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ON TYPE OF PERIODICITY AND ERGODICITY TO A CLASS OF INTEGRAL EQUATIONS WITH INFINITE DELAY

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Abstract. This work deals with new type of periodicity and ergodicity to a class of semilinear integral equations with infinite delay.

1. Introduction

Type of periodicity for evolutionary integral equations is one of most attractive topics in the qualitative theory of integral equations due to its mathematical interest and to their applications in physical science, economics, mathematical biology, engineering and many others subjects (see [12, 14, 15, 18, 19, 22, 28, 34, 47, 45, 46]).

Recently, to deal with delay equations and related topics, the concept of compact almost automorphic functions emerged (see [6, 21, 24, 25, 28]). The definition of compact almost automorphic functions was introduced by Fink [25] after previous work of Bochner, who introduced the concept of almost automorphic functions (see [9, 10]). Finally, the definition of pseudo compact almost automorphic functions was only recently introduced by Lizama and N'Guérérata in [42]. This notion is a generalization of pseudo almost automorphy introduced by Xiao et al. [41]. They established a general existence and uniqueness theorem of pseudo almost automorphic mild solutions to some abstract differential equations (see [3, 20, 36, 37, 38]).

Firstly, we study the existence of pseudo compact almost automorphic solution (see Definition 2.4) for the semilinear integral equations with infinite delay of the form

$$u(t) = \int_{-\infty}^{t} a(t-s)\big[Au(s) + f(s,u(s))\big]ds, \quad t \in \mathbb{R}, \tag{1.1}$$

where $a \in L^1([0,\infty))$, $A : D(A) \subset X \to X$ is the generator of an integral resolvent family defined on a complex Banach space X and $f : \mathbb{R} \times X \to X$ is a pseudo compact almost automorphic function in the first variable and satisfying suitable conditions in the second variable. We remark that equations of type (1.1) arise in the study of heat flow in materials of fading memory type (see [14, 45]); Cuevas and Lizama in [15] has proven the existence and uniqueness of an almost automorphic solution to equation (1.1) for each $f : \mathbb{R} \times X \to X$ almost automorphic in t,

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uniformly in $x \in X$, and satisfying diverse Lipschitz type conditions; Lizama and Henríquez in [28] has proven the existence of compact almost automorphic solutions to (1.1) for each perturbation $f(t,x)$ compact almost automorphic in t and satisfying suitable conditions in the second variable; the existence of pseudo compact almost automorphic solutions for abstract integral equations with infinite delay of type (1.1) remains an untreated topic in the literature; consequently should be widely investigated. The results obtained in this work can be considered the starting point toward such a direction. Some of them are extension of the results in [16, 28].

In 1980s, N'Guérérata [33] defined asymptotically almost automorphic functions as perturbation of almost automorphic functions by functions vanishing at infinite. Since then, those functions have generated lots of development and applications. There is an extensive literature on related topics, we refer the reader to [11, 23, 24]. As a general reference we quote the book [32].

Secondly, we use the machinery developed in Section 3 to give results on the existence of asymptotically compact almost automorphic solutions to the class of integral equations

$$u(t) = \int_0^t a(t-s)\big[Au(s) + f(s,u(s))\big]ds, \quad t \geq 0, \tag{1.2}$$

where a and A are as above, and $f : [0,\infty) \times X \to X$ is an asymptotically compact almost automorphic function. Observe that (1.1) can be viewed as the limiting equation of (1.2) see [47, Chapter III, Section 11.5] to obtain details on this assertion.

The literature concerning S-asymptotically ω-periodic functions with values in Banach spaces is very new and it is of interest in mathematics. Recently some interesting articles were published by Henríquez et al. [29, 30], Nicola and Pierri [40], Cuevas and de Souza [18, 19], de Andrade and Cuevas [5], de Andrade et al. [4], Caicedo and Cuevas [12]. In [16], Cuevas and Lizama have studied the existence of S-asymptotically ω-periodic solution to the semilinear Volterra equation

$$u'(t) = \int_0^t a(t-s)Au(s)ds + F(t,u(t)), \quad t \geq 0,$$

$$u(0) = u_0 \in X.$$

Since phase space approach is of interest by itself, among the classical result in this field one should mention Hino et al. [31]; we study the above problem to the abstract neutral integro-differential equation

$$\frac{d}{dt}D(t,u_t) = \int_0^t a(t-s)AD(s,u_s)ds + F(t,u_t), \quad t \geq 0, \tag{1.3}$$

$$u_0 = \phi \in \mathcal{B}, \tag{1.4}$$

where the history $u_t : (-\infty,0] \to X$ is defined by $u_t(\theta) = u(t+\theta)$, belongs to some abstract space $\mathcal{B}$ defined axiomatically; $D(t,\xi) = \xi(0) + G(t,\xi)$, $\xi \in \mathcal{B}$, and $F, G : [0,\infty) \times \mathcal{B} \to X$ are appropriate functions.

We will now present a summary of this work. The second section provides the definitions and preliminaries results to be used in theorems stated and proved in this article. In particular in Subsection 2.1, we recall the notion of asymptotically compact almost automorphy and pseudo compact almost automorphy. In Subsection 2.2 we review some of the standard properties of integral resolvent families, which are of fundamental importance in this work. It is well known that the study of composition of two functions with special properties is so important for deep investigations. In Subsection 2.3 we deal with results involving the convolution of strongly continuous function, associated to equation (1.1), with special type of functions. In subsection 2.4, we give a compactness criterion in $C_h(Z)$ (resp., $C_{h^*}(Z)$) (see Lemmas 2.14 and 2.15). In Subsection 2.5 (resp., Subsection 2.6) we present some of the basic facts on phase space (resp., S-asymptotically ω-periodic functions).

The third section is divided in two parts. In the first part, Subsection 3.1, we obtain sufficient conditions to existence and uniqueness of a pseudo compact almost automorphic mild solution of the linear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)\big[Au(s) + f(s)\big]ds, \quad t \in \mathbb{R}, \tag{1.5}$$

provided A is the generator of an integral resolvent family (see Theorem 3.1). We observe that under the assumptions that $a \in L^1([0,\infty)) \cap L^\infty([0,\infty))$ is completely positive and that $\int_0^\infty ta(t)dt < +\infty$, it was proved by Clément and Da Prato [14] that problem (1.5) is equivalent to

$$u(t) + \frac{d}{dt}\left(\alpha u(t) + \int_{-\infty}^{t} k(t-s)u(s)ds\right) = \left(\int_0^\infty a(\tau)d\tau\right)\big(Au(t) + f(t)\big), \; t \in \mathbb{R}, \tag{1.6}$$

for some $\alpha > 0$ and $k \in L^1([0,\infty))$ nonnegative and nonincreasing. Properties of the solutions of this linear integro-differential equation have been studied in several contexts, e. g. existence and regularity [13], maximal regularity [14], compact almost automorphy [28].

In the second part, we obtain very general results on the existence of pseudo compact almost automorphic mild solutions to the integral equation (1.1). Finally, in fourth section, we establish sufficient conditions for the existence of asymptotically compact almost automorphic (resp., S-asymptotically ω-periodic) solutions to the equation (1.2) (resp., equation (1.3)-(1.4)). The results in the present work are a contribution to the study of qualitative properties of the integral equations with infinite delay and they can open a new line of applications. It is worth noting that our assumptions are very natural and we have tested in the pratical context. The reader can see for instance Corollary 3.2, 3.8, 4.3, 4.15 and 4.18 as well as Examples 3.5 and 3.10 and other remarks in some of the authors previous papers.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this work. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two Banach spaces. The notation $\mathcal{B}(X,Y)$ stands for the space of bounded linear operators from X into Y endowed with the uniform operator topology, and we abbreviate to $\mathcal{B}(X)$, whenever $X = Y$.

2.1. **Compact almost automorphic type functions and ergodicity.** Let us describe the basic properties of the class of compact almost automorphic functions. The notation $C(\mathbb{R}, X)$ and $C_b(\mathbb{R}, X)$ stand for the collection of all continuous functions from $\mathbb{R}$ into X and the Banach space of all bounded continuous functions from $\mathbb{R}$ into X endowed with the uniform convergence topology, respectively. $C_0([0,\infty), X)$ denotes the Banach space of all bounded continuous functions from $[0,\infty)$ into X which vanishing at infinity equipped with sup norm in $[0,\infty)$. Similarly, $C_0([0,\infty)\times Y, X)$ denotes the space of all continuous functions $f : [0,\infty)\times Y \to X$ such that $\lim_{t\to\infty} f(t,x) = 0$ uniformly for x in any compact subset of Y.

Definition 2.1. (i) A continuous function $f : \mathbb{R} \to X$ is said to be compact almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n),$$

uniformly on compact subsets of $\mathbb{R}$ and the collection of such functions will be denoted by $AA_c(X)$.

(ii) A continuous function $f : \mathbb{R}\times Y \to X$ is said to be compact almost automorphic if $f(t,x)$ is compact almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of Y. Denote by $AA_c(Y;X)$ the set of all such functions.

Definition 2.2. A continuous function $f : [0,\infty) \to X$ (resp., $f : [0,\infty)\times Y \to X$) is said to be asymptotically compact almost automorphic if it admits a decomposition $f = g+\phi$ on $[0,\infty)$ where $g \in AA_c(X)$ (resp., $g \in AA_c(Y;X)$) and $\phi \in C_0([0,\infty), X)$ (resp., $\phi \in C_0([0,\infty)\times Y; X)$). Denote by $AAA_c(X)$ (resp., $AAA_c(Y;X)$) the set of all such functions.

Denote by $P_0(X)$ the set of all bounded continuous function $\xi : \mathbb{R} \to X$ which vanishing mean value, that is,

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\xi(t)\| dt = 0.$$

Example 2.3 ([37, Example 2.5])**.** *Let us consider the function* $\phi(t) = \max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^2}\}$, $t \in \mathbb{R}$. *We can see that* $\phi \in P_0(\mathbb{R})$. *In fact, for any* $r > 0$, *set* $l = [|\sqrt{r}|] + 1$. *Then we shall calculate the mean value of* ϕ *by elementary means*

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \phi(t)dt &= \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \max_{-l\le k\le l}\{e^{-(t\pm k^2)^2}\}dt \\ &\le \lim_{r\to\infty} \frac{l}{r}\int_{-\infty}^{\infty} e^{-t^2} dt \\ &= \lim_{r\to\infty} \frac{l\sqrt{\pi}}{2r} = 0. \end{aligned}$$

Similarly, we define $P_0(Y, X)$ as the collection of all bounded continuous functions $\xi : \mathbb{R} \times Y \to X$ satisfying

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\xi(t, y)\| dt = 0,$$

uniformly for y in any bounded subset of Y.

Definition 2.4. A bounded continuous function $f : \mathbb{R} \to X$ (resp., $f : \mathbb{R}\times Y \to X$) is said to be pseudo compact almost automorphic if it admits a decomposition $f = g + \phi$, where $g \in AA_c(X)$ (resp., $g \in AA_c(Y;X)$) and $\phi \in P_0(X)$ (resp., $\phi \in P_0(Y;X)$). Denote by $PAA_c(X)$ (resp., $PAA_c(Y;X)$) the set of all such functions. $PAA_c(X)$ is a Banach space with the sup norm.

We have the following continuous inclusions (see [42])

$$AA_c(X) \subset AAA_c(X) \subset PAA_c(X) \subset C_b(\mathbb{R}, X).$$

Definition 2.5. A continuous function $f : \mathbb{R} \times Y \to X$ (resp., $f : [0,\infty) \times Y \to X$) is called uniformly continuous on bounded sets K of Y uniformly for $t \in \mathbb{R}$ (resp., $t \in [0,\infty)$) if for every $\epsilon > 0$ and every bounded subset K of Y, there exists $\delta_{\epsilon,K} > 0$ such that $\|f(t, x) - f(t, y)\| \le \epsilon$ for all $t \in \mathbb{R}$ (resp., $t \in [0,\infty)$) and all $x, y \in K$ such that $\|x - y\| \le \delta_{\epsilon,K}$.

Lemma 2.6 ([22]). *Let $f \in AAA_c(Y, X)$ be given and let $f(t, y)$ be uniformly continuous in any bounded subset $K \subset Y$ uniformly for $t \in [0,\infty)$. If $u \in AAA_c(Y)$ then $f(\cdot, u(\cdot)) \in AAA_c(X)$.*

Lemma 2.7. *Let $f \in PAA_c(X, X)$ be given and assume that there exists a constant $L_f > 0$ such that*

$$\|f(t, x) - f(t, y)\| \le L_f\|x - y\|, \quad \forall\ t \in \mathbb{R},\ \forall\ x, y \in X. \tag{2.1}$$

If $h \in PAA_c(X)$ then $f(\cdot, h(\cdot)) \in PAA_c(X)$.

Lemma 2.8. *Let $f \in PAA_c(Y, X)$ be given and let $f(t, y)$ be uniformly continuous in any bounded subset $K \subset Y$ uniformly for $t \in \mathbb{R}$. If $u \in PAA_c(Y)$ then $f(\cdot, u(\cdot)) \in PAA_c(X)$.*

Proof. Since $f \in PAA_c(Y, X)$ and $u \in PAA_c(Y)$, we have by definition that $f = g + \xi$ and $u = u_1 + u_2$, where $g \in AA_c(Y, X)$, $\xi \in P_0(Y, X)$, $u_1 \in AA_c(Y)$ and $u_2 \in P_0(Y)$. Since f is bounded, $f(\cdot, u(\cdot)) \in C_b(\mathbb{R}, X)$. Now we decompose f as follows

$$f(\cdot, u(\cdot)) = g(\cdot, u_1(\cdot)) + f(\cdot, u(\cdot)) - f(\cdot, u_1(\cdot)) + \xi(\cdot, u_1(\cdot)).$$

By [22, Lemma 3.1] and [41, Theorem 2.2], we can see that $g(\cdot, u_1(\cdot)) \in AA_c(X)$. Thus, to prove our result is sufficient to show that $f(\cdot, u(\cdot))-f(\cdot, u_1(\cdot))$ and $\xi(\cdot, u_1(\cdot))$ belongs to $P_0(X)$. The proof of this assertion makes use of a similar argument used by Liang et al. in [37, Theorem 2.4, p. 1497], which ends the proof. □

2.2. **Integral resolvent family.** We recall that the Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is given by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

where the integral is absolutely convergent for $Re\lambda > \omega$. Furthermore, the notation $\rho(A)$ stands for the resolvent set of A. In order to establish an operator theoretical approach to equation (1.1), we consider the following definition (cf. [43]).

Definition 2.9. Let A be a closed linear operator with domain $D(A) \subseteq X$. We say that A is the generator of an integral resolvent if there exist $\omega \geq 0$ and a strongly continuous function $S : [0, \infty) \to \mathcal{B}(X)$ such that $\{1/\hat{a}(\lambda) : Re\lambda > \omega\} \subseteq \rho(A)$ and

$$(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt, \quad Re\lambda > \omega, \quad x \in X.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

Because of the uniqueness of the Laplace transform, an integral resolvent family with $a(t) \equiv 1$ is the same as a C_0-semigroup whereas that an integral resolvent family with $a(t) = t$ corresponds to the concept of sine family, see [8, Section 3.15].

We can establish several relations between the integral resolvent family and its generator (see [28, Proposition 2.2].

Proposition 2.10. *Let $S(t)$ be the integral resolvent family on X with generator A. Then the following properties hold:*

(a) *$S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.*

(b) *Let $x \in D(A)$ and $t \geq 0$. Then*

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds.$$

(c) *Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$ and*

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)xds.$$

In particular, $S(0) = a(0)I$.

2.3. **Miscellaneous.** In this subsection, we present some properties of convolution.

We introduce the following integrability assumption for strongly continuous functions $S : [0, \infty) \to \mathcal{B}(X)$.

(INT) There exists $\phi \in L^1([0, \infty))$ such that $\|S(t)\| \leq \phi(t)$ for all $t \in [0, \infty)$.

Remark 2.11. We note that conditions of type **(INT)** have been previously considered in the literature (see [28, 47]).

Lemma 2.12. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded linear operators that satisfies assumption* **(INT)**. *If $f \in PAA_c(X)$ and $w : \mathbb{R} \to X$ is given by*

$$w(t) = \int_{-\infty}^t S(t-s)f(s)ds.$$

Then $w \in PAA_c(X)$.

Proof. Let $f = g + \xi$ be the decomposition of f, where $g \in AA_c(X)$ and $\xi \in P_0(X)$. Then we can write

$$w(t) = \int_{-\infty}^{t} S(t-s)g(s)ds + \int_{-\infty}^{t} S(t-s)\xi(s)ds := G(t) + \Xi(t).$$

By [28, Lemma 3.1] we have that $G \in AA_c(X)$. To complete the proof, we show that $\Xi \in P_0(X)$. For $T > 0$ we see that

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left\|\int_{0}^{\infty} S(s)\xi(t-s)ds\right\| dt &\leq \frac{1}{2T}\int_{-T}^{T}\int_{0}^{\infty}\|S(s)\|\|\xi(t-s)\|dsdt \\ &\leq \int_{0}^{\infty}\phi(s)\left(\frac{1}{2T}\int_{-T}^{T}\|\xi(t-s)\|dt\right)ds \\ &= \int_{0}^{\infty}\Psi_T(s)ds,\end{aligned}$$

where $\Psi_T(s) = \frac{1}{2T}\int_{-T}^{T}\|\xi(t-s)\|dt$, $s \geq 0$. We can see that $\Psi_T(s) \to 0$ as $T \to \infty$. Next, using the Lebesgue dominated convergence it follows that

$$\frac{1}{2T}\int_{-T}^{T}\|\Xi(t)\|dt \to 0$$

as $T \to 0$. This finishes the proof. □

Lemma 2.13 ([35])**.** *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded linear operators that satisfies assumption* **(INT)**. *If $f \in AAA_c(X)$ and $\tilde{w} : [0,\infty) \to X$ is given by*

$$\tilde{w}(t) = \int_{0}^{t} S(t-s)f(s)ds.$$

Then $\tilde{w} \in AAA_c(X)$.

Proof. For the reader's convenience we give the proof. If $f = g + \nu$, where $g \in AA_c(X)$ and $\nu \in C_0([0,\infty), X)$. Then

$$\tilde{w}(t) = \int_{-\infty}^{t} S(t-s)g(s)ds - \int_{-\infty}^{0} S(t-s)g(s)ds + \int_{0}^{t} S(t-s)\nu(s)ds := G(t) + H(t),$$

where $G(t) = \int_{-\infty}^{t} S(t-s)g(s)ds$ and $H(t) = -\int_{-\infty}^{0} S(t-s)g(s)ds + \int_{0}^{t} S(t-s)\nu(s)ds$. By [28, Lemma 3.1], we have that $G \in AA_c(X)$. Next, let us show that $H \in C_0([0,\infty), X)$. Given $\epsilon > 0$ there exists a constant $L > 0$ such that $\int_{L}^{\infty}\phi(s)ds \leq \epsilon$ and $\|\nu(s)\| \leq \epsilon$ for all $s \geq L$. Then for all $t \geq 2L$, we deduce that

$$\begin{aligned}\|H(t)\| &\leq \int_{-\infty}^{0}\|S(t-s)\|\|g(s)\|ds + \int_{0}^{t}\|S(t-s)\|\|\nu(s)\|ds \\ &\leq \int_{-\infty}^{0}\phi(t-s)\|g(s)\|ds + \int_{t/2}^{t}\phi(t-s)\|\nu(s)\|ds + \int_{0}^{t/2}\phi(t-s)\|\nu(s)\|ds \\ &\leq \|g\|_\infty\int_{t}^{\infty}\phi(s)ds + \epsilon\int_{0}^{t/2}\phi(s)ds + \|\nu\|_\infty\int_{t/2}^{\infty}\phi(s)ds \\ &\leq (\|g\|_\infty + \|\phi\|_1 + \|\nu\|_\infty)\,\epsilon.\end{aligned}$$

Therefore, $\lim_{t\to\infty} H(t) = 0$, that is $H \in C_0([0,\infty), X)$. This completes the proof. □

2.4. **Two useful compactness criterion.** Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(X) = \{u \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0\}$$

endowed with the norm $\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$.

We need a very detailed knowledge of the relatively compact set in $C_h(X)$. We use the following result (see [28]).

Lemma 2.14. *A subset $K \subseteq C_h(X)$ is a relatively compact set if it verifies the following conditions:*

(c-1) *The set $K(t) = \{u(t) : u \in K\}$ is relatively compact in X for each $t \in \mathbb{R}$.*
(c-2) *The set K is equicontinuous.*
(c-3) *For each $\epsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leq \epsilon h(t)$ for all $u \in K$ and all $|t| > L$.*

Let $h^* : [0,\infty) \to [1,\infty)$ be a continuous function such that $h^*(t) \to \infty$ as $t \to \infty$. We consider the space $C_{h^*}(X) = \{u \in C([0,\infty), X) : \lim_{t\to\infty} \frac{u(t)}{h^*(t)} = 0\}$ endowed with the norm $\|u\|_{h^*} = \sup_{t\geq 0} \frac{\|u(t)\|}{h^*(t)}$.

Lemma 2.15 ([12])**.** *A subset $K \subseteq C_{h^*}(X)$ is a relatively compact set if it verifies the following conditions:*

(c-1)* *The set $K_b = \{u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0,b]; X)$ for all $b \geq 0$.*
(c-2)* $\lim_{t\to\infty} \frac{\|u(t)\|}{h^*(t)} = 0$ *uniformly for all $u \in K$.*

2.5. **Phase space axiomatic.** We employ the axiomatic definition of the phase space $\mathcal{B}$ introduced in Hino et al. [31]. Specifically, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm denoted $\| \cdot \|_{\mathcal{B}}$ and such that the following axioms hold:

(A) If $x : (-\infty, \sigma + a) \mapsto X$, $a > 0$, $\sigma \in \mathbb{R}$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

(i) $x_t \in \mathcal{B}$;
(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$;
(iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma\leq s\leq t} \| x(s) \| + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$; where $H > 0$ is a constant, K, $M : [0,\infty) \longmapsto [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A_1) If $x(\cdot)$ is the function as in (A), then x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

(C) If $(\psi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions with compact support and $\psi^n \longrightarrow \psi$, $n \to \infty$, in the compact-open topology, then $\psi \in \mathcal{B}$ and $\| \psi_n - \psi \|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Remark 2.16. Since $\mathcal{B}$ satisfies axiom (C), the space $C_b((-\infty,0],X)$ consisting of all continuous and bounded functions $\psi:(-\infty,0]\to X$, is continuously included in $\mathcal{B}$. Thus, there exists a constant $L\geq 0$ such that $\parallel \psi \parallel_{\mathcal{B}}\leq L\parallel \psi \parallel_\infty$, for every $\psi\in C_b((-\infty,0],X)$ (see [31, Proposition 7.1.1]).

Definition 2.17. Let $S(t):\mathcal{B}\to\mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta)=\varphi(0)$ on $[-t,0]$ and $S(t)\varphi(\theta)=\varphi(t+\theta)$ on $(-\infty,-t]$. The phase space $\mathcal{B}$ is called a fading memory space if $\parallel S(t)\varphi\parallel_{\mathcal{B}}\to 0$ as $t\to\infty$ for every $\varphi\in\mathcal{B}$ with $\varphi(0)=0$.

Remark 2.18. In this work we suppose the existence of a constant $\mathcal{K}>0$ such that $max\{K(t),M(t)\}\leq\mathcal{K}$ for each $t\geq 0$. Observe that this condition is verified, for example, if $\mathcal{B}$ is a fading memory space , see, e.g. ([31, Proposition 7.1.5]) for details.

Example 2.19 (The phase space $\mathcal{C}_r\times L^p(\rho,X)$)**.** *Let $r\geq 0$, $1\leq p<\infty$ and let $\rho:(-\infty,-r]\to\mathbb{R}$ be a non-negative mensurable function which satisfies the conditions $(g-5)-(g-6)$ in the terminology of Hino et al.* [31]*. Briefly, this means that ρ is locally integrable and there exists a non-negative locally bounded function $\gamma(\cdot)$ on $(-\infty,0]$ such that $\rho(\xi+\theta)\leq\gamma(\xi)\rho(\theta)$ for all $\xi\leq 0$ and $\theta\in(-\infty,-r]\setminus N_\varepsilon$, where $N_\varepsilon\subset(-\infty,-r]$ is a set whose Lebesgue measure zero. We denote by $\mathcal{B}$=$\mathcal{C}_r\times L^p(g,X)$ the set of all functions $\varphi:(-\infty,0]\to X$ such that φ is continuous in $[-r,0]$, Lebesgue measurable in $(-\infty,-r]$ and $\rho\parallel\varphi\parallel^p$ is Lebesgue integrable in $(-\infty,-r)$. The seminorm in $\mathcal{C}_r\times L^p(\rho,X)$ is defined as follows:*

$$\parallel \varphi \parallel_{\mathcal{B}}=\sup_{\theta\in[-r,0]}\parallel \varphi(\theta)\parallel_X+\left(\int_{-\infty}^{-r}\rho(\theta)\parallel\varphi(\theta)\parallel_X^p\,d\theta\right)^{\frac{1}{p}}.$$

From preceding conditions, the space $\mathcal{B}=\mathcal{C}_r\times L^p(\rho,X)$ satisfies axioms (A), $(A1)$ and (B). Moreover, when $r=0$, and $p=2$, it is possible to choose $H=1$, $K(t)=1+\left(\int_{-t}^{0}\rho(\theta)d\theta\right)^{\frac{1}{2}}$ and $M(t)=\gamma(-t)^{\frac{1}{2}}$ for $t\geq 0$ (see [31, Theorem 1.3.8]*). Note that if conditions $(g-6)-(g-7)$ of* [31] *hold, then $\mathcal{B}$ is a fading memory space (see* [31, Example 7.1.8]*).*

Lemma 2.20 ([22])**.** *If $u\in AA_c(X)$, then the function $s\to u_s$ belongs to $AA_c(\mathcal{B})$. Moreover, if $\mathcal{B}$ is a fading memory space and $u\in C(\mathbb{R},X)$ is such that $u_0\in\mathcal{B}$ and $u|_{[0,\infty)}\in AAA_c(X)$ then $t\to u_t\in AAA_c(\mathcal{B})$.*

2.6. **S-asymptotically ω-periodic functions.**

Definition 2.21 ([29])**.** A function $f\in C_b([0,\infty);X)$ is called S-asymptotically periodic if there exists $\omega>0$ such that $\lim_{t\to\infty}\big(f(t+\omega)-f(t)\big)=0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic. Denote by $SAP_\omega(X)$ the set of all such functions.

We note that $SAP_\omega(X)$ is a Banach space with the supnorm. In [29] is was shown the surprising fact that the property $\lim_{t\to\infty}(f(t+\omega)-f(t))=0$ does not characterize asymptotically ω-periodic functions, that is, bounded and continuous functions which admits the decomposition $f=g+\phi$, where g is ω-periodic and $\lim_{t\to\infty}\phi(t)=0$ (see also [40]).

Definition 2.22 ([29]). Let X, Y be two Banach spaces. A continuous function $f : [0,\infty) \times X \to Y$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded subset K of X, the set $\{f(t,x) : \ t \geq 0, \ x \in K\}$ is bounded and $\lim_{t\to\infty} \big(f(t+\omega, x) - f(t,x)\big) = 0$, uniformly in $x \in K$.

Definition 2.23 ([29]). Let X, Y be two Banach spaces. A continuous function $f : [0,\infty) \times X \to Y$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded subset K of X, there exist $L_{\epsilon,\ K} \geq 0$ and $\delta_{\epsilon,\ K} > 0$ such that $\|f(t,x) - f(t,y)\| \leq \epsilon$ for all $t \geq L_{\epsilon,\ K}$ and all $x, y \in K$ with $\|x - y\| \leq \delta_{\epsilon,\ K}$.

Lemma 2.24 ([29]). *Let $f : [0,\infty) \times Y \to X$ be uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u : [0,\infty) \to Y$ is an S-asymptotically ω-periodic function, then the function $v(t) = f(t, u(t)) \in SAP_\omega(X)$.*

Lemma 2.25 ([30]). *Assume that $\mathcal{B}$ is a fading memory space. Let $u : \mathbb{R} \to X$ such that $u_0 \in \mathcal{B}$ and $u|_{[0,\infty)} \in SAP_\omega(X)$ then $t \to u_t$ belongs to $SAP_\omega(\mathcal{B})$.*

3. Pseudo compact almost automorphic solutions

3.1. **The linear case.** In this subsection we examine the existence and uniqueness of pseudo compact almost automorphic solutions to the inhomogeneous linear integral equation (1.5).

We have the following result.

Theorem 3.1. *Let $a \in L^1([0,\infty))$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying assumption* **(INT)**. *If f is a pseudo compact almost automorphic function with values in $D(A)$, then the unique bounded solution of equation (1.5) is pseudo compact almost automorphic.*

Proof. Let $u(t)$ be the function given by

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s) ds, \quad t \in \mathbb{R}.$$

Since the values $f(t) \in D(A)$, it follows that $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see e.g. [47, Proposition 1.2]). Using Fubini's theorem and Proposition 2.1 (b) we obtain that

$$\begin{aligned} \int_{-\infty}^{t} a(t-s) A u(s) ds &= \int_{-\infty}^{t} \int_{0}^{t-\tau} a(t-\tau-s) A S(s) f(\tau) ds d\tau \\ &= u(t) - \int_{-\infty}^{t} a(t-\tau) f(\tau) d\tau, \end{aligned}$$

which establishes that $u(\cdot)$ is the solution of equation (1.5). Applying Lemma 2.12, we infer that u is pseudo compact almost automorphic. $\square$

We have the following result for the scalar case.

Corollary 3.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a pseudo compact almost automorphic function, $a\in L^1([0,\infty))$, and let $\rho>0$ be a real number. If the solution $S_\rho(t)$ of the one-dimensional equation*

$$S_\rho(t)=a(t)-\rho\int_0^t a(t-s)S_\rho(s)ds, \tag{3.1}$$

satisfies $|S_\rho(t)|\le\phi_\rho(t)$, with $\phi_\rho\in L^1([0,\infty))$, then the equation

$$u(t)=\int_{-\infty}^t a(t-s)[-\rho u(s)+f(s)]ds,\quad t\in\mathbb{R}, \tag{3.2}$$

has a unique pseudo compact almost automorphic solution.

Remark 3.3. In [15, Corollary 3.7] or [28, Corollary 3.5] the authors provide a wide class of kernels $a(t)$ such that $|S_\rho(t)|\le\phi_\rho(t)$, with $\phi_\rho\in L^1([0,\infty))$.

Remark 3.4. We have recovered [15, Corollary 4.3] and [28, Corollary 4.4] as an immediate consequence of Corollary 3.2.

Example 3.5. *Consider $a(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta t}$, where $\beta>1$ and $1<\alpha<2$. Then we can check that $S_\rho(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\rho t)e^{-\beta t}$ is the solution of Eq. (3.1), where $E_{\alpha,\alpha}$ denotes the generalized Mittag-Leffler function (see* [26]*) which is defined by*

$$E_{\alpha,\beta}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta>0,\ z\in\mathbb{C}.$$

Using the explicit description of $S_\rho(t)e^{\beta t}$ given in [7, Corollary 3.7] *we can show that $|S_\rho(t)|\le\phi_\rho(t)$, with $\phi_\rho\in L^1([0,\infty))$. We conclude that the equation*

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-s)^{\alpha-1}e^{-\beta(t-s)}\left(-\rho u(s)+f(s)+\max_{k\in\mathbb{Z}}\{e^{-(s\pm k^2)^2}\}\right)ds,\ t\in\mathbb{R},$$

has a unique pseudo compact almost automorphic solution, whenever f is compact almost automorphic.

3.2. **The semilinear case.** In this section, we are concerned with the study of existence of pseudo compact almost automorphic solutions for equation (1.1).

Definition 3.6. Let A be the generator of an integral resolvent family $\{S(t)\}_{t\ge0}$. A continuous function $u:\mathbb{R}\to X$ satisfying the integral equation

$$u(t)=\int_{-\infty}^t S(t-s)f(s,u(s))ds,\ \ \forall t\in\mathbb{R}, \tag{3.3}$$

is called a *mild* solution on $\mathbb{R}$ to the equation (1.1).

Theorem 3.7. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\ge0}$ that satisfies the assumption* **(INT)**. *Let $f:\mathbb{R}\times X\to X$ be a pseudo compact almost automorphic function that satisfies the Lipschitz condition (2.1) with $L_f<\|\phi\|_1^{-1}$. Then equation (1.1) has a unique pseudo compact almost automorphic mild solution.*

Proof. We define the operator $F : PAA_c(X) \to PAA_c(X)$ by

$$(F\psi)(t) := \int_{-\infty}^{t} S(t-s)f(s,\psi(s))\,ds, \quad t \in \mathbb{R}. \tag{3.4}$$

In view of Lemmas 2.7 and 2.12 the map F is well defined. Moreover, for $\psi_1, \psi_2 \in PAA_c(X)$ we have:

$$\begin{aligned} \|F\psi_1 - F\psi_2\|_\infty &\le L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|S(\tau)\|\|\psi_1(t-\tau) - \psi_2(t-\tau)\|d\tau \\ &\le L_f\|\phi\|_1\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

This proves that F is a contraction, so there exists a unique $u \in PAA_c(X)$ mild solution of Eq. (1.1). □

An immediate consequence of Theorem 3.1 and [15, Corollary 3.7 (a)] is the following result for the scalar equation.

Corollary 3.8. *Let $\rho > 0$ be a real number. Suppose $a \in L^1([0,\infty))$ is a positive, nonincreasing and log-convex function and let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a pseudo compact almost automorphic function that satisfies (2.1) with $L_f < \|S_\rho\|_1^{-1}$, where S_ρ is the solution of Eq. (3.1), then the semilinear equation*

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s,u(s))]ds, \quad t \in \mathbb{R},$$

has a unique pseudo compact almost automorphic solution.

Remark 3.9. A similar result as the previous corollary was obtained in [15] (resp., [28]) in the case of f to be almost automorphic (resp., compact almost automorphic) function that satisfies (2.1) with $L_f < \|S_\rho\|_1^{-1}$. We note that if a is as above, then there is $S_\rho \in L^1([0,\infty)) \cap C([0,\infty))$ satisfying the linear equation (3.1).

Example 3.10. *We consider $g : \mathbb{R} \to \mathbb{R}$ a pseudo compact almost automorphic function, $\nu \in \mathbb{R}$ and $\beta \ge 1$. We examine the existence and uniqueness of pseudo compact almost automorphic solution to the integro-differential equation*

$$u(t,x) = \int_{-\infty}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\big[u_{xx}(s,x) + \nu g(s)u(s,x)\big]ds, \quad t\in\mathbb{R},\ x\in[0,1], \tag{3.5}$$

$$u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}.$$

To obtain a formulation as an abstract evolutionary integral equation like (1.1), we choose $X = L^2[0,1]$ and define an operator A by means of $Au(x) = u_{xx}(x)$ with domain $D(A) = \{u \in X : u_{xx} \in X, u(0) = u(1) = 0\}$. It is well known that A generates a bounded analytic semigroup with $0 \in \rho(A)$. Since $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$ satisfies all conditions of [47, Corollary 10.1] *it follows that A generates an integral resolvent family with the property* **(INT)**. *We define $f(s,u) = \nu g(s)u$, $u \in X$; we can show that f verify (2.1) with $L_f = |\nu|\|g\|_\infty$. If $|\nu|$ is small enough, then Eq. (3.5) has a unique pseudo compact almost automorphic solution.*

In the next three results, we introduce different type of Lipschitz conditions.

Theorem 3.11. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies the assumption* **(INT)** *with ϕ a decreasing function and $\phi_0 = \sum_{m=0}^{\infty} \phi(m) < +\infty$. Let $f : \mathbb{R} \times X \to X$ be a pseudo compact almost automorphic function that satisfies the following Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \quad \forall\ x,y \in X, \quad \forall\ t \in \mathbb{R}. \tag{3.6}$$

where $L \in C_b(\mathbb{R})$ is such that

$$\|L\|_M := \sup_{t\in\mathbb{R}} \int_t^{t+1} L(s)ds < +\infty. \tag{3.7}$$

Then equation (1.1) has a unique pseudo compact almost automorphic mild solution whenever $\|L\|_M\phi_0 < 1$.

Proof. Since ϕ is a decreasing function such that $\sum_{m=0}^{\infty} \phi(m) < \infty$ we have that $\phi \in L^1([0,\infty))$ and hence $S(t)$ is integrable. Let $\psi_1, \psi_2 \in PAA_c(X)$, for the operator F defined by the formulae (3.4) we have

$$\begin{aligned} \|F\psi_1(t) - F\psi_2(t)\| &\leq \left(\sum_{m=0}^{\infty} \int_{t-(m+1)}^{t-m} \phi(t-s)L(s)ds \right) \|\psi_1 - \psi_2\|_\infty \\ &\leq \left(\sum_{m=0}^{\infty} \phi(m) \int_{t-(m+1)}^{t-m} L(s)ds \right) \|\psi_1 - \psi_2\|_\infty \\ &\leq \|L\|_M\phi_0\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

Therefore, F is a $\|L\|_M\phi_0$-contraction, which finish the proof. □

Remark 3.12. We note that conditions of type (3.6) has been previously considered in the literature in the study of pseudo almost periodic (resp., almost automorphic) solution of semilinear evolutions equations in [2, 17] (resp., integral equations on the line [15]). Note that we essentially recover Cuevas and Lizama's Theorem 4.6 [15] as a corollary of previous result in the case of f to be an almost automorphic in t uniformly in $x \in X$ satisfying (3.6) and (3.7).

An integral resolvent family $\{S(t)\}_{t\geq 0}$ is said to be uniformly bounded if there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Theorem 3.13. *Assume that A generates an uniformly bounded integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption* **(INT)**. *Let $f : \mathbb{R} \times X \to X$ be a pseudo compact almost automorphic function that satisfies (3.6) with $L \in L^1(\mathbb{R}) \cap C_b(\mathbb{R})$. Then equation (1.1) has a unique pseudo compact almost automorphic mild solution.*

Proof. We define the operator F as in (3.4). Let ψ_1, ψ_2 be in $PAA_c(X)$. We have the following estimate

$$\|F^n\psi_1 - F^n\psi_2\|_\infty \leq \frac{(M||L||_1)^n}{n!}\|\psi_1 - \psi_2\|_\infty.$$

Since $\dfrac{(M||L||_1)^n}{n!} < 1$ for n sufficiently large, by fixed point iteration method F has a unique fixed point $u \in PAA_c(X)$. This completes the proof. □

Remark 3.14. We obtain [28, Theorem 4.5] as a corollary of Theorem 3.13.

We establish a version of Theorem 3.7 which, on one side, enable us to consider locally Lipschitz perturbations for the equation (1.1) and, on the other side, is an extension of the results in [28]. We have the following result.

Theorem 3.15. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption* **(INT)**. *Let $f : \mathbb{R}\times X \to X$ be a pseudo compact almost automorphic function and assume that there is a nondecreasing function $L : [0,\infty) \to [0,\infty)$ such that for each positive number R, and $x, y \in X$, $\|x\| \leq R$, $\|y\| \leq R$, we have*

$$\|f(t,x) - f(t,y)\| \leq L(R)\|x-y\|, \quad \forall\, t \in \mathbb{R},$$

with $\limsup_{R\to\infty} \|\phi\|_1 L(R) < 1$. Then Eq. (1.1) has a unique pseudo compact almost automorphic mild solution.

Proof. We define F by (3.4). We consider $R > 0$ such that $L(R)\|\phi\|_1 R + \|F(0)\|_\infty \leq R$. Let B_R be the closed ball $\{\psi \in PAA_c(X) : \|\psi\|_\infty \leq R\}$. A straightforward computation shows that $F : B_R \to B_R$ is well defined. We claim that F has a unique fixed point in B_R. In fact, let $\psi_1, \psi_2 \in B_R$. Then we obtain that

$$\begin{aligned}\|F\psi_1(t) - F\psi_2(t)\| &\leq \int_{-\infty}^{t} \phi(t-s)L(R)\|\psi_1(s) - \psi_2(s)\|ds \\ &\leq L(R)\|\phi\|_1\|\psi_1 - \psi_2\|_\infty.\end{aligned}$$

Then F is a contraction on the ball B_R. This completes the proof. ☐

To establish our next result we consider functions f that satisfies the following boundedness condition.

(B0) There exists a continuous nondecreasing function $W : [0,\infty) \to [0,\infty)$ such that $\|f(t,x)\| \leq W(\|x\|)$ for all $t \in \mathbb{R}$ and $x \in X$.

Theorem 3.16. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption* **(INT)**. *Let $f : \mathbb{R}\times X \to X$ be a pseudo compact almost automorphic function that satisfies assumption* **(B0)** *and the following conditions:*

- (B1) *$f(t,x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$.*
- (B2) *For each $\nu \geq 0$, $\lim_{|t|\to\infty} \frac{1}{h(t)} \int_{-\infty}^{t} \phi(t-s)W(\nu h(s))ds = 0$, where h is given in Lemma 2.14.*
- (B3) *For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|v-u\|_h \leq \delta$ implies that*

$$\int_{-\infty}^{t} \phi(t-s)\|f(s,v(s)) - f(s,u(s))\|ds \leq \varepsilon$$

for all $t \in \mathbb{R}$. We set

$$\beta(\nu) := \left\| \int_{-\infty}^{\cdot} \phi(\cdot - s)W(\nu h(s))ds \right\|_h .$$

- (B4) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(B5) *For all* $a, b \in \mathbb{R}$, $a \leq b$, *and* $r > 0$, *the set* $\{f(s,x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ *is relatively compact in* X.

Then equation (1.1) has a pseudo compact almost automorphic mild solution.

Proof. For the sake of brevity we give just a sketch of proof (see [28] for details). We define the operator F on $C_h(X)$ as in (3.4). It follows from condition (B2) that F is well defined. Using (B3) we have that the map F is continuous. Let $V = F(B_r(C_h(X)))$, where $B_r(Z)$ denotes the closed ball with center in 0 and radius r in the space Z. Taking into account The Mean Value Theorem for the Bochner's integral (see [39, Lemma 2.13]) and condition (B5), for each $\epsilon > 0$ we can choose $a \geq 0$ and a relatively compact set K such that

$$V(t) \subseteq a\overline{c_0(K)} + B_\varepsilon(X),$$

where $c_0(K)$ denotes the convex hull of K and $V(t)$ is defined as Lemma 2.14 (c-1). Hence $V(t)$ is a relatively compact subset of X for each $t \in \mathbb{R}$. Put now $v = F(u)$, $u \in B_r(C_h(X))$. Condition (B5) and the following formulae

$$\begin{aligned} v(t+s) - v(t) &= \int_0^s S(\xi) f(t+s-\xi, u(t+s-\xi))d\xi \\ &+ \int_0^a (S(\xi+s) - S(\xi)) f(t-\xi, u(t-\xi))d\xi \\ &+ \int_a^\infty (S(\xi+s) - S(\xi)) f(t-\xi, u(t-\xi))d\xi. \end{aligned}$$

Imply that V is equicontinuous. Applying condition (B2), we can show that $\lim_{|t|\to\infty} \frac{v(t)}{h(t)} = 0$ uniformly for $u \in B_r(C_h(X))$. Hence V satisfies (c-1), (c-2) and (c-3) of Lemma 2.14, which completes the proof that V is relatively compact in $C_h(X)$. Hence F is a completely continuous map.

If $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda F(u^\lambda)$ for some $0 < \lambda < 1$, from the estimate $\|u^\lambda\|_h \leq \beta(\|u^\lambda\|_h)$, which is established by elementary means and under condition (B4), we conclude that the set $\{u^\lambda : u^\lambda = \lambda F(u^\lambda), \lambda \in (0,1)\}$ is bounded. It follows from Lemma 2.8 and 2.12 that $F(PAA_c(X)) \subseteq PAA_c(X)$ and, consequently, we have that $F : \overline{PAA_c(X)} \to \overline{PAA_c(X)}$ is completely continuous. Applying the Leray-Schauder alternative theorem ([27, Theorem 6.5.4]), we completes the proof. □

Remark 3.17. We note that conditions like of the preceding result has been previously considered in the literature (cf. [1, 2, 12, 16, 20, 28]).

Corollary 3.18. *Assume that* A *generates an integral resolvent family* $\{S(t)\}_{t\geq 0}$ *satisfying assumption* **(INT)**. *Let* $f : \mathbb{R} \times X \to X$ *be a pseudo compact almost automorphic function that satisfies the Hölder type condition*

$$\|f(t,x) - f(t,y)\| \leq C\|x-y\|^\alpha, \quad 0 < \alpha < 1,$$

for all $x, y \in X$, $t \in \mathbb{R}$, *where* $C > 0$ *is a constant. Moreover, assume the following conditions:*

(i) $f(t,0) = q$.

(ii) $\sup_{t\in\mathbb{R}} \int_{-\infty}^t \phi(t-s)h(s)^\alpha ds < +\infty$, *where* h *is given in Lemma 2.14.*

(iii) *For all* $a, b \in \mathbb{R}$, $a \leq b$, *and* $r > 0$, *the set* $\{f(s,x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ *is relatively compact in* X.
Then Eq. (1.1) has a pseudo compact almost automorphic mild solution.

4. Asymptotically compact almost automorphic solutions

Initially in this Section , we examine sufficient conditions for the existence of asymptotically compact almost automorphic solution of Eq. (1.2). The reader can perceive that, repeating most parts of our proofs, the same type of results as the preceding theorems hold for the semilinear integral equation (1.2). We note that a key ingredient to proof the results are the Lemmas 2.6 and 2.13. For convenience, we will give the statements of results some of them without proof. The straightforward changes in the details may safely be left to the reader.

Definition 4.1. Let A be the generator of an integral resolvent family $\{S(t)\}_{t\geq 0}$. A continuous function $u : [0,\infty) \to X$ satisfying the integral equation

$$u(t) = \int_0^t S(t-s)f(s,u(s))ds, \ \ \forall\ t \geq 0, \tag{4.1}$$

is called a mild solution to equation (1.2).

Theorem 4.2. *Assume that* A *generates an integral resolvent family* $\{S(t)\}_{t\geq 0}$ *that satisfies assumption* **(INT)**. *Let* $f : [0,\infty) \times X \to X$ *be an asymptotically compact almost automorphic function that satisfies the Lipschitz condition (2.1) for all* $t \in [0,\infty)$ *with* $L_f < \|\phi\|_1^{-1}$. *Then equation (1.2) has a unique asymptotically compact almost automorphic mild solution.*

The following result is of more practical use and it follows from [47, Corollary 10.1].

Corollary 4.3. *Suppose* $a(t)$ *is completely monotonic and satisfies* $a(\infty) = \lim_{t\to\infty} a(t) > 0$. *Assume that* A *generates a bounded analytic* C_0*-semigroup and* $0 \in \rho(A)$. *Let* $f : [0,\infty)\times X \to X$ *be an asymptotically compact almost automorphic function that satisfies a* L_f*-Lipschitz condition with* L_f *small enough. Then Eq. (1.2) has a unique asymptotically compact almost automorphic mild solution.*

Remark 4.4. Let $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$. Then $a(t)$ satisfies the hypotheses of the previous corollary if $\beta \geq 1$.

Theorem 4.5. *Assume that* A *generates an uniformly bounded integral resolvent family* $\{S(t)\}_{t\geq 0}$ *that satisfies assumption* **(INT)**. *Let* $f : [0,\infty) \times X \to X$ *be an asymptotically compact almost automorphic function that satisfies the Lipschitz condition (3.6) for all* $t \geq 0$ *with* $L \in C_b([0,\infty)) \cap L^1([0,\infty))$. *Then equation (1.2) has a unique asymptotically compact almost automorphic mild solution.*

Proof. We define the operator $F : AAA_c(X) \to AAA_c(X)$ by

$$(F\psi)(t) = \int_0^t S(t-s)f(s,\psi(s))ds, \quad t \geq 0. \tag{4.2}$$

In view of Lemmas 2.6 and 2.13, F is well defined. Define a new norm $|||\psi||| := \sup_{t\geq 0}\{v(t)\|\psi(t)\|\}$, where $v(t) := e^{-k\int_0^t L(s)ds}$ and k is a fixed positive constant greather tham $M = \sup_{t\geq 0}\|S(t)\|$. Let $\psi_1, \psi_2 \in AAA_c(X)$, then we have that

$$\begin{aligned}
v(t)\|F\psi_1(t) - F\psi_2(t)\| &\leq M\int_0^t v(t)L(s)\|\psi_1(s) - \psi_2(s)\|ds \\
&\leq M\int_0^t v(t)v(s)^{-1}L(s)v(s)\|\psi_1(s) - \psi_2(s)\|ds \\
&\leq M|||\psi_1 - \psi_2|||\int_0^t v(t)v(s)^{-1}L(s)ds \\
&= \frac{M}{k}|||\psi_1 - \psi_2|||\int_0^t ke^{k\int_t^s L(\tau)d\tau}L(s)ds \\
&= \frac{M}{k}|||\psi_1 - \psi_2|||\int_0^t \frac{d}{ds}\left(e^{k\int_t^s L(\tau)d\tau}\right)ds \\
&= \frac{M}{k}[1 - e^{-k\int_0^t L(\tau)d\tau}]|||\psi_1 - \psi_2||| \\
&\leq \frac{M}{k}|||\psi_1 - \psi_2|||.
\end{aligned}$$

Hence, since $M/k < 1$, F has a unique fixed point in $AAA_c(X)$. □

Remark 4.6. We can use the same argument of proof of the previous result to prove Theorem 3.13 by defining a new norm $|||\phi||| = \sup_{t\in\mathbb{R}}\{v(t)\|\phi(t)\|\}$, where $v(t)$ is given in the proof of Theorem 4.5.

Similarly to the previous section for situations where the perturbation f is not Lipschitz continuous we need the following boundedness condition.

($\mathbf{B_0^*}$) There exists a continuous nondecreasing function $W : [0,\infty) \to [0,\infty)$ such that $\|f(t,x)\| \leq W(\|x\|)$ for all $t \geq 0$ and $x \in X$.

Theorem 4.7. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption* **(INT)**. *Let $f : [0,\infty) \times X \to X$ be an asymptotically compact almost automorphic function that satisfies assumption* ($\mathbf{B_0^*}$) *and the following conditions:*

($\mathbf{B_1^*}$) *$f(t,x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $t \geq 0$.*

($\mathbf{B_2^*}$) *For each $\nu \geq 0$, $\lim_{t\to\infty}\frac{1}{h^*(t)}\int_0^t \phi(t-s)W(\nu h^*(s))ds = 0$, where h^* is given in Lemma 2.15.*

($\mathbf{B_3^*}$) *For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_{h^*}(X)$, $\|v-u\|_{h^*} \leq \delta$ implies that*

$$\int_0^t \phi(t-s)\|f(s,v(s)) - f(s,u(s))\|ds \leq \varepsilon, \quad \forall\ t \geq 0.$$

We set

$$\beta^*(C) := \|\int_0^{\cdot} \phi(\cdot - s)W(Ch^*(s))ds\|_{h^*}.$$

($\mathbf{B_4^*}$) $\liminf_{r\to\infty}\frac{r}{\beta^*(r)} > 1$.

($\mathbf{B_5^*}$) *For all* $a, b \in [0, \infty)$, $a \leq b$, *and* $r > 0$, *the set* $\{f(s, x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ *is relatively compact in* X.
Then equation (1.2) has an asymptotically compact almost automorphic mild solution.

Corollary 4.8. *Assume that* A *generates an integral resolvent family* $\{S(t)\}_{t\geq 0}$ *satisfying assumption* **(INT)**. *Let* $f : [0, \infty) \times X \to X$ *be an asymptotically compact almost automorphic function that satisfies the Hölder type condition*

$$\|f(t, x) - f(t, y)\| \leq C\|x - y\|^{\alpha}, \quad 0 < \alpha < 1,$$

for all $x, y \in X$, $t \geq 0$, *where* $C > 0$ *is a constant. Moreover, assume the following conditions:*

(i) $f(t, 0) = q$.

(ii) $\sup_{t\in[0,\infty)} \int_0^t \phi(t - s)h^*(s)^{\alpha} ds < +\infty$, *where* h^* *is given in Lemma 2.15.*

(iii) *For all* $a, b \in [0, \infty)$, $a \leq b$, *and* $r > 0$, *the set* $\{f(s, x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ *is relatively compact in* X.

Then Eq. (1.1) has an asymptotically compact almost automorphic mild solution.

4.1. **About the problem (1.3)-(1.4).** Next, we concentrate in discuss the existence of S-asymptotically ω-periodic solution to the problem (1.3)-(1.4). The considerations in [16] motivates the following definition.

Definition 4.9. Let A be the generator of a solution operator $\{S(t)\}_{t\geq 0}$ (cf. [16, Definition 2.1]). A continuous function $u : \mathbb{R} \to X$ is called a mild solution of (1.3)-(1.4) if $u_0 = \phi$ and

$$u(t) = S(t)(\phi(0) + G(0, \phi)) - G(t, u_t) + \int_0^t S(t - s)F(s, u_s)ds, \quad t \geq 0.$$

Definition 4.10 ([47])**.** A strongly mensurable family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly integrable if $\|T\|_1 = \int_0^\infty \|T(t)\|dt < +\infty$.

We also recall the following concept, studied for resolvent families in [44].

Definition 4.11. A strongly continuous family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly stable if $\|T(t)\| \to 0$ as $t \to \infty$.

Definition 4.12 ([29])**.** A strongly continuous family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be strongly S-asymptotically ω-periodic if there is $\omega > 0$ such that $T(\cdot)x$ is S-asymptotically ω-periodic for all $x \in X$.

Lemma 4.13 ([16])**.** *Suppose that* A *generates an uniformly integrable solution operator* $\{S(t)\}_{t\geq 0}$ *and let* $f \in SAP_\omega(X)$ *. Then*

$$\int_0^t S(t - s)f(s)ds \in SAP_\omega(X).$$

Theorem 4.14. *Suppose* A *generates an uniformly integrable solution operator* $\{S(t)\}_{t\geq 0}$, *which is in addition strongly S-asymptotically* ω*-periodic and that* $\mathcal{B}$ *is a fading memory space. Let* $F, G : [0, \infty) \times \mathcal{B} \to X$ *be two functions uniformly S-asymptotically* ω*-periodic on bounded sets such that*

$$\|G(t, \phi) - G(t, \psi)\| \leq L_G\|\phi - \psi\|_{\mathcal{B}}, \quad \phi, \psi \in \mathcal{B},\ t \geq 0, \tag{4.3}$$

$$\|F(t,\phi) - F(t,\psi)\| \le L_F \|\phi - \psi\|_{\mathcal{B}}, \quad \phi, \psi \in \mathcal{B},\ t \ge 0. \tag{4.4}$$

If $(L_G + \|S\|_1 L_F)\mathcal{K} < 1$*, where* $\mathcal{K}$ *is the constant given in Remark 2.18, then (1.3)-(1.4) has a unique S-asymptotically* ω*-periodic mild solution.*

Proof. We set $SAP_{\omega,0}(X) = \{x \in SAP_\omega(X) : x(0) = 0\}$. It is clear that $SAP_{\omega,0}(X)$ is a closed subspace of $SAP_\omega(X)$. We next identify the elements $x \in SAP_{\omega,0}(X)$ with its extension to $\mathbb{R}$ given by $x(\theta) = 0$ for $\theta \le 0$. Moreover, we denote by $y(\cdot)$ the function defined by $y_0 = \phi$ and $y(t) = S(t)\phi(0)$ for $t \ge 0$. We observe that by hypothese $y|_{[0,\infty)} \in SAP_\omega(X)$. Since $\mathcal{B}$ is a fading memory space, it follows from Lemma 2.25 that the function $t \to y_t$ belongs to $SAP_\omega(\mathcal{B})$. Next, we define the map Γ on the space $SAP_{\omega,0}(X)$ by $(\Gamma x)_0 = 0$ and

$$(\Gamma x)(t) = S(t)G(0,\phi) - G(t, x_t + y_t) + \int_0^t S(t-s)F(s, x_s + y_s)ds, \quad t \ge 0. \tag{4.5}$$

We observe that $S(\cdot)G(0,\phi) \in SAP_\omega(X)$. Again taking in account that $\mathcal{B}$ is a fading memory space and Lemma 2.25, we have that the function $s \to x_s + y_s$ belongs to $SAP_\omega(\mathcal{B})$. In view of F and G are asymptotically uniformly continuous on bounded sets, by Lemma 2.24, we conclude that the functions $s \to F(s, x_s + y_s)$ and $t \to G(t, x_t + y_t)$ belong to $SAP_\omega(X)$. From Lemma 4.13, we infer that Γ is a map from $SAP_{\omega,0}(X)$ into $SAP_{\omega,0}(X)$. Furthermore, we have the estimate

$$\begin{aligned}
\|\Gamma x(t) - \Gamma z(t)\| &\le L_G \|z_t - x_t\|_{\mathcal{B}} + L_F \int_0^t \|S(t-s)\| \|x_s - z_s\|_{\mathcal{B}} ds \\
&\le \left(L_G + L_F \int_0^t \|S(s)\| ds \right) \mathcal{K} \|x - z\|_\infty \\
&\le (L_G + \|S\|_1 L_F)\mathcal{K}\|x - z\|_\infty,
\end{aligned}$$

which proves that Γ is a contraction. We conclude that Γ has a unique fixed point $x \in SAP_{\omega,0}(X)$. Defining $u(t) = y(t) + x(t)$ for $t \in \mathbb{R}$, we can confirm that $u \in SAP_\omega(X)$ is a mild solution of (1.3)-(1.4). This completes the proof. □

Corollary 4.15. *Suppose that* $a(t)$ *is* 1*-regular (see* [16, Definition 2.8]*) and* A *generates a parabolic (see* [16, Definition 3.3]*) and uniformly integrable solution operator* $\{S(t)\}_{t \ge 0}$ *and* $\mathcal{B}$ *is a fading memory space. Let* $F : [0,\infty) \times \mathcal{B} \to X$ *be uniformly S-asymptotically* ω*-periodic on bounded sets such that condition (4.4) holds with* $L_F < (\|S\|_1 \mathcal{K})^{-1}$*, where* $\mathcal{K}$ *is the constant given in Remark 2.18, then the problem*

$$u'(t) = \int_0^t a(t-s)Au(s)ds + F(t, u_t), \quad t \ge 0, \tag{4.6}$$

$$u_0 = \phi, \tag{4.7}$$

has a unique S-asymptotically ω*-periodic mild solution.*

Proof. Since $a(t)$ is 1-regular and A generates a parabolic and uniformly integrable solution operator, we obtain by main result in [44] that $\{S(t)\}_{t \ge 0}$ is uniformly stable. In particular, $\{S(t)\}_{t \ge 0}$ is S-asymptotically ω-periodic for any $\omega > 0$. The result is now a consequence of Theorem 4.14. □

Corollary 4.16. *Suppose that $a(t)$ is 1-regular and A generates a parabolic and uniformly integrable solution operator $\{S(t)\}_{t\geq 0}$ and $\mathcal{B}$ is a fading memory space. Let $F : [0,\infty)\times\mathcal{B}\to X$ be an asymptotically compact almost automorphic function such that (4.4) holds with $L_F < (\|S\|_1\mathcal{K})^{-1}$, then (4.6)-(4.7) has a unique asymptotically compact almost automorphic mild solution.*

We need introduce the following condition:
Condition **(S1)**: The functions $F, G : [0,\infty)\times\mathcal{B}\to X$ are uniformly S-asymptotically ω-periodic on bounded sets and assume that there are continuous and nondecreasing functions $L_F, L_G : [0,\infty)\to[0,\infty)$ such that for each positive number R, and $\phi,\psi\in\mathcal{B}$, $\|\phi\|_{\mathcal{B}}\leq R$, $\|\psi\|_{\mathcal{B}}\leq R$, we have

$$\|G(t,\phi)-G(t,\psi)\|\leq L_G(R)\|\phi-\psi\|_{\mathcal{B}},\quad t\geq 0, \tag{4.8}$$

$$\|F(t,\phi)-F(t,\psi)\|\leq L_F(R)\|\phi-\psi\|_{\mathcal{B}},\quad t\geq 0, \tag{4.9}$$

where $L_F(0)=L_G(0)=0$ and $F(t,0)=G(t,0)=0$ for every $t\geq 0$.
We have the following result.

Theorem 4.17. *Suppose that A generates an uniformly bounded and integrable solution operator $\{S(t)\}_{t\geq 0}$, which is in addition strongly S-asymptotically ω-periodic; $\mathcal{B}$ is a fading memory space and that* **(S1)** *is fulfilled, then there is $\epsilon>0$ such that for each ϕ satisfying $\|\phi\|_{\mathcal{B}}\leq\epsilon$, there is a unique S-asymptotically ω-periodic mild solution of (1.3)-(1.4).*

Proof. One can easily take $R>0$ and $\lambda\in(0,1)$ so that

$$\begin{aligned} H \;&:=\; \|S\|_\infty L_G(\lambda R)\lambda + L_G((1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R)(1+(\|S\|_\infty H+1)\lambda)\mathcal{K}\\ &+\; \|S\|_1 L_F((1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R)(1+(\|S\|_\infty H+1)\lambda)\mathcal{K}<1,\end{aligned}$$

where $\mathcal{K}$ is the constant appearing in Remark 2.18. We affirm that the assertion holds for $\epsilon=\lambda R$. In fact, let ϕ be such that $\|\phi\|_{\mathcal{B}}\leq\epsilon$. Again we identify the elements $x\in SAP_{\omega,0}(X)$ with its extension to $\mathbb{R}$ given by $x_0=0$ and we define the space

$$\mathcal{D}_R := \{x\in SAP_{\omega,0}(X) : \|x\|_\infty\leq R\}$$

endowed with the metric defined by $d(u,v)=\|u-v\|_\infty$. We also define the operator Γ on the space $\mathcal{D}_R$ by $(\Gamma x)_0=0$ and (4.5). In a similar way as proof of Theorem 4.14 it follows that Γ is well defined. Moreover, we have the estimate

$$\begin{aligned}\|(\Gamma x)(t)\| &\leq \|S\|_\infty L_G(\lambda R)\lambda R\\ &+ L_G((1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R)(1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R\\ &+ \|S\|_1 L_F((1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R)(1+(\|S\|_\infty H+1)\lambda)\mathcal{K}R\\ &= HR\leq R.\end{aligned}$$

Therefore $\Gamma(\mathcal{D}_R)\subset\mathcal{D}_R$. on the other hand, for $x,z\in\mathcal{D}_R$, we see that

$$\begin{aligned}\|(\Gamma x)(t)-(\Gamma z)(t)\| &\leq \frac{H-\|S\|_\infty L_G(\lambda R)\lambda}{1+(\|S\|_\infty H+1)\lambda}\|z-x\|_\infty\\ &\leq H\|z-x\|_\infty,\end{aligned}$$

which shows that Γ is a contraction from $\mathcal{D}_R$ into $\mathcal{D}_R$. The assertion is now a consequence of the contraction mapping principle. □

Corollary 4.18. *Suppose that $a(t)$ is 1-regular and that A generates a parabolic and uniformly integrable solution operator $\{S(t)\}_{t\geq 0}$ and that $\mathcal{B}$ is a fading memory space. Let $F : [0,\infty)\times\mathcal{B}\to X$ be an asymptotically compact almost automorphic function so that (4.9) holds with $L_F(0) = 0$ and $F(t,0) = 0$ for every $t \geq 0$, then for each ϕ satisfying $\|\phi\| \leq \epsilon$ there is a unique asymptotically compact almost automorphic mild solution of (4.6)-(4.7).*

We need to introduce the following condition:
Condition **(S1)***: The functions $F, G : [0,\infty)\times\mathcal{B}\to X$ are uniformly S-asymptotically ω-periodic on bounded sets and assume that there are nondecreasing functions $L_F, L_G : [0,\infty)\to[0,\infty)$ such that for each positive number R, and $\psi_1,\psi_2\in\mathcal{B}$, $\|\phi_1\|_{\mathcal{B}}\leq R$, $\|\psi_2\|_{\mathcal{B}}\leq R$, we have (4.8) and (4.9) with

$$\limsup_{R\to\infty}\mathcal{K}L_G(\mathcal{K}(R+(\|S\|_\infty H+1)\|\phi\|_{\mathcal{B}})+\mathcal{K}\|S\|_1 L_F(\mathcal{K}(R+(\|S\|_\infty H+1)\|\phi\|_{\mathcal{B}})<1,$$

where $\mathcal{K}$ is the constant appearing in Remark 2.18.

Theorem 4.19. *Suppose that A generates an uniformly bounded and integrable solution operator $\{S(t)\}_{t\geq 0}$, which is in addition strongly S-asymptotically ω-periodic; $\mathcal{B}$ is a fading memory space and* **(S1)*** *is fulfilled. Then there is a unique S-asymptotically ω-periodic mild solution of (1.3)-(1.4).*

Theorem 4.20. *Suppose that A generates an uniformly bounded and integrable solution operator $\{S(t)\}_{t\geq 0}$, which is strongly S-asymptotically ω-periodic and that $\mathcal{B}$ is a fading memory space. In addition the following conditions hold.*

(a) *The functions $F, G : [0,\infty)\times\mathcal{B}\to X$ are uniformly S-asymptotically ω-periodic on bounded sets and F is asymptotically uniformly continuous on bounded sets.*

(b) *There is a continuous nondecreasing function $W : [0,\infty)\to[0,\infty)$ such that $\|F(t,\psi)\|\leq W(\|\psi\|_{\mathcal{B}})$ for all $t\geq 0$ and $\psi\in\mathcal{B}$.*

(c) *There is a constant $L_G > 0$ such that*

$$\|G(t,h_\#(t)\psi_1)-G(t,h_\#(t)\psi_2)\|\leq L_G\|\psi_1-\psi_2\|_{\mathcal{B}},$$

for all $t\geq 0$ and $\psi_1,\psi_2\in\mathcal{B}$, where h^* is as Lemma 2.15 and $h_\#(t)=\sup_{0\leq\tau\leq t}h^*(\tau)$.

(d) *For each $\nu>0$,*

$$\lim_{t\to\infty}\frac{1}{h^*(t)}\int_0^t\|S(t-s)\|W(\nu h_\#(s))ds=0.$$

(e) *For each $\epsilon>0$ there is $\delta>0$ such that for every $u,v\in C_h^*(X)$, $\|u-v\|_{h^*}\leq\delta$ implies*

$$\int_0^t\|S(t-s)\|\|F(s,u_s)-F(s,v_s)\|ds\leq\epsilon,$$

for all $t\geq 0$.

(f) *For all $a,b\in[0,\infty)$, $a\leq b$ and $r>0$ the set $\{F(s,\psi) : s\in[a,b],\psi\in\mathcal{B},\|\psi\|_{\mathcal{B}}\leq r\}$ is relatively compact in X.*

We set $\nu(\xi):=\mathcal{K}(\xi+(\|S\|_\infty H+1)\|\varphi\|_{\mathcal{B}})$ and $\beta(\xi):=\left\|\int_0^{\cdot}\|S(\cdot-s)\|W(\nu(\xi)h_\#(s))ds\right\|_{h^*}$, where H and $\mathcal{K}$ are the constants given in Axiom (A) and Remark 2.18, respectively.

(g) $L_G\mathcal{K} + \liminf_{\xi\to\infty} \frac{\beta(\xi)}{\xi} < 1.$

Then the problem (1.3)-(1.4) has an S-asymptotically ω-periodic mild solution.

Proof. We identify the elements $v \in C_{h^*}(X)$ with its extension to $\mathbb{R}$ given by $v(\theta) = 0$ for $\theta \leq 0$. We define the operator Γ on $C_{h^*}(X)$ by $(\Gamma x)_0 = 0$ and (4.5); we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $\Gamma_1 u(t) = S(t)G(0,\varphi) - G(t, u_t + y_t)$, $t \geq 0$, $(\Gamma_1 u)_0 = 0$, $\Gamma_2 u(t) = \int_0^t S(t-s)F(s, u_s + y_s)ds$, $t \geq 0$, $(\Gamma_2 u)_0 = 0$, where $y(\cdot)$ is the function defined in the proof of Theorem 4.14. For $u \in C_{h^*}(X)$, we have that

$$\| \Gamma_1 u(t) \| \leq \|S\|_\infty \| G(0,\varphi) \| + L_G\mathcal{K}(\| u \|_{h^*} + (\|S\|_\infty H + 1) \| \varphi \|_{\mathcal{B}}) + \| G(\cdot, 0) \|_\infty,$$

hence Γ_1 is $C_{h^*}(X)$-valued. On the other hand Γ_1 is a $L_G\mathcal{K}$-contraction. We next show that Γ_2 is $C_{h^*}(X)$-valued. For $u \in C_{h^*}(X)$, we have that

$$\frac{\| \Gamma_2 u(t) \|}{h^*(t)} \leq \frac{1}{h^*(t)} \int_0^t \|S(t-s)\| W(\nu(\| u \|_{h^*}) h_{\#}(s)) ds.$$

It follows from condition (d) that $\Gamma_2 : C_{h^*}(X) \to C_{h^*}(X)$. Using (e) we have that the map Γ_2 is continuous. We next show that Γ_2 is completely continuous. Let $V = \Gamma_2(B_r(C_{h^*}(X)))$ and $v = \Gamma_2 u$ for $u \in B_r(C_{h^*}(X))$. Initially, we prove that $V_b(t)$ is relatively compact subset of X for each $t \in [0,b]$. We get $v(t) \in t\overline{c_0(K)}$, where $K = \{S(s)F(\xi, h_{\#}(\xi)\psi) : 0 \leq s \leq t, 0 \leq \xi \leq t; \| \psi \|_{\mathcal{B}} \leq \nu(r)\}$. Using the fact that $S(\cdot)$ is strongly continuous and the condition (f), we infer that $c_0(K)$ is a relatively compact set, which establishes our assertion. The following decomposition is responsible for that fact that V_b is equicontinuous.

$$\begin{aligned} v(t+s) - v(t) &= \int_t^{t+s} S(t+s-\xi)F(\xi, u_\xi + y_\xi)d\xi \\ &+ \int_0^t (S(\xi+s) - S(\xi))F(t-\xi, u_{t-\xi} + y_{t-\xi})d\xi. \end{aligned}$$

Finally, applying condition (d), we can show that

$$\lim_{t\to\infty} \frac{\| v(t) \|}{h^*(t)} = 0,$$

and this convergence is independent of $u \in B_r(C_{h^*}(X))$. Hence V satisfies conditions (c-1)* and (c-2)* of Lemma 2.15, which completes the proof that V is a relatively compact set in $C_{h^*}(X)$. Hence Γ_2 is completely continuous. Taking into account condition (a). It follows from Lemmas 2.24, 2.25 and 4.13 that $\Gamma_i(SAP_{\omega,0}(X)) \subseteq SAP_{\omega,0}(X)$, i=1,2, where we identify the elements $x \in SAP_{\omega,0}(X)$ with its extension to $\mathbb{R}$ given by $x_0 = 0$. Hence $\Gamma(SAP_{\omega,0}(X)) \subseteq SAP_{\omega,0}(X)$ and $\Gamma_2 : \overline{SAP_{\omega,0}(X)} \to \overline{SAP_{\omega,0}(X)}$ is completely continuous. Putting $B_r := B_r(\overline{SAP_{\omega,0}(X)})$. We claim that there is $r > 0$ such that $\Gamma(B_r) \subseteq B_r$. In fact, if we assume that this assertion is false, then for all $r > 0$, we can choose $u^r \in B_r$

and $t^r \geq 0$ such that $\frac{\|\Gamma u^r(t^r)\|}{h^*(t^r)} > r$. Observe that a standard computation yields

$$\begin{aligned}\| \Gamma u^r(t^r) \| &\leq \|S\|_\infty \| G(0,\varphi) \| + L_G\mu(r) + ||G(\cdot,0)||_\infty \\ &\quad + \int_0^{t^r} \|S(t^r - s)\| W(\nu(r)h_\#(s))ds.\end{aligned}$$

Thus $1 \leq L_G\mathcal{K} + \liminf_{r\to\infty} \frac{\beta(r)}{r}$, which is contrary to condition (g). We have that Γ_1 is a contraction on B_r and $\overline{\Gamma_2(B_r)}$ is a compact set. It follows from [39, Corollary 4.3.2] that Γ has a fixed point $u \in \overline{SAP_{\omega,0}(X)}$. Let $(u_n)_n$ be sequence in $SAP_{\omega,0}(X)$ that converges to u. We see that $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $[0,\infty)$. This implies that $u \in SAP_{\omega,0}(X)$, and this finishes the proof. □

Remark 4.21. A natural expectation is that most of the results discussed in this subsection to be valid for a large class of equations of type (1.3) with nonlocal conditions. For the sake of brevity we leave the details as an exercise to the reader.

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