following are new:

Definition 2.3. A γ-*convex space* $(E, D; \gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\gamma : D \times D \multimap E$ with nonempty values $\gamma(a, b)$ for any $a, b \in D$.

For any $D' \subset D$, the γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\gamma D' := \bigcup\{\gamma(a, b) \mid a, b \in D'\} \subset E.$$

A subset X of E is called a *γ-convex subset* of $(E, D; \gamma)$ relative to D' if for any $a, b \in D'$, we have $\gamma(a, b) \subset X$, that is, $\mathrm{co}_\gamma D' \subset X$.

In case $E \supset D$, let $(E \supset D; \gamma) := (E, D; \gamma)$ and let $(E; \gamma) := (E, E; \gamma)$.

Example. (1) Every abstract convex space $(E, D; \Gamma)$ is a γ-convex space with $\gamma(a, b) := \Gamma\{a, b\}$ for any $a, b \in D$.

(2) Suppose that X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [11]. Let $\gamma(a, b) := conv_H\{a, b\}$ for $a, b \in X$. Then $(H \supset X; \gamma)$ is a γ-convex space.

Definition 2.4. Let $(E, D; \gamma)$ be a γ-convex space. If a map $G : D \multimap E$ satisfies

$$\gamma(a, b) \subset G(a) \cup G(b) \qquad \text{for any} \quad a, b \in D,$$

then G is called a 2-*KKM map*.

Definition 2.5. For a γ-convex space $(E, D; \gamma)$ and a given family $\mathcal{F}(E)$ of nonempty subsets of E, the 2-*KKM principle* with respect to $\mathcal{F}(E)$ is the statement that, for any 2-KKM map $G : D \to \mathcal{F}(E)$, the family $\{G(y)\}_{y \in D}$ has the nonempty intersection.

Example. (1) For any γ-convex space $(E, D; \gamma)$, let $\mathcal{F}(E) := \{E\}$. Then the 2-KKM principle holds.

(2) Later we will see that a hyperconvex γ-metric space $(E, D; \gamma)$ satisfies the 2-KKM principle with respect to the family $\mathcal{A}(E)$ of admissible sets; see Section 3, Theorem 3.10.

In our recent works [19–21], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the KKM principle. Motivated by this situation, we will seek consequences of the 2-KKM principle when it is available.

3. 2-KKM maps in metric spaces

Let (M, d) be a metric space. Motivated by [9] and others, for a bounded subset $A \subset M$, we set

$\mathrm{ad}(A) := \bigcap \{B \mid B \text{ is a closed ball such that } A \subset B\}$.

$\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

For $x \in M$ and $\varepsilon > 0$, let

$$B(x, \varepsilon) := \{y \in M \mid d(x, y) \le \varepsilon\} \text{ and } N(x, \varepsilon) := \{y \in M \mid d(x, y) < \varepsilon\}.$$

We introduce new definitions:

Definition 3.1. A triple $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map having admissible values.

A Γ-*convex* subset of $(M, D; \Gamma)$ relative to some $D' \subset D$ is said to be *subadmissible* by some authors.

Example. For any metric space (M, d), we give examples of metric spaces $(M, D; \Gamma)$ and KKM maps on them.

(1) For any nonempty set $D \subset M$, we have $(M \supset D; \Gamma)$ where $\Gamma_A := \mathrm{ad}(A)$ for each $A \in \langle D \rangle$; see [9]. A map $G : D \multimap M$ is called a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

(2) Let D be a nonempty set. For each $A := \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$, choose a set $B := \{x_0, x_1, \dots, x_n\} \in \langle M \rangle$ and define $\Gamma_A := \mathrm{ad}(B)$. Then $(M, D; \Gamma)$ becomes a metric space. For this metric space, the so-called generalized $gKKM$ mapping in [3] simply becomes a KKM map.

The following originates from [1].

Definition 3.2. A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

Example. It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Therefore, $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex metric spaces.

Results of Aronszajn and Panitchpakti [1] and Isbell [8] are combined in the following:

Lemma 3.3. *A hyperconvex metric space is complete and (freely) contractible.*

Lemma 3.4. *An admissible subset of a hyperconvex metric space is hyperconvex.*

Definition 3.5. An abstract convex space $(H, D; \Gamma)$ is called simply a *hyperconvex metric space* if (H, d) is a hyperconvex metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(H)$ is a map having admissible values such that

$$A, B \in \langle D \rangle,\ A \subset B \text{ implies } \Gamma_A \subset \Gamma_B.$$

There should be no confusion between a hyperconvex metric space $H = (H, d)$ and $(H, D; \Gamma)$. Note that every Γ_A is contractible and hence $(H, D; \Gamma)$ becomes an H-space and a G-convex space.

Therefore, the KKM theory on hyperconvex metric spaces are simle consequences of the well-established H-space theory and G-convex space theory. For example, the following typical theorem follows from the corresponding one for G-convex spaces in [7]:

Theorem 3.6. *Let $(H, D; \Gamma)$ be a hyperconvex metric space and $G : D \multimap H$ a map such that*

(1) G has closed [resp., open] values; and

(2) G is a KKM map.

Then $\{G(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\Gamma_N \cap \bigcap_{z \in N} G(z) \neq \emptyset$.)

Further, if

(3) $\bigcap_{z \in A} \overline{G(z)}$ *is compact for some* $A \in \langle D \rangle$,

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Example. (1) As a consequence of Theorem 3.6, we obtain Khamsi's KKM theorem for a particular Γ and for particular KKM maps with finitely closed values; see [9]. In fact, by replacing the original topology of H by its finitely generated extension, we can eliminate "finitely".

(2) From Theorem 3.6, we obtain another particular forms in [2, Theorems 2 and 3], [3, Theorem 4], and [4, Theorem 2].

It is well-known that any family of closed balls in a hyperconvex metric space has nonempty intersection whenever each two members of the family intersects. More precisely, we have the following due to Penot [10, p.406]:

Lemma 3.7. *Let* (H, d) *be a hyperconvex metric space and* $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{A}(H)$. *If for each* $\alpha, \beta \in I$, $\quad A_\alpha \cap A_\beta \neq \emptyset$, *then* $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$.

Note that Lemma 3.7 simply tells that, for a family of admissible subsets of a hyperconvex metric space, the finite intersection property implies the whole intersection property.

We introduce another γ-convex spaces:

Definition 3.8. A *γ-metric space* $(M, D; \gamma)$ consists of a metric space (M, d), a nonempty set D, and a map $\gamma : D \times D \to \mathcal{A}(M)$ such that $\gamma(a, b) = \gamma(b, a)$ for $a, b \in D$.

Sometimes, we use the conventions $(a, b) := \{a, b\}$ and $\{a\} := a$.

Example. (1) Any metric space (M, d) can be made into a γ-metric space $(M; \gamma)$ by defining $\gamma(x, y) := \mathrm{ad}(\{x, y\})$ for $x, y \in M$.

(2) Any metric space $(M, D; \Gamma)$ can be made into a γ-metric space $(M, D; \gamma)$ by defining $\gamma := \Gamma|_{D \times D} : D \times D \to \mathcal{A}(M)$.

(3) Any hyperconvex metric space $(H, D; \Gamma)$ is a γ-convex space $(H, D; \gamma)$, called a *hyperconvex γ-metric space*, with $\gamma(a, b) = \Gamma\{a, b\}$ for $a, b \in D$.

Definition 3.9. For a γ-metric space $(M, D; \gamma)$, a map $G : D \multimap M$ is called a 2-*KKM map* if

$$\gamma(a, b) \subset G(a) \cup G(b) \quad \text{for each } a, b \in D.$$

Example. Our definition of 2-KKM maps unifies various similar ones in [2–4].

(1) [2, 4] Let M be a metric space and $X \subset M$. A map $G : X \multimap M$ is called a 2-KKM map if for each $x_1, x_2 \in X$,

$$x_1 \in G(x_1), \quad x_2 \in G(x_2), \quad \text{and} \quad \mathrm{ad}(\{x_1, x_2\}) \subset G(x_1) \cup G(x_2).$$

This reduces to our definition by putting $D := X$ and $\gamma(x, y) := \mathrm{ad}(\{x, y\})$.

(2) [2, 4] Let X be a nonempty set and Y a metric space. A map $G : X \multimap Y$ is called a generalized 2-KKM map if for each $x_1, x_2 \in X$, there exists $y_1, y_2 \in Y$ such that

$$y_1 \in G(x_1), \quad y_2 \in G(x_2), \quad \text{and} \quad \mathrm{ad}(\{y_1, y_2\}) \subset G(x_1) \cup G(x_2).$$

This reduces to our 2-KKM map by putting $D := X$, $M := Y$, and $\gamma(x_1, x_2) := \mathrm{ad}(\{y_1, y_2\})$.

(3) [3] Let Z be a nonempty set and Y a metric space. A map $G : Z \multimap Y$ is called a generalized 2-${}_gKKM$ map if for each $z_1, z_2 \in Z$, there exists $y_1, y_2 \in Y$ such that

$$y_1 \in G(z_1), \quad y_2 \in G(z_2), \quad \text{and} \quad \mathrm{ad}(\{y_1, y_2\}) \subset G(z_1) \cup G(z_2).$$

This definition is same to the preceding one.

The following is a 2-KKM theorem:

Theorem 3.10. *Let $(H, D; \Gamma)$ be a hyperconvex metric space. If $G : D \to \mathcal{A}(H)$ is a 2-KKM map, then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

Proof 1. For any $z_1, z_2 \in D$, let $D' := \{z_1, z_2\}$. Consider the hyperconvex metric space $(H, D'; \Gamma')$ where $\Gamma' := \Gamma|_{\langle D' \rangle}$. Then $G' := G|_{D'} : D' \to \mathcal{A}(H)$ is a KKM map. By Theorem 3.6, we have $G(z_1) \cap G(z_2) = G'(z_1) \cap G'(z_2) \neq \emptyset$. Since each $G(z) \in \mathcal{A}(H)$, by Lemma 3.7, we have the conclusion. □

Proof 2. The hyperconvex metric space $(H, D; \Gamma)$ can be regarded as a hyperconvex γ-metric space $(H, D; \gamma)$ with $\gamma(a, b) = \mathrm{ad}(\{a, b\})$ for each $a, b \in D$. Since any element of $\mathcal{A}(H)$ is contractible by Lemma 3.3 and hence connected. Suppose that $G(a) \cap G(b) = \emptyset$ for some $a, b \in D$. Since $a \in G(a)$, $b \in G(b)$, $G(a)$ and $G(b)$ are closed and $\gamma(a, b) \in \mathcal{A}(H)$ is connected, it is impossible that $\gamma(a, b) \subset G(a) \cup G(b)$. Therefore $G(a) \cap G(b) \neq \emptyset$ for all $a, b \in D$. Now we apply Lemma 3.7. □

Note that Theorem 3.10 unifies the main results in [2, Theorem 4], [3, Theorem 5], and [4, Theorem 3].

Corollary 3.11. *Let X be an admissible subset of a hyperconvex metric space $(H \supset X; \Gamma)$. If $G : X \to \mathcal{A}(X)$ is a 2-KKM map, then G has a fixed point.*

Proof. By Lemma 3.4, X itself is a hyperconvex metric space. Hence $\bigcap_{x \in X} G(x) \neq \emptyset$ by Theorem 3.10. Then any point in the intersection is fixed under G. □

Remark. Any KKM map on a metric space $(M, D; \Gamma)$ is 2-KKM. The converse does not hold; see [2].

Question. For a hyperconvex metric space $(H, D; \Gamma)$, is there any $\mathcal{A}(H)$-valued 2-KKM map $G : D \to \mathcal{A}(H)$, which is not a KKM map?

We have a partial solution to this problem:

Theorem 3.12. *Let $(H, D; \Gamma)$ be a hyperconvex metric space and $G : D \to \mathcal{A}(H)$ a 2-KKM map. Then there is a hyperconvex metric space $(H, D; \Gamma')$ for which G is a KKM map.*

Proof. Since G is a 2-KKM map, by Theorem 3.10, there exists an $x_* \in H$ such that $x_* \in \bigcap_{z \in D} G(z)$. Define a map $\Gamma' : \langle D \rangle \to \mathcal{A}(H)$ by $\Gamma'_A := \{x_*\}$ for each $A \in \langle D \rangle$. Then $\Gamma'_A \subset G(A)$ and hence G is a KKM map on the hyperconvex metric space $(H, D; \Gamma')$. □

Note that Theorem 3.10 follows from Theorems 3.6 and 3.12. Consequently, main results in [2–4] follow from our G-convex space theory.

4. Equivalents of the 2-KKM principle

For a γ-convex space $(E, D; \gamma)$, let $\mathcal{F}(E)$ be a family of nonempty subsets of E. Consider the following statements:

(0) The 2-KKM principle. *For any 2-KKM map $G : D \to \mathcal{F}(E)$, we have*

$$\bigcap_{z \in D} G(z) \neq \emptyset.$$

(I) The matching property. *Let $S : D \multimap E$ be a map satisfying*

(1.1) *for each $z \in D$, $E \setminus S(z) \in \mathcal{F}(E)$; and*

(1.2) $E = \bigcup_{z \in D} S(z)$.

Then there exist $a, b \in D$ such that

$$\gamma(a, b) \cap S(a) \cap S(b) \neq \emptyset.$$

(II) Another whole intersection property. *Let $S : D \multimap E$, $T : E \multimap E$ be maps satisfying*

(2.1) *for each $z \in D$, $S(z) \in \mathcal{F}(E)$;*

(2.2) *for each $y \in E$, $\mathrm{co}_\gamma(D \setminus S^-(y)) \subset E \setminus T^-(y)$; and*

(2.3) *$x \in T(x)$ for each $x \in E$.*

Then $\bigcap_{z \in D} S(z) \neq \emptyset$.

(III) The geometric or the section property. *Let $A \subset D \times E$, $B \subset E \times E$ maps satisfying*

(3.1) *for each $z \in D$, $\{y \in E \mid (z, y) \in A\} \in \mathcal{F}(E)$;*

(3.2) *for each $y \in E$, $\mathrm{co}_\gamma\{z \in D \mid (z, y) \notin A\} \subset \{x \in E \mid (x, y) \notin B\}$; and*

(3.3) *$(x, x) \in B$ for each $x \in E$.*

Then there exists an $x_0 \in E$ such that $D \times \{x_0\} \subset A$.

(IV) Another geometric property. *For any sets $A \subset D \times E$, $B \subset E \times E$ satisfying*

(4.1) *for each $z \in D$, $\{y \in E \mid (z, y) \notin A\} \in \mathcal{F}(E)$;*

(4.2) *for each $y \in E$, $\mathrm{co}_\gamma\{z \in D \mid (z, y) \in A\} \subset \{x \in E \mid (x, y) \in B\}$; and*

(4.3) *$D \times E \subset A$.*

Then there exists an $x_0 \in E$ such that $(x_0, x_0) \in B$.

(V) The Fan-Browder fixed point property. *Let $S : D \multimap E$, $T : E \multimap E$ be maps satisfying*

(5.1) *for each $z \in D$, $E \setminus S(z) \in \mathcal{F}(E)$;*

(5.2) *for each $y \in E$, $\mathrm{co}_\gamma S^-(y) \subset T^-(y)$; and*

(5.3) $E = \bigcup_{z \in D} S(z)$.

Then T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$.

(VI) Existence of maximal elements. *Let $S : E \multimap D$, $T : E \multimap E$ be maps satisfying*

(6.1) *for each $z \in D$, $E \setminus S^-(z) \in \mathcal{F}(E)$;*

(6.2) *for each* $x \in E$, $\mathrm{co}_\gamma S(x) \subset T(x)$*; and*
(6.3) *for each* $x \in E$, $x \notin T(x)$.
Then there exists an $x \in E$ *such that* $S(x) = \emptyset$.

(VII) Analytic formulation. *Let* $A \subset C$ *be sets and* $\phi : D \times E \to C$ *be a function such that*
(7.1) *for each* $a, b \in D$ *and* $y \in \gamma(a, b)$*, we have* $\phi(a, y) \in A$ *or* $\phi(b, y) \in A$*; and*
(7.2) *for each* $z \in D$, $\{y \in E \mid \phi(z, y) \in A\} \in \mathcal{F}(E)$.
Then there exists a $y_0 \in E$ *such that*

$$\phi(z, y_0) \in A \quad \text{for all } z \in D.$$

(VIII) Minimax inequality. *Let* $\phi : D \times E \to \overline{\mathbb{R}}$ *be an extended real-valued function and* $\alpha \in \overline{\mathbb{R}}$ *such that*
(8.1) *for each* $a, b \in D$ *and* $y \in \gamma(a, b)$, $\min\{\phi(a, y), \phi(b, y)\} \le \alpha$*, and*
(8.2) *for each* $z \in D$, $\{y \in E \mid \phi(z, y) \le \alpha\} \in \mathcal{F}(E)$.
Then (i) *there exists a* $y_0 \in E$ *such that*

$$\phi(z, y_0) \le \alpha \quad \text{for all } z \in D; \text{ and}$$

(ii) *if* $E = D$ *and* $\alpha = \sup\{\phi(x, x) \mid x \in E\}$*, then we have the minimax inequality:*

$$\inf_{y \in E} \sup_{x \in E} \phi(x, y) \le \sup_{x \in E} \phi(x, x).$$

(IX) Analytic alternative. *Let* $A, B \subset C$ *sets and* $f : D \times E \to C$, $g : E \times E \to C$ *functions. Suppose that*
(9.1) *for each* $y \in E$, $\mathrm{co}_\gamma\{z \in D \mid f(z, y) \notin A\} \subset \{x \in E \mid g(x, y) \notin B\}$*; and*
(9.2) *for each* $z \in D$*, the set* $\{y \in E \mid f(z, y) \in A\} \in \mathcal{F}(E)$.
Then either
(a) *there exists a* $y_0 \in E$ *such that* $f(z, y_0) \in A$ *for all* $z \in D$*; or*
(b) *there exists an* $\hat{x} \in E$ *such that* $g(\hat{x}, \hat{x}) \notin B$.

(X) Analytic alternative. *Let* $\alpha, \beta \in \mathbb{R}$*, and* $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ *extended real-valued functions. Suppose that*
(10.1) *for each* $y \in E$, $\mathrm{co}_\gamma\{z \in D \mid f(z, y) > \alpha\} \subset \{x \in E \mid g(x, y) > \beta\}$*, and*
(10.2) *for each* $z \in D$*, the set* $\{y \in E \mid f(z, y) \le \alpha\} \in \mathcal{F}(E)$.
Then either
(a) *there exists a* $y_0 \in E$ *such that* $f(z, y_0) \le \alpha$ *for all* $z \in D$*; or*
(b) *there exists an* $\hat{x} \in E$ *such that* $g(\hat{x}, \hat{x}) > \beta$.

(XI) Minimax inequality. *Under the hypothesis of* (X)*, if*

$$\alpha = \beta = \sup\{g(x, x) \mid x \in E\},$$

then (c) *there exists a* $y_0 \in E$ *such that*

$$f(z, y_0) \le \sup_{x \in E} g(x, x) \quad \text{for all } z \in D; \text{ and}$$

(d) *we have the following minimax inequality*

$$\inf_{y \in E} \sup_{z \in D} f(z, y) \le \sup_{x \in E} g(x, x).$$

Now we show that (0)-(XI) are mutually equivalent and hold for hyperconvex metric spaces:

Theorem 4.1 (Characterizations of the 2-KKM principle)**.** (1) *For a γ-convex space $(E, D; \gamma)$, the statements* (0)-(XI) *are mutually equivalent.*

(2) *For a hyperconvex γ-metric space $(E, D; \gamma)$ and the admissible class $\mathcal{F}(E) := \mathcal{A}(E)$,* (0)-(XI) *hold.*

Proof. (0) $\Longrightarrow$ (I). Let $G : D \multimap E$ be a map given by $G(z) := E \setminus S(z) \in \mathcal{F}(E)$ for $z \in D$. Suppose, on the contrary to the conclusion, that for any $a, b \in D$, we have $\gamma(a, b) \cap S(a) \cap S(b) = \emptyset$; that is, $\gamma(a, b) \subset G(a) \cup G(b)$. Then $G : D \to \mathcal{F}(E)$ is a 2-KKM map. Since $(E, D; \gamma)$ satisfies the 2-KKM principle (0), there exists a $\widehat{y} \in \bigcap_{z \in D} G(z) = \bigcap_{z \in D}(E \setminus S(z)) \neq \emptyset$. Hence $\widehat{y} \notin S(z)$ for all $z \in D$. This violates condition (1.2).

(I) $\Longrightarrow$ (II). Suppose that $\bigcap_{z \in D} S(z) = \emptyset$, that is, $E = \bigcup_{z \in D}(E \setminus S(z))$. Then (1.1) and (1.2) are satisfied for $E \setminus S(z)$ instead of $S(z)$. Hence, by (I), there exist $a, b \in D$ and

$$y_0 \in \gamma(a, b) \cap (E \setminus S(a)) \cap (E \setminus S(b)) \neq \emptyset.$$

Since $y_0 \in E \setminus S(z) \Leftrightarrow y_0 \notin S(z) \Leftrightarrow z \notin S^-(y_0)$ for $z = a, b$, we have $\{a, b\} \subset D \setminus S^-(y_0)$. Since

$$y_0 \in \gamma(a, b) \subset \mathrm{co}_\gamma(D \setminus S^-(y_0)) \subset E \setminus T^-(y_0)$$

by (2.2), we have $y_0 \notin T^-(y_0)$ or $y_0 \notin T(y_0)$, which violates (2.3).

(II) $\Longrightarrow$ (III). For each $z \in D$, let $S(z) := \{y \in E \mid (z, y) \in A\}$. Then (3.1) $\Rightarrow$ (2.1). Moreover, for each $x \in E$, let $T(x) := \{y \in E \mid (x, y) \in B\}$. Then (3.2) $\Rightarrow$ (2.2). Further (3.3) $\Rightarrow$ (2.3). Therefore, by (II), we have

$$\bigcap_{z \in D} S(z) = \bigcap_{z \in D} \{y \in E \mid (z, y) \in A\} \neq \emptyset.$$

Hence there exists an $x_0 \in E$ such that $(z, x_0) \in A$ for all $z \in D$; that is, $D \times \{x_0\} \subset A$.

(III) $\Longrightarrow$ (IV). Consider (III) replacing (A, B) by their respective complements (A^c, B^c). Then (3.1) and (3.2) are satisfied by (4.1) and (4.2). Since (4.3) is the negation of the conclusion of (III), we should have the negation of (3.3). Therefore, the conclusion follows.

(IV) $\Longrightarrow$ (V). Let A and B be the graphs of S and T, resp. Then (5.1) - (5.3) imply (4.1) - (4.3). Therefore, by (IV), there exists an $x_0 \in E$ such that $(x_0, x_0) \in B$, that is, T has a fixed point $x_0 \in E$.

(V) $\Longrightarrow$ (VI). Suppose that $S(x) \neq \emptyset$ for all $x \in E$. Then E is covered by $S^-(z)$'s, $z \in D$. Consider (V) replacing S, T by S^-, T^-, resp. Then all of the requirements of (V) are satisfied. Therefore, there exists an $x_0 \in E$ such that $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. But this violates (6.3).

(VI) $\Longrightarrow$ (0). Let $G : D \multimap E$ be a 2-KKM map with $\mathcal{F}(E)$ values. Suppose that $\bigcap_{z \in D} G(z) = \emptyset$. Define a map $S : E \multimap D$ by $S^-(z) := E \setminus G(z)$ for $z \in D$ and a map $T : E \multimap E$ by $\mathrm{co}_\gamma S(x) = T(x)$ for each $x \in E$. Then the requirements (6.1) and (6.2) hold. Moreover, $\bigcap_{z\in D} G(z) = \emptyset$ implies that $E = \bigcup_{z\in D} S^-(z)$, which violates the conclusion of (VI). Hence there exists an $x_0 \in E$ such that $x_0 \in T(x_0)=$ $\mathrm{co}_\gamma S(x_0)$. So, there exist $a, b \in S(x_0)$ such that $x_0 \in \gamma(a, b) \subset \mathrm{co}_\gamma S(x_0)$. Therefore, $x_0 \in S^-(a)\cap S^-(b)$ or $x_0 \notin G(a)\cup G(b)$. Hence $\Gamma\{a, b\} \not\subset G(a)\cup G(b)$ and G is not a 2-KKM map, a contradiction. Therefore $(E, D; \gamma)$ satisfies the KKM principle (0).

(0) $\Longrightarrow$ (VII). Let $G(z) := \{y \in E \mid \phi(z, y) \in A\} \in \mathcal{F}(E)$ for $z \in D$ by (7.2). We show that (7.1) implies that G is a 2-KKM map. Suppose, on the contrary, that there exist $a, b \in D$ such that $\gamma(a, b) \not\subset G(a) \cup G(b)$. Choose a $y \in \gamma(a, b)$ such that $y \notin G(a)\cup G(b)$, whence $\phi(z, y) \notin A$ for $z = a, b$. This contradicts (7.1). Therefore, by (0), there exists a $y_0 \in E$ such that $y_0 \in G(z)$ for all $z \in D$, that is, $\phi(z, y_0) \in A$ for all $z \in D$.

(VII) $\Longrightarrow$ (VIII). Put $A := [-\infty, \alpha]$ and $C := \overline{\mathbb{R}}$. Then (i) follows from (VII) and it is clear that (ii) follows from (i).

(VIII) $\Longrightarrow$ (0). Define $\phi : D \times E \to \overline{\mathbb{R}}$ by

$$\phi(z, y) := \begin{cases} 0 & \text{if } y \in G(z); \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times E$. Put $\alpha = 0$ in (VIII). Since $G : D \to \mathcal{F}(E)$ is a 2-KKM map, we have (8.1). In fact, for each $a, b \in D$ and a $y \in \gamma(a, b)$, we have $y \in \gamma(a, b) \subset$ $G(a) \cup G(b)$ and hence $y \in G(a)$ or $y \in G(b)$. Then $\phi(a, y) = 0$ or $\phi(b, y) = 0$, that is, $\min\{\phi(z, y) \mid z = a, b\} = 0$. Hence (8.1) holds. Note (8.2) holds since $\{y \in E \mid \phi(z, y) \le 0\} = G(z) \in \mathcal{F}(E)$ for each $z \in D$. Therefore, by (VIII), there exists a $y_0 \in E$ such that $\phi(z, y_0) = 0$ for all $z \in D$; that is, $y_0 \in \bigcap\{G(z) \mid z \in D\}$. Therefore (0) holds.

(VII) $\Longrightarrow$ (IX). Assume that (b) does not hold, that is, $g(x, x) \in B$ for all $x \in E$. Then (9.1) implies (7.1). In fact, suppose that there exist $a, b \in D$ and a $y \in \gamma\{a, b\}$ such that $f(a, y), f(b, y) \notin A$, that is, $a, b \in \{z \in D \mid f(z, y) \notin A\}$. Hence, by (9.1), we have $\gamma(a, b) \subset \{x \in E \mid g(x, y) \notin B\}$. Since $y \in \gamma(a, b)$, we have $g(y, y) \notin B$, which contradicts our assumption. Now, by (VII) with $\phi = f$, we have the conclusion (a).

(IX) $\Longrightarrow$ (X). Put $C := \overline{\mathbb{R}}$, $A := [-\infty, \alpha]$, and $B := [-\infty, \beta]$ in (IX).

(X) $\Longrightarrow$ (XI). Clear.

(XI) $\Longrightarrow$ (II). Define real-valued functions $f : D \times E \to \mathbb{R}$ and $g : E \times E \to \mathbb{R}$ by

$$f(z, y) := \begin{cases} 0 & \text{if } \ y \in S(z); \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times E$ and

$$g(x, y) = \begin{cases} 0 & \text{if } \ y \in T(x); \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in E \times E$. Put $\alpha = \beta = 0$. Then (2.1) clearly implies (10.2). We claim that (2.2) implies (10.1). In fact, for any $y \in E$, $a, b \in \{z \in D \mid f(z, y) > 0\} = \{z \in$ $D \mid f(z, y) = 1\} = D \setminus S^-(y)$ implies $\gamma(a, b) \subset E \setminus T^-(y) = \{x \in E \mid g(x, y) = 1\} =$

$\{x \in E \mid g(x,y) > 0\}$. Hence (10.1) holds. Moreover, (2.3) implies $g(x,x) = 0$ for all $x \in E$ by the definition of g, that is, $\sup\{g(x,x) \mid x \in E\} = 0$. Therefore, by (XI), there exists a $y_0 \in E$ such that

$$f(z, y_0) \leq \sup_{x \in E} g(x,x) = 0 \quad \text{for all } z \in D.$$

Hence $f(z, y_0) = 0$ for all $z \in D$, that is, $y_0 \in S(z)$ for all $z \in D$. Therefore,

$$\bigcap \{S(z) \mid z \in D\} \neq \emptyset.$$

This completes our proof of Theorem 4.1. □

5. Applications of the particular 2-KKM principle

In this section, for the particular case $X := E = D$, we consider further properties of γ-convex spaces $(X;\gamma)$ satisfying the 2-KKM principle. We are mainly concerned with minimax inequalities and variational inequalities.

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X : f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

Similarly, for a γ-convex spaces $(X;\gamma)$ and a family $\mathcal{F}(X)$ of nonempty subsets of E, a function $f : X \to \overline{\mathbb{R}}$ is said to be $\mathcal{F}(X)$-*l.s.c.* if $\{x \in X \mid f(x) \leq r\} \in \mathcal{F}(X)$ for each $r \in \overline{\mathbb{R}}$. Alternatively, $\mathcal{F}(X)$-*u.s.c.* is also defined.

For a γ-convex space $(X \supset D;\gamma)$, an extended real-valued function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is γ-convex for each $r \in \overline{\mathbb{R}}$.

Consider the following statements for γ-convex spaces $(X;\gamma)$ satisfying the 2-KKM principle (0) with respect to a family $\mathcal{F}(X)$ of nonempty subsets of X:

(XII) Minimax inequality. *Let $f, g : X \times X \to \overline{\mathbb{R}}$ be extended real-valued functions and $\alpha, \beta \in \overline{\mathbb{R}}$. Suppose that*

(12.1) *for each $y \in X$, $\mathrm{co}_\gamma\{x \in X \mid f(x,y) > \alpha\} \subset \{x \in X \mid g(x,y) > \beta\}$, and*

(12.2) *for each $x \in X$, $\{y \in X \mid f(x,y) \leq \alpha\} \in \mathcal{F}(X)$.*

Then (a) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \leq \alpha \quad \text{for all } x \in X; \text{ and}$$

(b) *we have the following minimax inequality*

$$\inf_{y \in X} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} g(x,x).$$

(XIII) Minimax inequality. *Let $f, g : X \times X \to \overline{\mathbb{R}}$ be functions and $\alpha \in \mathbb{R}$ such that*

(13.1) *for any $x, y \in X$, $f(x,y) \leq g(x,y)$ and $g(x,x) \leq \alpha$;*

(13.2) *for each $x \in X$, $\{y \in X \mid f(x,y) \leq \alpha\} \in \mathcal{F}(X)$; and*

(13.3) *for each $y \in X$, $\{x \in X \mid g(x,y) > \alpha\}$ is γ-convex on X.*

Then (i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \leq \alpha \quad \text{for all } x \in X;$$

(ii) *if* $\alpha := \sup_{x\in X} g(x,x)$, *then*

$$\inf_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} g(x,x).$$

(XIV) Minimax inequality. *Let* $f, g : X \times X \to \overline{\mathbb{R}}$ *be functions such that*

(14.1) $f(x,y) \le g(x,y)$ *for each* $(x,y) \in X \times X$,

(14.2) *for each* $x \in X$, $f(x,\cdot)$ *is* $\mathcal{F}(X)$*-l.s.c.; and*

(14.3) *for each* $y \in X$, $g(\cdot,y)$ *is quasiconcave on* E.

Then we have

$$\inf_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} g(x,x).$$

(XV) Variational inequality. *Let* $p, q : X \times X \to \overline{\mathbb{R}}$ *and* $h : X \to \mathbb{R}$ *functions satisfying*

(15.1) $p(x,y) \le q(x,y)$ *for each* $(x,y) \in X \times X$, *and* $q(x,x) \le 0$ *for all* $x \in X$;

(15.2) *for each* $x \in X$, $p(x,\cdot) + h(\cdot)$ *is* $\mathcal{F}(X)$*-l.s.c.; and*

(15.3) *for each* $y \in X$, $q(\cdot,y) - h(\cdot)$ *is quasiconcave on* X.

Then there exists a $y_0 \in X$ *such that*

$$p(x,y_0) + h(y_0) \le h(x) \quad \textit{for all } x \in X.$$

(XVI) Variational inequality. *Let* $f, g : X \times X \to \overline{\mathbb{R}}$ *be functions satisfying*

(16.1) *for any* $x, y \in X$, $f(x,y) \le g(x,y)$;

(16.2) *for each* $x \in X$, $\{y \in X \mid f(x,y) \le f(y,y)\} \in \mathcal{F}(X)$; *and*

(16.3) *for each* $y \in X$, $\{x \in X \mid g(x,y) < g(y,y)\}$ *is is* γ*-convex.*

Then (i) *there exists a* $y_0 \in X$ *such that*

$$f(x,y_0) \ge f(y_0,y_0) \quad \textit{for all } x \in X; \textit{ and}$$

(ii) *we have*

$$\sup_{y\in X}\inf_{x\in X} f(x,y) \ge \inf_{x\in X} f(x,x).$$

(XVII) Variational inequality. *Let* $f, g : X \times X \to \mathbb{R}$ *be functions satisfying*

(17.1) $f \le g$ *on the diagonal* $\Delta := \{(x,x) \mid x \in X\}$ *and* $g \le f$ *on* $(X \times X) \setminus \Delta$;

(17.2) *for each* $x \in X$, $y \mapsto g(y,y) - g(x,y)$ *is* $\mathcal{F}(X)$*-l.s.c. on* X*; and*

(17.3) *for each* $y \in X$, $x \mapsto f(x,y)$ *is quasiconcave on* X.

Then there exists a $y_0 \in X$ *such that*

$$f(y_0,y_0) \ge f(x,y_0) \quad \textit{for all } x \in X.$$

Theorem 5.1. (1) *For a* γ*-convex space* $(X;\gamma)$ *satisfying the 2-KKM principle* (0) *with* $X := E = D$, *the statements* (XII)-(XVII) *hold.*

(2) *For a hyperconvex* γ*-metric space* $(X;\gamma)$ *and the admissible class* $\mathcal{F}(X) := \mathcal{A}(X)$, (XII)-(XVII) *hold.*

Proof. (XII) The particular case (XI) for $E = D = X$.

(XIII) $\Longleftarrow$ (XII). Note that (13.1) and (13.3) imply (12.1) and that (13.2) implies (12.2). Therefore, by (XII) with $\alpha = \beta$, we have the conclusion (i). If

$\sup_{x\in X} g(x,x) = +\infty$, then the inequality in the conclusion (ii) holds automatically. If $\alpha = \sup_{x\in X} g(x,x) < +\infty$, then by (XII), we have the conclusion (ii).

(XIV) $\Longleftarrow$ (XIII). Clear.

(XV) $\Longleftarrow$ (XIV). Let

$$f(x,y) := p(x,y) + h(y) - h(x), \; g(x,y) := q(x,y) + h(y) - h(x)$$

for $(x,y) \in X \times X$. Then f and g satisfy the requirements of (XIII). Furthermore, $g(x,x) = q(x,x) \le 0$ for all $x \in X$. Therefore, by (XIV), the conclusion follows.

(XVI) $\Longleftarrow$ (XIII). In (XIII), put $\alpha = 0$ and replace $f(x,y)$ and $g(x,y)$ by $f(y,y) - f(x,y)$ and $g(y,y) - g(x,y)$, resp. Then we have the conclusion.

(XVII) $\Longleftarrow$ (XVI). Define $p, q : X \times X \to \mathbb{R}$ by

$$p(x,y) := f(y,y) - f(x,y), \; q(x,y) := g(y,y) - g(x,y).$$

Then (1) $p(x,y) \le q(x,y)$ and $q(x,x) = 0$ for all $x, y \in X$ by (17.1). Moreover (2) for each $x \in X$, $\{y \in X \mid p(x,y) \le p(y,y)\} = \{y \in X \mid f(y,y) \le f(x,y)\} \in \mathcal{F}(X)$ by (17.2). Further, for each $y \in X$, $\{x \in X \mid q(x,y) < q(y,y)\} = \{x \in X \mid g(y,y) < g(x,y)\}$ is γ-convex by (17.3). Therefore, by (XVI) with (p,q) instead of (f,g), we have a $y_0 \in X$ such that $p(x,y_0) \ge p(y_0,y_0)$ for all $x \in X$. Note that

$$\begin{aligned} p(x,y_0) \ge p(y_0,y_0) \;&\Leftrightarrow\; f(y_0,y_0) - f(x,y_0) \ge f(y_0,y_0) - f(y_0,y_0) = 0 \\ &\Leftrightarrow\; f(y_0,y_0) \ge f(x,y_0) \text{ for all } x \in X. \end{aligned}$$

This completes the proof of Theorem 5.1. □

Remark. From (XV) or (XVII) we can obtain the Fan type best approximation theorems.

6. Further results related to the 2-KKM principle

In this section, we show that our results can be applied to those in [2–4]:

Theorem 6.1 (A fixed point theorem). *For any γ-convex space $(E \supset D; \gamma)$ satisfying the 2-KKM principle with respect to a family $\mathcal{F}(E)$, any 2-KKM map $G : D \to \mathcal{F}(E)$ has a fixed point.*

Proof. By the 2-KKM principle, we have $\bigcap_{z\in D} G(z) \neq \emptyset$. Hence any point in $\bigcap_{z\in D} G(z)$ is a fixed point of G. □

Corollary 6.2 ([2, Theorem 5]). *Let X be an admissible subset of a hyperconvex metric space M and $F : X \multimap X$ be a generalized 2-KKM map with admissible values. Then F has a fixed point.*

Proof. Note that X itself is a hyperconvex metric space by Lemma 3.4. Hence it satisfies the 2-KKM principle with respect to the admissible family $\mathcal{A}(X)$. Moreover, F is a 2-KKM map (see Example (2) just before Theorem 3.10). Now, by Theorem 6.1, F has a fixed point. □

Corollary 6.3 ([2, Corollary 1]; [4, Corollary 1]). *Let X be a nonempty subset of a hyperconvex metric space M and $F : X \to \mathcal{A}(X)$ be a generalized 2-KKM map. Then F has a fixed point.*

Proof. Note that any closed ball $B_X(x, \varepsilon)$ in X is $B(x, \varepsilon) \cap X$, the intersection of X and a closed ball in M. □

Theorem 6.4 (A minimax theorem)**.** *Let X be a hyperconvex metric space and $f : X \times X \to \overline{\mathbb{R}}$ be an extended real-valued function and $\alpha := \sup_{x\in X} \inf_{y\in X} f(x, y)$ such that*

(1) *for each $a, b \in X$ and $y \in \gamma(a, b)$, $\min\{f(a, y), f(b, y)\} \le \alpha$, and*

(2) *for each $x \in X$, $\{y \in X \mid f(x, y) \le \alpha\} \in \mathcal{A}(X)$.*

Then (i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \le \alpha \quad \text{for all } x \in X; \text{ and}$$

(ii) *we have the minimax equality:*

$$\inf_{y\in X} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in X} f(x, y).$$

Proof. Put $\alpha := \sup_{x\in X} \inf_{y\in X} f(x, y)$ in (VIII). Then we have

$$\inf_{y\in X} \sup_{x\in X} f(x, y) \le \sup_{x\in X} \inf_{y\in X} f(x, y).$$

Since

$$\inf_{y\in X} \sup_{x\in X} f(x, y) \ge \sup_{x\in X} \inf_{y\in X} f(x, y)$$

is trivially true, we have the equality. □

This is a correct generalized form of [2, Theorem 8].

Remark. Theorem 5-8 in [2] are main results and they are generalized by Theorem 4.1, (V), (X), and Theorem 6.4, resp. For the results in [3, 4], we may apply our method and obtain new correct general versions. For example, (V) contains [4, Theorems 4 and 6] and (X) corrects [4, Theorem 8]. But we will stop here.

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