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AN ESTIMATION OF EXACT PENALTY IN CONSTRAINED OPTIMIZATION

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Abstract. We use the penalty approach in order to study constrained minimization problems in infinite dimensional spaces. A penalty function is said to have the exact penalty property if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem. In our recent work we established the exact penalty property for a large class of inequality-constrained minimization problems. In the present paper we improve this result and obtain an estimation of the exact penalty.

1. Introduction

Penalty methods are an important and useful tool in constrained optimization. See, for example, [1-12, 14-28] and the references mentioned there. In this paper we use the penalty approach in order to study constrained minimization problems in infinite dimensional spaces. A penalty function is said to have the exact penalty property [3, 5, 11, 17] if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem.

The notion of exact penalization was introduced by Eremin [14] and Zangwill [26] for use in the development of algorithms for nonlinear constrained optimization. Since that time, exact penalty functions have continued to play a key role in the theory of mathematical programming [4, 6, 16, 18-21]. For a detailed historical review of the literature on exact penalization see [3, 5, 11].

In [28] and here we study the exact penalty property for a large class of inequality-constrained minimization problems

$$f(x) \to \min \text{ subject to } x \in A$$

where

$$A = \{x \in X : \ g_i(x) \le c_i \text{ for } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, the constraint functions g_i, $i = 1, \dots, n$ are convex and lower semicontinuous and the objective function f belongs to a class of functions described in Section 2. It is shown in [28] that this class of objective functions is a convex cone in the vector space of all functions on X. It includes the set of all convex bounded from below semicontinuous functions $f : X \to R^1$ which satisfy the growth condition $\lim_{||x|| \to \infty} f(x) = \infty$ and the set of all functions f on X which satisfy the growth condition above and which are

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Lipschitzian on all bounded subsets of X. It should be mentioned that if f belongs to this class of functions and $g : R^1 \to R^1$ is an increasing Lipschitzian function, then $g \circ f$ also belongs to it.

We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \to \min, \ z \in X$$

where $\gamma > 0$ is a penalty. In [28] we established the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. This is a novel approach in the penalty type methods.

Consider a minimization problem $h(z) \to \min$, $z \in X$ where $h : X \to R^1$ is a lower semicontinuous bounded from below function. If the space X is infinite-dimensional, then the existence of solutions of the problem is not guaranteed and in this situation we consider δ-approximate solutions. Namely, $x \in X$ is a δ-approximate solution of the problem $h(z) \to \min$, $z \in X$, where $\delta > 0$, if $h(x) \le \inf\{h(z) : \ z \in X\} + \delta$.

Since in this paper and in [28] we consider minimization problems in a general Banach space the existence of their solutions is not guaranteed. Therefore we are interested in approximate solutions of the unconstrained penalized problem and in approximate solutions of the corresponding constrained problem. In [28] under a mild assumption (see (2.5)) we establish (see Theorem 2.1 of the present paper) the existence of a constant $\Lambda_0 > 0$ such that the following property holds:

For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on ϵ such that if x is a $\delta(\epsilon)$-approximate solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0, then there exists an ϵ-approximate solution y of the corresponding constrained problem such that $||y - x|| \le \epsilon$.

It was shown in [28] that this property implies that any exact solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0, is an exact solution of the corresponding constrained problem. Therefore the result obtained in [28] also includes the classical penalty result as a special case. In the present paper we improve the main result of [28] and obtain an estimation of the exact penalty Λ_0.

2. The main result

We use the convention that $\lambda \cdot \infty = \infty$ for all $\lambda \in (0, \infty)$, $\lambda + \infty = \infty$ and $\max\{\lambda, \infty\} = \infty$ for any real number λ and that supremum over empty set is $-\infty$. For each real number λ put $\lambda_+ = \max\{\lambda, 0\}$.

We use the following notation and definitions.

Let $(X, ||\cdot||)$ be a Banach space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \le r\}.$$

For each function $f : X \to R^1 \cup \{\infty\}$ and each nonempty set $A \subset X$ put

$$\mathrm{dom}(f) = \{x \in X : \ f(x) < \infty\},$$

$$\inf(f) = \inf\{f(z) : \ z \in X\}$$

and

$$\inf(f;A) = \inf\{f(z) :\ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ set

$$d(x,B) = \inf\{||x-y|| :\ y \in B\}. \tag{2.1}$$

Let n be a natural number. For each $\kappa \in (0,1)$ denote by Ω_κ the set of all $\gamma = (\gamma_1,\dots,\gamma_n) \in R^n$ such that

$$\kappa \le \min\{\gamma_i :\ i = 1,\dots,n\} \text{ and } \max\{\gamma_i :\ i = 1,\dots,n\} = 1. \tag{2.2}$$

Let $g_i : X \to R^1 \cup \{\infty\}$, $i = 1,\dots,n$ be convex lower semicontinuous functions and $c = (c_1,\dots,c_n) \in R^n$. Set

$$A = \{x \in X :\ g_i(x) \le c_i \text{ for all } i = 1,\dots,n\}. \tag{2.3}$$

Let $f : X \to R^1 \cup \{\infty\}$ be a bounded from below lower semicontinuous function which satisfies the following growth condition

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2.4}$$

We suppose that there is $\tilde{x} \in X$ such that

$$g_j(\tilde{x}) < c_j \text{ for all } j = 1,\dots,n \text{ and } f(\tilde{x}) < \infty. \tag{2.5}$$

In this paper we consider the following constrained minimization problem

$$f(x) \to \min \text{ subject to } x \in A. \tag{P}$$

In view of (2.5) $A \neq \emptyset$ and $\inf(f;A) < \infty$.

For each vector $\gamma = (\gamma_1,\dots,\gamma_n) \in (0,\infty)^n$ define

$$\psi_\gamma(z) = f(z) + \sum_{i=1}^n \gamma_i \max\{g_i(z) - c_i, 0\},\ z \in X. \tag{2.6}$$

Clearly for each $\gamma \in (0,\infty)^n$ the function $\psi_\gamma : X \to R^1 \cup \{\infty\}$ is bounded from below and lower semicontinuous and satisfies $\inf(\psi_\gamma) < \infty$. We associate with problem (P) the corresponding family of unconstrained minimization problems

$$\psi_\gamma(z) \to \min,\ z \in X \tag{P_γ}$$

where $\gamma \in (0,\infty)^n$.

In this paper we assume that there exists a function $h : X \times \mathrm{dom}(f) \to R^1 \cup \{\infty\}$ such that the following assumptions hold:

(A1) $h(z,y)$ is finite for each $y, z \in \mathrm{dom}(f)$ and $h(y,y) = 0$ for each $y \in \mathrm{dom}(f)$.

(A2) For each $y \in \mathrm{dom}(f)$ the function $h(\cdot,y) \to R^1 \cup \{\infty\}$ is convex.

(A3) For each $z \in \mathrm{dom}(f)$ and each $r > 0$

$$\sup\{h(z,y) :\ y \in \mathrm{dom}(f) \cap B(0,r)\} < \infty.$$

(A4) For each $M > 0$ there exists $M_1 > 0$ such that for each $y \in X$ satisfying $f(y) \le M$ there exists a neighborhood V of y in X such that if $z \in V$, then

$$f(z) - f(y) \le M_1 h(z,y).$$

Remark 2.1. Note that if f is convex, then assumptions (A1)-(A4) hold with $h(z, y) = f(z) - f(y)$, $z \in X$, $y \in \operatorname{dom}(f)$. In this case $M_1 = 1$ for all $M > 0$. If the function f is finite-valued and Lipschitzian on all bounded subsets of X, then assumptions (A1)-(A4) hold with $h(z, y) = ||z - y||$ for all $z, y \in X$.

Let $\kappa \in (0, 1)$. The main result of [28] (Theorem 2.1 stated below) imply that if λ is sufficiently large, then any solution of problem $(P_{\lambda\gamma})$ with $\gamma \in \Omega_\kappa$ is a solution of problem (P). Note that if the space X is infinite-dimensional, then the existence of solutions of problems $(P_{\lambda\gamma})$ and (P) is not guaranteed. In this case Theorem 2.1 implies that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on ϵ such that the following property holds:

If $\lambda \geq \Lambda_0$, $\gamma \in \Omega_\kappa$ and if x is a δ-approximate solution of $(P_{\lambda\gamma})$, then there exists an ϵ-approximate solution y of (P) such that $||y - x|| \leq \epsilon$.

Here Λ_0 is a positive constant which does not depend on ϵ.

It should be mentioned that we deal with penalty functions whose penalty parameters for constraints $g_1, \dots, g_n$ are $\lambda\gamma_1, \dots, \lambda\gamma_n$ respectively, where $\lambda > 0$ and $(\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$ for a given $\kappa \in (0, 1)$. Note that the vector $(1, 1, \dots, 1) \in \Omega_\kappa$ for any $\kappa \in (0, 1)$. Therefore our results also includes the case $\gamma_1 = \cdots = \gamma_n = 1$ where one single parameter λ is used for all constraints. Note that sometimes it is an advantage from numerical consideration to use penalty coefficients $\lambda\gamma_1, \dots, \lambda\gamma_n$ with different parameters γ_i, $i = 1, \dots, n$. For example, in the case when some of the constrained functions are very "small" and some of the constraint functions are very "large".

The next theorem is the main result of [28].

Theorem 2.1. *Let $\kappa \in (0, 1)$. Then there exists a positive number Λ_0 such that for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

If $\gamma \in \Omega_\kappa$, $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$\psi_{\lambda\gamma}(x) \leq \inf(\psi_{\lambda\gamma}) + \delta,$$

then there exists $y \in A$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; A) + \epsilon.$$

Note that Theorem 2.1 is just an existence result and it does not provide any estimation of the constant Λ_0. In the present paper we improve Theorem 2.1 and obtain an estimation of the exact penalty Λ_0.

By (2.4) and (2.5) there is $M > 0$ such that

$$\text{if } y \in X \text{ satisfies } f(y) \leq |f(\tilde{x})| + 1, \text{ then } ||y|| < M. \tag{2.7}$$

In view of (2.7),

$$||\tilde{x}|| < M. \tag{2.8}$$

By (A4) there is $M_1 > 0$ such that the following property holds:

(P1) for each $y \in X$ satisfying $f(y) \leq |f(\tilde{x})| + 1$ there is a neighborhood V of y in X such that $f(z) - f(y) \leq M_1 h(z, y)$ for all $z \in V$.

It follows from (2.4), (2.5) and (A3) that there is $M_2 > 0$ such that

$$\sup\{h(\tilde{x}, z) : \ z \in X \text{ and } f(z) \leq f(\tilde{x}) + 1\} \leq M_2.$$

Remark 2.2. If the function f is convex, then in view of Remark 2.1, we choose $h(z, y) = f(z) - f(y)$ for all $z \in X$ and all $y \in \mathrm{dom}(f)$ with $M_1 = 1$ for all $M > 0$ and then

$$\sup\{h(\tilde{x}, z) : \; z \in X \text{ and } f(z) \le f(\tilde{x}) + 1\}$$
$$\le \sup\{f(\tilde{x}) - f(z) : \; z \in X \text{ and } f(z) \le f(\tilde{x}) + 1\} = f(\tilde{x}) - \inf(f).$$

Thus in this case M_2 can be any positive number such that $M_2 \ge f(\tilde{x}) - \inf(f)$.

If the function f is finite-valued and Lipschitzian on bounded subests of X, then in view of Remark 2.1, we choose $h(z, y) = ||z - y||$ for all $z, y \in X$ and M_1 is a Lipschitz constant of the restriction of f to $B(0, M)$. In this case

$$\sup\{h(\tilde{x}, z) : \; z \in X \text{ and } f(z) \le f(\tilde{x}) + 1\} \le \sup\{||\tilde{x} - z|| : \; z \in B(0, M)\} \le 2M$$

and $M_2 = M$.

Let $\kappa \in (0, 1)$ Choose $\Lambda_0 > 1$ such that

$$\kappa \sum_{i=1}^{n} (c_i - g_i(\tilde{x})) > \max\{2\Lambda_0^{-1} M_1 M_2, \; 8\Lambda_0^{-2} M^2\}. \tag{2.9}$$

We will prove the following result.

Theorem 2.2. *For each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

If $\gamma \in \Omega_\kappa$, $\lambda \ge \Lambda_0$ and if $x \in X$ satisfies

$$\psi_{\lambda\gamma}(x) \le \inf(\psi_{\lambda\gamma}) + \delta,$$

then there exists $y \in A$ such that

$$||y - x|| \le \epsilon \text{ and } f(y) \le \inf(f; A) + \epsilon.$$

3. Proof of Theorem 2.2

We show that the following property holds:

(P2) For each $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon)$ such that for each $\lambda \ge \Lambda_0$, each $\gamma \in \Omega_\kappa$ and each $x \in X$ which satisfies

$$\psi_{\lambda\gamma}(x) \le \inf(\psi_{\lambda\gamma}) + \delta$$

there is $y \in A$ for which

$$||y - x|| \le \epsilon \text{ and } \psi_{\lambda\gamma}(y) \le \psi_{\lambda\gamma}(x).$$

(It is easy to see that (P2) implies the validity of Theorem 2.2).

Assume the contrary. Then there exist

$$\epsilon \in (0, 1), \; \gamma = (\gamma_1, \ldots, \gamma_n) \in \Omega_\kappa, \; \lambda \ge \Lambda_0 \text{ and } \bar{x} \in X \tag{3.1}$$

such that

$$\psi_{\lambda\gamma}(\bar{x}) \le \inf(\psi_{\lambda\gamma}) + 2^{-1}\epsilon\Lambda_0^{-1} \tag{3.2}$$

and

$$\{y \in B(\bar{x}, \epsilon) \cap A : \; \psi_{\lambda\gamma}(y) \le \psi_{\lambda\gamma}(\bar{x})\} = \emptyset. \tag{3.3}$$

It follows from (3.2) and Ekeland's variational principle [13] that there is $\bar{y} \in X$ such that

$$\psi_{\lambda\gamma}(\bar{y}) \le \psi_{\lambda\gamma}(\bar{x}), \tag{3.4}$$

$$||\bar{y} - \bar{x}|| \le 2^{-1}\epsilon \tag{3.5}$$

and

$$\psi_{\lambda\gamma}(\bar{y}) \le \psi_{\lambda\gamma}(z) + \Lambda_0^{-1}||z - \bar{y}|| \text{ for all } z \in X. \tag{3.6}$$

By (3.3)-(3.5),

$$\bar{y} \notin A. \tag{3.7}$$

Set

$$\begin{aligned} I_1 &= \{i \in \{1, \dots, n\} : \ g_i(\bar{y}) > c_i\}, \\ I_2 &= \{i \in \{1, \dots, n\} : \ g_i(\bar{y}) = c_i\}, \\ I_3 &= \{i \in \{1, \dots, n\} : \ g_i(\bar{y}) < c_i\}. \end{aligned} \tag{3.8}$$

In view of (3.7) and (3.8),

$$I_1 \ne \emptyset. \tag{3.9}$$

It follows from (2.3), (2.6), (3.1), (3.2), (3.4), (3.8) and (3.9) that

$$\inf\{f(z) : \ z \in A\} = \inf\{\psi_{\lambda\gamma}(z) : \ z \in A\} \ge \inf(\psi_{\lambda\gamma})$$

$$\ge \psi_{\lambda\gamma}(\bar{x}) - 1 \ge \psi_{\lambda\gamma}(\bar{y}) - 1 = f(\bar{y}) + \sum_{i \in I_1} \lambda\gamma_i(g_i(\bar{y}) - c_i) - 1. \tag{3.10}$$

Together with (3.8) and (2.5) this relation implies that

$$f(\bar{y}) \le \inf\{f(z) : \ z \in A\} + 1 \le f(\tilde{x}) + 1. \tag{3.11}$$

By (3.11) and (2.7),

$$||\bar{y}|| < M. \tag{3.12}$$

It follows from (P1), (3.11) and (3.12) that there exists an open neighborhood V of $\bar{y}$ in X such that

$$V \subset B(0, M), \tag{3.13}$$

$$f(z) - f(\bar{y}) \le M_1 h(z, \bar{y}) \text{ for each } z \in V. \tag{3.14}$$

Since the functions $g_i, \ i = 1, \dots, n$ are lower semicontinuous it follows from (3.8) that there exists a positive number $r < 1$ such that for each $y \in B(\bar{y}, r)$

$$g_i(y) > c_i \text{ for each } i \in I_1. \tag{3.15}$$

It follows from (3.11), (2.5), (3.4), (3.2), (3.15) and (3.6) that for each $z \in B(\bar{y}, r) \cap \mathrm{dom}(f)$

$$\sum_{i \in I_1} \lambda\gamma_i(g_i(z) - c_i) + \sum_{i \in I_2 \cup I_3} \lambda\gamma_i \max\{g_i(z) - c_i, 0\}$$

$$- \sum_{i \in I_1} \lambda\gamma_i(g_i(\bar{y}) - c_i) - \sum_{i \in I_2 \cup I_3} \lambda\gamma_i \max\{g_i(\bar{y}) - c_i, 0\}$$

$$= \psi_{\lambda\gamma}(z) - \psi_{\lambda\gamma}(\bar{y}) - f(z) + f(\bar{y}) \geq -\Lambda_0^{-1}||\bar{y} - z|| - f(z) + f(\bar{y}).$$

This inequality implies that for each $z \in B(\bar{y}, r)$

$$\sum_{i \in I_1} \gamma_i g_i(z) + \sum_{i \in I_2 \cup I_3} \gamma_i \max\{g_i(z) - c_i, 0\}$$

$$- \sum_{i \in I_1} \gamma_i g_i(\bar{y}) - \sum_{i \in I_2 \cup I_3} \gamma_i \max\{g_i(\bar{y}) - c_i, 0\}$$

$$+ \lambda^{-1}(f(z) - f(\bar{y})) \geq -\Lambda_0^{-2}||\bar{y} - z||.$$

In view of this inequality, (3.13) and (3.14) for each $z \in B(\bar{y}, r) \cap V$

$$\sum_{i \in I_1} \gamma_i g_i(z) + \sum_{i \in I_2 \cup I_3} \gamma_i \max\{g_i(z) - c_i, 0\} + \lambda^{-1} M_1 h(z, \bar{y}) + \Lambda_0^{-2}||z - \bar{y}|| \tag{3.16}$$

$$\geq \sum_{i \in I_1} \gamma_i g_i(\bar{y}) + \sum_{i \in I_2 \cup I_3} \gamma_i \max\{g_i(\bar{y}) - c_i, 0\}.$$

By (A2) the function

$$\sum_{i \in I_1} \gamma_i g_i(z) + \sum_{i \in I_2 \cup I_3} \gamma_i \max\{g_i(z) - c_i, 0\} + \lambda^{-1} M_1 h(z, \bar{y}) + \Lambda_0^{-2}||z - \bar{y}||, \ z \in X$$

is convex. Together with the equality $h(\bar{y}, \bar{y}) = 0$ (see (A1)) this implies that (3.16) is true for all $z \in X$.

Since (3.16) holds for $z = \tilde{x}$ it follows from (2.5), (3.1), (2.2), (3.1) and (3.8) that

$$\sum_{i \in I_1} \gamma_i g_i(\tilde{x}) + \lambda^{-1} M_1 h(\tilde{x}, \bar{y}) + \Lambda_0^{-2}||\tilde{x} - \bar{y}|| \geq \sum_{i \in I_1} \gamma_i g_i(\bar{y}) > \sum_{i \in I_1} \gamma_i c_i.$$

Together with (3.11), (2.8), (3.12), (3.1), (2.2) and the choice of M_2 (see Section 2) this implies that

$$4\Lambda_0^{-2} M^2 + \Lambda_0^{-1} M_1 \sup\{h(\tilde{x}, z) : \ z \in X \text{ and } f(z) \leq f(\tilde{x}) + 1\}$$

$$\geq \Lambda_0^{-2} 4M^2 + \Lambda_0^{-1} M_1 (h(\tilde{x}, \bar{y})_+) \geq \sum_{i \in I_1} \gamma_i (c_i - g_i(\tilde{x}))$$

$$\geq \kappa \sum_{i=1}^{n} (c_i - g_i(\tilde{x}))$$

and

$$\kappa \sum_{i=1}^{n} (c_i - g_i(\tilde{x})) \leq 4\Lambda_0^{-2} M^2 + \Lambda_0^{-1} M_1 M_2.$$

This contradicts (2.9). The contradiction we have reached proves Theorem 2.2.

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