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STRONG AND WEAK CONVERGENCE THEOREMS FOR ASYMPTOTICALLY STRICT PSEUDOCONTRACTIVE MAPPINGS IN INTERMEDIATE SENSE

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Dedicated to Professor Wataru Takahashi

Abstract. In this paper, we first prove the strong convergence of viscosity approximation method for a modified Mann iteration process for asymptotically strict pseudocontractive mappings in intermediate sense, and then prove the strong convergence of general CQ algorithm for asymptotically strict pseudocontractive mappings in intermediate sense. We extend the concept of asymptotically strict pseudocontractive mappings in intermediate sense to Banach space setting, called nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense. We establish the weak convergence theorems for a fixed point of a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense which is not necessarily Lipschitzian.

1. Introduction

The concept of an asymptotically nonexpansive mapping is introduced by Goebel and Kirk [13] as an important generalization of nonexpansive mappings. They proved that every asymptotically nonexpansive mapping defined on a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point. An iterative method for the approximation of fixed points of asymptotically nonexpansive mappings is developed by Schu [27]. He established the following weak convergence theorem.

Theorem 1.1 ([27]). *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\delta \le \alpha_n \le 1 - \delta$ for all $n \ge 1$ and for some $\delta > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1, \tag{1.1}$$

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converges weakly to a fixed point of T.

Several iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been further studied in the recent past; See for example [7, 6, 5, 9, 10, 20, 27, 28, 29] and references therein.

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck et al. [2] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal et al. [1], Bruck et al. [2], Chidume et al. [11], Kim and Kim [16] and several others. Recently, Kim and Xu [18] introduced the following concept of asymptotically κ-strict pseudocontractive mappings in the setting of Hilbert spaces.

Definition 1.2. Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim\limits_{n\to\infty} \gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2, \tag{1.2}$$

for all $n \ge 1$ and for all $x, y \in C$.

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\left\{\frac{\kappa+\sqrt{1+(1-\kappa)\gamma_n}}{1+\kappa} : n \ge 1\right\}$.

Very recently, Sahu et al. [26] defined the following concept of asymptotically κ-strict pseudocontractive mappings in intermediate sense, which are not necessarily Lipschitzian.

Definition 1.3. A mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim\limits_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2\right) \le 0. \tag{1.3}$$

Letting

$$c_n := \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2)\right\}.$$

Then $c_n \ge 0$ $(\forall n \ge 1)$, $c_n \to 0$ $(n \to \infty)$ and (1.3) becomes

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2 + c_n, \quad \forall n \ge 1, \tag{1.4}$$

for all $x, y \in C$.

Whenever $c_n = 0$ for all $n \ge 1$ in (1.4), then T is an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$.

The weak convergence of modified Mann iteration process (1.1) and strong convergence of further modification of (1.1), that is, the following CQ method for (1.1), for asymptotically κ-strict pseudocontractive mappings in intermediate sense have been studied in [26].

The CQ method:

$$\begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(u), \ \forall n \ge 1, \end{cases}$$

where $F(T)$ is assumed to be nonempty and bounded, $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty$.

Furthermore, Martinez-Yanes and Xu [22] derived the strong convergence result of CQ method for Halpern iteration process for a nonexpansive mapping defined on a nonempty closed convex subset of a real Hilbert space.

On the other hand, the viscosity approximation method for finding a fixed point of a given nonexpansive mapping is initiated by Moudafi [23]. He proved the strong convergence of the sequence generated by his method to a unique solution of some variational inequality. It is further studied by several authors; See for example [4, 33] and references therein. Some related methods for variational inequalities and fixed points can be found in [3, 5, 8] and references therein.

The paper is organized as follows: In Section 2, we recall the useful definitions and lemmas. In Section 3, we study the strong convergence of the viscosity approximation method for the modified Mann iteration process (1.1) for the class of asymptotically κ-strict pseudocontractive mappings in intermediate sense. In Section 4, CQ algorithm is extended to develop the general CQ algorithm by Halpern iteration method. The strong convergence of the sequence generated by this general CQ algorithm is also studied. In Section 5, we extend the concept of asymptotically strict pseudocontractive mappings in intermediate sense to Banach space setting, called nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense. We first extend demiclosedness principle for nearly asymptotically κ-strict pseudocontractive mappings in intermediate sense in the setting of Banach spaces, and then establish the weak convergence theorems for a fixed point of a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense which is not necessarily Lipschitzian.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and its dual is denoted by X^*. The normalized duality mapping J from X into the family of nonempty weak* compact subsets of X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. Recall that the norm of X is Gâteaux differentiable (and X is said to be smooth) if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in X : \|x\| = 1\}$. X is smooth if and only if J is single-valued. In this case, $J : X \to X^*$ is continuous from the strong topology of X to the weak* topology of X^*, that is, norm-to-weak* continuous. The norm is called uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. Furthermore, the norm is called uniformly Fréchet differentiable (and X is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. Recall that if for every $\varepsilon > 0$ with $0 \le \varepsilon \le 2$, the modulus $\delta_X(\varepsilon)$ of convexity of a Banach space X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1-\frac{\|x+y\|}{2} : \|x\|\le 1,\ \|y\|\le 1,\ \|x-y\|\ge\varepsilon\right\}.$$

A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$. A Banach space X is said to be strictly convex if $\|(x + y)/2\| < 1$ for each $x, y \in U$ with $x \neq y$. Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. A uniformly convex Banach space is strictly convex and reflexive. The converse implication is false. A discussion on these and related concepts can be found in [12].

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ, the duality map $J_\varphi : X \to 2^{X^*}$ is defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in X.$$

We say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (that is, if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$). For instance, for each $1 < p < \infty$, the space l^p has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$; See [19] for more details.

In the case where $\varphi(t) = t$ for all $t \in [0,\infty)$, the associated duality reduces to the normalized duality map J. Now, set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \ge 0.$$

Then it can be easily seen that Φ is convex. It is also known that $J_\varphi(x)$ is the subdifferential $\partial\Phi(\|x\|)$ of the convex function $\Phi(\|\cdot\|)$ at x, that is,

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in X.$$

We adopt the following notations:

(i) $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence;
(ii) $\omega_w(\{x_n\}) = \{x \in X : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$;

(iii) $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of a self-mapping T on a set C.

Lemma 2.1 ([31]). *Let X be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|), \tag{2.2}$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in X : \|z\| \le r\}$. In particular, in the case when $X = H$ a Hilbert space and $g(t) = t^2$, $\forall t \in [0, \infty)$, then

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,$$

for all $x, y \in H$ and $t \in [0, 1]$.

The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [19].

Lemma 2.2. *Let X be a real Banach space and J_φ be the duality map associated with the gauge φ.*

(i) *For all $x, y \in X$ and $j \in J_\varphi(x + y)$,*

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, j\rangle.$$

In particular, for all $x, y \in X$ and $j \in J(x + y)$,

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j\rangle.$$

(ii) *Assume that J_φ is weakly continuous. Then for any sequence $\{x_n\}$ in X which converges weakly to a point $\hat{x}$, we have for all $y \in X$,*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - \hat{x}\|) + \Phi(\|y - \hat{x}\|). \tag{2.3}$$

In this case, X satisfies Opial's condition; that is, the weak convergence to $\hat{x}$ of $\{x_n\}$ implies that

$$\limsup_{n\to\infty} \|x_n - y\| > \limsup_{n\to\infty} \|x_n - \hat{x}\|, \quad \forall y \in X,\ y \ne \hat{x}.$$

We need some facts and tools which are listed as lemmas below:

Lemma 2.3 ([25, 28]). *Let $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality:*

$$\delta_{n+1} \le \beta_n\delta_n + \gamma_n, \quad \forall n \ge 1.$$

If $\beta_n \ge 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Lemma 2.4 ([32]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \bar{\alpha}_n)s_n + \bar{\alpha}_n\bar{\beta}_n + r_n, \quad \forall n \ge 1,$$

where $\{\bar{\alpha}_n\}, \{\bar{\beta}_n\}$ and $\{r_n\}$ satisfy the conditions: $\{\bar{\alpha}_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty}\bar{\alpha}_n = \infty$, $\limsup_{n\to\infty}\bar{\beta}_n \le 0$ and $r_n \ge 0$, $\sum_{n=1}^{\infty} r_n < \infty$. Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([1, Proposition 2.4]). *Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Let H be a real Hilbert space with inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

The following lemma can be found in any standard book on functional analysis.

Lemma 2.6. *Let C be a nonempty closed convex subset of a Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.7. *Let H be a real Hilbert space. Then the following statements hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;

(b) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$ *for all* $t \in [0,1]$ *and for all* $x, y \in H$;

(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

Lemma 2.8 ([22]). *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$ and given also a real number $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 2.9 ([26, Lemma 2.6]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-\kappa}\left(\kappa\|x - y\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x - y\|^2 + (1-\kappa)c_n}\right)$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 2.10 ([26, Lemma 2.7]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.11 ([14, Demiclosedness Principle]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x = y$.*

The following proposition is a generalization of for asymptotically κ-strict pseudocontractive mappings in intermediate sense in the setting of Hilbert spaces.

Proposition 2.12 (Demiclosedness Principle [26, Proposition 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Proposition 2.13 ([26, Proposition 3.2]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Remark 2.14. Propositions 2.12 and 2.13 give some basic properties of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Moreover, Proposition 2.12 extends the demiclosedness principles studied for certain classes of nonlinear mappings in Gornicki [15], Kim and Xu [18], Marino and Xu [21] and Xu [31].

3. Hybrid viscosity approximation method for modified Mann iteration process

In this section, we prove the strong convergence of viscosity approximation method for modified Mann iteration process (1.1) for an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction mapping with contractive constant $\alpha \in (0, 1)$, $S : C \to C$ a nonexpansive mapping, and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(S) \cap F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0, 1]$ such that $0 < \delta \leq \beta_n \leq 1 - \kappa$,*

$$\sum_{n=1}^{\infty} \beta_n c_n < \infty, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \alpha_n \to 0.$$

Let $\{x_n\}$ be a sequence in C generated by the following viscosity approximation method for Mann iteration process:

$$\left\{ \begin{array}{l} x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)Sy_n, \ \forall n \geq 1. \end{array} \right. \tag{3.1}$$

If $\|x_{n+1} - x_n\| \to 0$ and $\|y_n - Sy_n\| \to 0$, then $\{x_n\}$ converges strongly to a unique solution $\tilde{x}$ in $F(S) \cap F(T)$ to the following variational inequality:

$$\langle (I - f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in F(S) \cap F(T). \tag{3.2}$$

In other words, $\tilde{x} = P_{F(S)\cap F(T)} f(\tilde{x})$.

Proof. Let p be an arbitrary element in $F(S) \cap F(T)$. Using Lemma 2.7 (b), we obtain

(3.3)
$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n[(1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - T^n x_n\|^2 + c_n] \\ &\quad -\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\le (1+\gamma_n)\|x_n - p\|^2 - \beta_n(1-\beta_n-\kappa)\|x_n - T^n x_n\|^2 + \beta_n c_n \\ &\le (1+\gamma_n)\|x_n - p\|^2 + \beta_n c_n.\end{aligned}$$

For the rest of the proof we divide it into several steps.

Step 1. We claim that both the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, observe that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(f(y_n) - p) + (1-\alpha_n)(Sy_n - p)\|^2 \\ &\le (1-\alpha_n)\|Sy_n - p\|^2 + \alpha_n\|f(y_n) - p\|^2 \\ &\le (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\left(\|f(y_n) - f(p)\| + \|f(p) - p\|\right)^2 \\ &\le (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\left(\alpha\|y_n - p\| + \|f(p) - p\|\right)^2 \\ &= (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\left(\alpha\|y_n - p\| + (1-\alpha)\tfrac{\|f(p)-p\|}{1-\alpha}\right)^2 \\ &\le (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\left(\alpha\|y_n - p\|^2 + \tfrac{\|f(p)-p\|^2}{1-\alpha}\right) \\ &= (1-(1-\alpha)\alpha_n)\|y_n - p\|^2 + \alpha_n\tfrac{\|f(p)-p\|^2}{1-\alpha} \\ &\le \max\left\{\|y_n - p\|^2, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}.\end{aligned} \tag{3.4}$$

We claim that for all $n \ge 1$

$$\|x_{n+1} - p\|^2 \le \left(\prod_{j=1}^{n}(1+\gamma_j)\right)\left(\sum_{i=1}^{n}\beta_i c_i + \max\left\{\|x_1 - p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right). \tag{3.5}$$

As a matter of fact, whenever $n = 1$, from (3.3) and (3.4) we have

$$\begin{aligned}\|x_2 - p\|^2 &\le \max\left\{\|y_1 - p\|^2, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\ &\le \max\left\{(1+\gamma_1)\|x_1 - p\|^2 + \beta_1 c_1, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\ &\le \max\left\{(1+\gamma_1)\|x_1 - p\|^2 + \beta_1 c_1, (1+\gamma_1)\tfrac{\|f(p)-p\|^2}{(1-\alpha)^2} + \beta_1 c_1\right\} \\ &= (1+\gamma_1)\max\left\{\|x_1 - p\|^2, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} + \beta_1 c_1 \\ &\le (1+\gamma_1)\left(\max\left\{\|x_1 - p\|^2, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} + \beta_1 c_1\right) \\ &= \left(\prod_{j=1}^{1}(1+\gamma_j)\right)\left(\sum_{i=1}^{1}\beta_i c_i + \max\left\{\|x_1 - p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right).\end{aligned}$$

Assume that (3.5) holds for some $n \geq 1$. Consider the case of $n+1$. From (3.3) and (3.4), we obtain

$$
\begin{aligned}
&\|x_{n+2}-p\|^2 \\
&\leq \max\left\{\|y_{n+1}-p\|^2, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\
&\leq \max\left\{(1+\gamma_{n+1})\|x_{n+1}-p\|^2+\beta_{n+1}c_{n+1}, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\
&\leq \max\left\{(1+\gamma_{n+1})\left(\prod_{j=1}^{n}(1+\gamma_j)\right)\left(\sum_{i=1}^{n}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right)\right. \\
&\quad \left. +\beta_{n+1}c_{n+1}, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\
&= \max\left\{\left(\prod_{j=1}^{n+1}(1+\gamma_j)\right)\left(\sum_{i=1}^{n}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right)\right. \\
&\quad \left. +\beta_{n+1}c_{n+1}, \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\
&\leq \max\left\{\left(\prod_{j=1}^{n+1}(1+\gamma_j)\right)\left(\sum_{i=1}^{n+1}\beta_i c_i+\max\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\}\right), \tfrac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\} \\
&\leq \max\left\{\left(\prod_{j=1}^{n+1}(1+\gamma_j)\right)\left(\sum_{i=1}^{n+1}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right),\right. \\
&\quad \left.\left(\prod_{j=1}^{n+1}(1+\gamma_j)\right)\left(\sum_{i=1}^{n+1}\beta_i c_i+\frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right)\right\} \\
&= \left(\prod_{j=1}^{n+1}(1+\gamma_j)\right)\left(\sum_{i=1}^{n+1}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right).
\end{aligned}
$$

This shows that (3.5) holds for the case of $n+1$. By induction, (3.5) holds for all $n \geq 1$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n c_n < \infty$, from (3.5) we deduce that for all $n \geq 1$,

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left(\prod_{j=1}^{n}(1+\gamma_j)\right)\left(\sum_{i=1}^{n}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right) \\
&\leq \exp\left(\sum_{j=1}^{n}\gamma_j\right)\left(\sum_{i=1}^{n}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right) \\
&\leq \exp\left(\sum_{j=1}^{\infty}\gamma_j\right)\left(\sum_{i=1}^{\infty}\beta_i c_i+\max\left\{\|x_1-p\|^2, \frac{\|f(p)-p\|^2}{(1-\alpha)^2}\right\}\right).
\end{aligned}
$$

This implies that $\{x_n\}$ is bounded and so is $\{y_n\}$ by virtue of (3.3).

Step 2. We claim that $\|x_n - Tx_n\| \to 0$. Indeed, it follows from (3.1) that

$$\|x_{n+1}-y_n\| = \|\alpha_n(f(y_n)-y_n)+(1-\alpha_n)(Sy_n-y_n)\| \leq \alpha_n\|f(y_n)-y_n\|+\|Sy_n-y_n\|.$$

Since $f, S : C \to C$ are Lipschitzian, $\|Sy_n - y_n\| \to 0$ and $\alpha_n \to 0$, from the boundedness of $\{y_n\}$ we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.6}$$

Since

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

and $\|x_{n+1} - x_n\| \to 0$, it follows from (3.6) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.7}$$

Since

$$\|T^n x_n - x_n\| = \frac{1}{\beta_n}\|y_n - x_n\| \le \frac{1}{\delta}\|y_n - x_n\|,$$

from (3.7), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.8}$$

Since $\|x_{n+1} - x_n\| \to 0$, $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$ and $T : C \to C$ is uniformly continuous, we obtain from Lemma 2.10 that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Step 3. We claim that

$$\limsup_{n\to\infty} \langle \tilde{x} - y_n, \tilde{x} - f(\tilde{x})\rangle \le 0, \tag{3.9}$$

where $\tilde{x} = P_{F(S)\cap F(T)} f(\tilde{x})$.

Indeed, by Proposition 2.13, $F(T)$ is closed and convex, and so is $F(S) \cap F(T)$. Since $f : C \to C$ is a contraction with contractive constant $\alpha \in (0, 1)$ and $P_{F(S)\cap F(T)} : H \to F(S)\cap F(T)$ is a nonexpansive mapping, the composite mapping $P_{F(S)\cap F(T)} f : C \to F(S) \cap F(T)$ $(\subseteq C)$ is a contraction with contractive constant $\alpha \in (0, 1)$. By Banach contraction principle, there exists a unique $\tilde{x} \in C$ such that $\tilde{x} = P_{F(S)\cap F(T)} f(\tilde{x})$, equivalently, $\tilde{x}$ is the unique solution in $F(S)\cap F(T)$ to the following variational inequality:

$$\langle (I - f)\tilde{x}, x - \tilde{x}\rangle \ge 0, \quad \forall x \in F(S) \cap F(T).$$

Take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \tilde{x} - y_n, \tilde{x} - f(\tilde{x})\rangle = \lim_{k\to\infty} \langle \tilde{x} - y_{n_k}, \tilde{x} - f(\tilde{x})\rangle. \tag{3.10}$$

We may assume that $y_{n_k} \rightharpoonup \bar{x}$. It follows from Lemma 2.11 and $\|y_n - Sy_n\| \to 0$ that $\bar{x} \in F(S)$. From (3.7) and $y_{n_k} \rightharpoonup \bar{x}$ it follows that $x_{n_k} \rightharpoonup \bar{x}$. Since T is uniformly continuous and $\|x_n - Tx_n\| \to 0$ by Step 2, we have $\|x_n - T^m x_n\| \to 0$ for all $m \ge 1$. By Proposition 2.12, we obtain $\bar{x} \in F(T)$. Consequently, we deduce that $\bar{x} \in F(S) \cap F(T)$ and so from (3.2) we obtain

$$\langle (I - f)\tilde{x}, \bar{x} - \tilde{x}\rangle \ge 0.$$

Therefore, we derive from (3.10)

$$\limsup_{n\to\infty} \langle \tilde{x} - y_n, \tilde{x} - f(\tilde{x})\rangle = \lim_{k\to\infty} \langle \tilde{x} - y_{n_k}, \tilde{x} - f(\tilde{x})\rangle = \langle \tilde{x} - \bar{x}, \tilde{x} - f(\tilde{x})\rangle \le 0$$

as required.

Step 4. We claim that $\|x_n - \tilde{x}\| \to 0$ as $n \to \infty$. Indeed, observe that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|(1-\alpha_n)(Sy_n - \tilde{x}) + \alpha_n(f(y_n) - \tilde{x})\|^2 \\
&= (1-\alpha_n)^2\|Sy_n - \tilde{x}\|^2 + \alpha_n^2\|f(y_n) - \tilde{x}\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle Sy_n - \tilde{x}, f(y_n) - \tilde{x}\rangle \\
&\le (1-2\alpha_n+\alpha_n^2)\|y_n - \tilde{x}\|^2 + \alpha_n^2\|f(y_n) - \tilde{x}\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle Sy_n - \tilde{x}, f(y_n) - f(\tilde{x})\rangle \\
&\quad +2\alpha_n(1-\alpha_n)\langle Sy_n - \tilde{x}, f(\tilde{x}) - \tilde{x}\rangle \\
&\le \big(1-2\alpha_n+\alpha_n^2+2\alpha\alpha_n(1-\alpha_n)\big)\,\|y_n - \tilde{x}\|^2 \\
&\quad +\alpha_n\left[2(1-\alpha_n)\langle Sy_n - \tilde{x}, f(\tilde{x}) - \tilde{x}\rangle + \alpha_n\|f(y_n) - \tilde{x}\|^2\right] \\
&= (1-\bar{\alpha}_n)\|y_n - \tilde{x}\|^2 + \bar{\alpha}_n\bar{\beta}_n,
\end{aligned} \tag{3.11}
$$

where $\bar{\alpha}_n = \alpha_n(2-\alpha_n-2\alpha(1-\alpha_n))$ and

$$\bar{\beta}_n = \frac{2(1-\alpha_n)\langle Sy_n - \tilde{x}, f(\tilde{x}) - \tilde{x}\rangle + \alpha_n\|f(y_n) - \tilde{x}\|^2}{2-\alpha_n-2\alpha(1-\alpha_n)}.$$

It is readily seen that $\bar{\alpha}_n \to 0$, $\sum_{n=1}^{\infty}\bar{\alpha}_n = \infty$, and $\limsup_{n\to\infty}\bar{\beta}_n \le 0$ by virtue of (3.9).

On the other hand, from (3.3) and (3.10), we get

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le (1-\bar{\alpha}_n)\|y_n - \tilde{x}\|^2 + \bar{\alpha}_n\bar{\beta}_n \\
&\le (1-\bar{\alpha}_n)\left[(1+\gamma_n)\|x_n - \tilde{x}\|^2 + \beta_n c_n\right] + \bar{\alpha}_n\bar{\beta}_n \\
&\le (1-\bar{\alpha}_n)\|x_n - \tilde{x}\|^2 + \bar{\alpha}_n\bar{\beta}_n + \gamma_n\|x_n - \tilde{x}\|^2 + \beta_n c_n \\
&= (1-\bar{\alpha}_n)\|x_n - \tilde{x}\|^2 + \bar{\alpha}_n\bar{\beta}_n + r_n,
\end{aligned}
$$

where $r_n = \gamma_n\|x_n - \tilde{x}\|^2 + \beta_n c_n$ for all $n \ge 1$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\beta_n c_n < \infty$ and $\{x_n\}$ is bounded, we know that $\sum_{n=1}^{\infty} r_n < \infty$. Therefore, in terms of Lemma 2.4, we conclude that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. □

As a consequence of Theorem 3.1, we derive the following corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction mapping with contractive constant $\alpha \in (0,1)$, $S : C \to C$ a nonexpansive mapping, and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense (in this case, $\gamma_n = 0$, $\forall n \ge 1$) such that $F(S) \cap F(T) \ne \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that $0 < \delta \le \beta_n \le 1-\kappa$,*

$$\sum_{n=1}^{\infty}\beta_n c_n < \infty,\ \sum_{n=1}^{\infty}\alpha_n = \infty,\ \text{and } \alpha_n \to 0.$$

Let $\{x_n\}$ be a sequence in C generated by (3.1). *If $\|x_{n+1} - x_n\| \to 0$ and $\|y_n - Sy_n\| \to 0$, then $\{x_n\}$ converges strongly to a unique solution $\tilde{x}$ in $F(S) \cap F(T)$ to the variational inequality* (3.2).

Corollary 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction mapping with contractive constant $\alpha \in (0,1)$ and T :*

$C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ such that $0 < \delta \leq \beta_n \leq 1 - \kappa$,

$$\sum_{n=1}^{\infty} \beta_n c_n < \infty, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \text{and } \alpha_n \to 0.$$

Let $\{x_n\}$ be a sequence in C generated by the following viscosity approximation method for Mann iteration process:

$$\left\{ \begin{array}{l} x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)y_n, \ \forall n \geq 1. \end{array} \right. \tag{3.12}$$

If $\|x_{n+1} - x_n\| \to 0$, then $\{x_n\}$ converges strongly to a unique solution $\tilde{x}$ in $F(T)$ to the following variational inequality

$$\langle (I-f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T). \tag{3.13}$$

In particular, whenever $f \equiv u$ $(\in C)$ a constant, $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Corollary 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction with contractive constant $\alpha \in (0,1)$ and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that $0 < \delta \leq \beta_n \leq 1 - \kappa$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$. Let $\{x_n\}$ be a sequence in C generated by* (3.12). *If $\|x_{n+1} - x_n\| \to 0$, then $\{x_n\}$ converges strongly to a unique solution $\tilde{x}$ in $F(T)$ to the variational inequality* (3.13).

Example 3.5. Let

$$H = \ell^2 = \left\{ \{a_j\}_{j=1}^{\infty} : \{a_j\}_{j=1}^{\infty} \text{ is a real sequence satisfying } \sum_{j=1}^{\infty} a_j^2 < \infty \right\}.$$

with its inner product $\langle .,. \rangle$ and norm $\| \cdot \|$ are defined as

$$\langle x, y \rangle = \sum_{j=1}^{\infty} a_j b_j \quad \text{and} \quad \|x\| = \left(\sum_{j=1}^{\infty} a_j^2 \right)^{\frac{1}{2}},$$

for all $x, y \in H$ with $x = \{a_j\}_{j=1}^{\infty}$ and $y = \{b_j\}_{j=1}^{\infty}$. Then ℓ^2 is a real Hilbert space. Also, let $C = \ell^2$ and

$$R_\infty = \left\{ \{a_j\}_{j=1}^{\infty} \in l_2 : \exists j_0 \geq 1 \text{ such that } a_j = 0, \forall j \geq j_0 + 1 \right\}.$$

Then $0 = \{0, 0, ...\} \in R_\infty$ and the closed hull of R_∞ is equal to ℓ^2, that is, $\overline{R_\infty} = l^2$. Furthermore, define the mappings $f, S, T : C \to C$ by

$$f(x) = \alpha x, \quad Sx = x$$

and

$$Tx = T(\{a_1, a_2, a_3, ...\}) = \{a_2, a_3, ...\}$$

for some $\alpha \in (0,1)$ and all $x \in \ell^2$ where $x = \{a_j\}_{j=1}^{\infty}$. Then T is a nonexpansive mapping and hence an asymptotically κ-strictly pseudocontractive mapping in intermediate sense with sequences $\gamma_n = 0$ and $c_n = 0$ for all $n \geq 0$ where $\kappa = 0$. It is easy to see that $0 = \{0, 0, ...\}$ is a unique fixed point of T in H and that

$$T^n x = \{a_{n+1}, a_{n+2}, a_{n+3}, ...\}.$$

Now, take $x_1 \in R_\infty$ arbitrarily. Then there exists an integer $j_0 \geq 1$ such that $x_1 = \left\{a_1^{(1)}, a_2^{(1)}, ..., a_{j_0}^{(1)}, 0, ...\right\}$. In this case, from the iterative scheme (3.1) in Theorem 3.1 we have

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = \alpha_n \alpha y_n + (1-\alpha_n) y_n, \quad \forall n \geq 1. \end{cases}$$

Since $x_1 = \left\{a_1^{(1)}, a_2^{(1)}, ..., a_{j_0}^{(1)}, 0, ...\right\}$, we know that $T^n x_n = 0$ for all $n \geq j_0$. Thus it follows that for all $n \geq j_0$

$$\begin{aligned} x_{n+1} &= (1-(1-\alpha)\alpha_n)y_n \\ &= (1-(1-\alpha)\alpha_n)[(1-\beta_n)x_n + \beta_n T^n x_n] \\ &= (1-(1-\alpha)\alpha_n)(1-\beta_n)x_n, \end{aligned}$$

and hence for $0 < \delta \leq \beta_n \leq 1$

$$\begin{aligned} \|x_{n+1}\| &= \|(1-(1-\alpha)\alpha_n)(1-\beta_n)x_n\| \\ &\leq (1-\beta_n)\|x_n\| \leq (1-\delta)\|x_n\| \leq ... \leq (1-\delta)^{n-j_0+1}\|x_{j_0}\|. \end{aligned}$$

So, $\{x_n\}$ converges strongly to $0 = \{0, 0, ...\} \in \{0\} = F(T) = F(S) \cap F(T)$. There is no doubt that $0 = \{0, 0, ...\}$ is a unique solution in $F(S) \cap F(T)$ to the following variational inequality:

$$\langle (I-f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(S) \cap F(T).$$

In other words, $\tilde{x} = P_{F(S)\cap F(T)} f(\tilde{x})$.

On the other hand, in terms of Theorem 3.4 of [26], we know that the sequence $\{x_n\}$ generated by the modified Mann iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1,$$

converges weakly to $0 = \{0, 0, ...\} \in \{0\} = F(T)$, where $0 < \delta \leq \alpha_n \leq 1 - \delta < 1$.

Remark 3.6. (a) Compare with the corresponding results studied by Kim and Xu [18], Marino and Xu [21], Sahu et al. [26] and Schu [27] , Theorem 3.1 is a strong convergence result.

(b) Compared with Theorem 3.4 of [26], Theorem 3.1 is more general in the following ways:

(i) Theorem 3.1 is a strong convergence one, but Theorem 3.4 of [26] is a weak convergence one;

(ii) Our problem of finding an element of $F(S) \cap F(T)$ is more general than the one of finding an element of $F(T)$ in [26, Theorem 3.4];

(iii) Our method for proving the boundedness and strong convergence of $\{x_n\}$ is very different from the one of proof in [26, Theorem 3.4], and hence is nontrivial;

(iv) The condition $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ in [26, Theorem 3.4] is replaced by our weaker one $0 < \delta \leq \beta_n \leq 1 - \kappa$ in Theorem 3.1.

If $F(S) \subset F(T)$ and T is not necessarily uniformly continuous but asymptotically κ-strict pseudocontractive mapping in intermediate sense, then we have the following result.

Theorem 3.7. *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let $f : C \to C$ be a contraction with contractive constant $\alpha \in (0,1)$, $S : C \to C$ be a nonexpansive mapping, and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \supset F(S)$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that $0 \leq \beta_n \leq 1 - \kappa$,*

$$\sum_{n=1}^{\infty} \beta_n c_n < \infty, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0 \text{ and } \beta_n \to 0.$$

If $\|x_{n+1} - x_n\| \to 0$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a unique solution $\tilde{x}$ in $F(S)$ to the following variational inequality:

$$\langle (I - f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{3.14}$$

In other words, $\tilde{x} = P_{F(S)} f(\tilde{x})$.

Proof. Since C is a nonempty closed convex bounded subset of a real Hilbert space H, it is clear that the concept of asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ is equivalent to the one of nearly asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. In this case, we know that $c_n = \theta_n$ and $g(t) = t^2, \ \forall t \in [0, \infty)$.

Let p be an arbitrary element in $F(S)$. AS in the proof of Theorem 3.1, by using Lemma 2.7 (ii) and the assumption $F(S) \subset F(T)$, we obtain

$$\|y_n - p\|^2 \leq (1 + \gamma_n)\|x_n - p\|^2 + \beta_n c_n, \quad \forall n \geq 1. \tag{3.15}$$

We claim that

$$\|y_n - Sy_n\| \to 0.$$

Indeed, it follows from (3.1) that

$$\begin{aligned} \|y_n - Sy_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| \\ &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| \\ &= \beta_n \|T^n x_n - x_n\| + \|x_n - x_{n+1}\| + \alpha_n \|f(y_n) - Sy_n\|. \end{aligned} \tag{3.16}$$

Since C is bounded, and $f : C \to C$, $S : C \to C$ and $T : C \to C$ all are self-mappings, $\{f(y_n)\}$, $\{Sy_n\}$ and $\{T^n x_n\}$ all are bounded. Hence, from (3.16) and $\alpha_n \to 0$, $\beta_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, we obtain $\|y_n - Sy_n\| \to 0$.

As in Step 3 in the proof of Theorem 3.1, we have

$$\limsup_{n \to \infty} \langle \tilde{x} - y_n, \tilde{x} - f(\tilde{x}) \rangle \leq 0, \tag{4.5}$$

where $\tilde{x} = P_{F(S)} f(\tilde{x})$.

Finally, as in Step 4 in the proof of Theorem 3.1, we obatin $x_n \to \tilde{x}$. □

From Theorem 3.7, we derive the following corollaries.

Corollary 3.8. *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let $f : C \to C$ be a contraction with contractive constant $\alpha \in (0,1)$, $S : C \to C$ be a nonexpansive mapping, and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ such that $F(T) \supset F(S)$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that $0 \leq \beta_n \leq 1-\kappa$,*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0 \text{ and } \beta_n \to 0.$$

If $\|x_{n+1} - x_n\| \to 0$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a unique solution $\tilde{x}$ in $F(S)$ to the variational inequality (3.14).

Proof. Note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ must be an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. In this case, $c_n = 0$ for all $n \geq 1$. Thus, from Theorem 3.7 we immediately obtain the desired result. □

Corollary 3.9. *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let $f : C \to C$ be a contraction with contractive constant $\alpha \in (0,1)$, $S : C \to C$ be a nonexpansive mapping, and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $F(T) \supset F(S)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0 \text{ and } \beta_n \to 0.$$

If $\|x_{n+1} - x_n\| \to 0$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a unique solution $\tilde{x}$ in $F(S)$ to the variational inequality (3.14).

Corollary 3.10. *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let $f : C \to C$ be a contraction with contractive constant $\alpha \in (0,1)$, and $S, T : C \to C$ be two nonexpansive mappings such that $F(T) \supset F(S)$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0 \text{ and } \beta_n \to 0.$$

If $\|x_{n+1} - x_n\| \to 0$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a unique solution $\tilde{x}$ in $F(S)$ to the variational inequality (3.14).

4. CQ method for modified Mann iteration process

Recall that in [26, Theorem 4.1], Sahu et al. established the strong convergence criteria on CQ method for modified Mann iteration process for an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ in a real Hilbert space H. In this section, CQ algorithm is extended to develop a general

CQ algorithm by virtue of Halpern iteration method. The main result of this section is the following which is more general than Theorem 4.1 in [26].

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping, and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(S) \cap F(T)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ such that $\alpha_n \to 0$ and $0 < \delta \le \beta_n \le 1 - \kappa$ for all $n \ge 1$. Let $\{x_n\}$ be a sequence in C generated by the following general CQ algorithm:*

$$(4.1) \quad \begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ z_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ y_n = \alpha_n u + (1-\alpha_n) S z_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \le \alpha_n \|u - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 + \theta_n \right\}, \\ Q_n = \left\{ z \in C : \langle x_n - z, u - x_n \rangle \ge 0 \right\}, \\ x_{n+1} = P_{C_n \cap Q_n}(u), \quad \forall n \ge 1, \end{cases}$$

where $\theta_n = c_n + \gamma_n \varDelta_n$ and $\Delta_n = \sup\left\{\|x_n - z\|^2 : z \in F(S) \cap F(T)\right\} < \infty$. If $\|z_n - Sz_n\| \to 0$, then the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}(u)$.

Proof. We divide the proof into the following six steps.

Step 1. We claim that C_n is closed and convex. Indeed, the defining inequality in C_n is equivalent to the inequality

$$2\langle \alpha_n u + (1-\alpha_n)x_n - y_n, z\rangle \le \alpha_n \|u\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \theta_n.$$

Thus, it is easy to see that C_n is closed and convex.

Step 2. We claim that $F(S) \cap F(T) \subset C_n$. Indeed, let $p \in F(S) \cap F(T)$. From (4.1), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\left[(1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - T^n x_n\|^2 + c_n\right] \\ &\quad -\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\le \|x_n - p\|^2 + \beta_n(\kappa - (1-\beta_n))\|x_n - T^n x_n\|^2 + c_n + \gamma_n\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 + c_n + \gamma_n \varDelta_n, \end{aligned}$$

and hence

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(u - p) + (1-\alpha_n)(Sz_n - p)\|^2 \\ &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|Sz_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\left[\|x_n - p\|^2 + c_n + \gamma_n \varDelta_n\right] \\ &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + c_n + \gamma_n \varDelta_n. \end{aligned}$$

Thus, $p \in C_n$.

Step 3. We claim that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \ge 1$. Indeed, it is sufficient to show that $F(S) \cap F(T) \subset Q_n$. We prove this by induction.

For $n = 1$, we have $F(S) \cap F(T) \subset C = Q_1$. Assume that $F(S) \cap F(T) \subset Q_n$ for some $n \ge 1$. Since x_{n+1} is the projection of u onto $C_n \cap Q_n$, it follows that

$$\langle x_{n+1} - z, u - x_{n+1}\rangle \ge 0, \quad \forall z \in C_n \cap Q_n.$$

As $F(S) \cap F(T) \subset C_n \cap Q_n$, the last inequality holds, in particular for all $z \in F(S) \cap F(T)$. By the definition of Q_{n+1},

$$Q_{n+1} = \{z \in C : \langle x_{n+1} - z, u - x_{n+1} \rangle \geq 0\},$$

it follows that $F(S) \cap F(T) \subset Q_{n+1}$. By induction, we get

$$F(S) \cap F(T) \subset Q_n, \quad \forall n \geq 1.$$

Step 4. We claim that $\|x_n - x_{n+1}\| \to 0$. Indeed, by the definition of Q_n, we have $x_n = P_{Q_n}(u)$ and hence

$$\|u - x_n\| \leq \|u - y\|, \quad \forall y \in F(S) \cap F(T) \subset Q_n.$$

Note that the boundedness of $F(S) \cap F(T)$ implies that $\{\|x_n - u\|\}$ is bounded. Since $x_n = P_{Q_n}(u)$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|u - x_n\| \leq \|u - x_{n+1}\|.$$

Thus, $\{\|u - x_n\|\}$ is nondecreasing. Since $\{\|x_n - u\|\}$ is bounded, we obtain that $\lim_{n\to\infty} \|x_n - u\|$ exists.

Observe that $x_n = P_{Q_n}(u)$ and $x_{n+1} \in Q_n$ which imply that

$$\langle x_{n+1} - x_n, x_n - u \rangle \geq 0.$$

Using Lemma 2.7 (a), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - u - (x_n - u)\|^2 \\ &= \|x_{n+1} - u\|^2 - \|x_n - u\|^2 - \langle x_{n+1} - x_n, x_n - u \rangle \\ &\leq \|x_{n+1} - u\|^2 - \|x_n - u\|^2 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Step 5. We claim that $\|x_n - Tx_n\| \to 0$. Indeed, for $p \in F(S) \cap F(T)$, from Lemma 2.9 we have

$$\|T^n x_n - p\| \leq \frac{1}{1-\kappa} \left(\kappa \|x_n - p\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x_n - p\|^2 + (1-\kappa)c_n} \right).$$

This together with the boundedness of $\{x_n\}$, implies that $\{T^n x_n\}$ is bounded. Hence from the definition of z_n, it follows that $\{z_n\}$ is bounded.

Now by the definition of z_n, we have

$$\begin{aligned} \|x_n - T^n x_n\|^2 &= \beta_n^{-1} \|x_n - z_n\| \\ &\leq \beta_n^{-1} \left(\|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \right) \\ &\leq \delta^{-1} \left(\|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \right) \\ &\leq \delta^{-1} \left(\|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - z_n\| \right). \end{aligned} \tag{4.2}$$

Since $x_{n+1} \in C_n$, we have from $\alpha_n \to 0$

$$\|y_n - x_{n+1}\|^2 \leq \alpha_n \|u - x_{n+1}\|^2 + (1 - \alpha_n)\|x_n - x_{n+1}\|^2 + c_n + \gamma_n \mathit{\Delta}_n \to 0.$$

It follows from $\|Sz_n - z_n\| \to 0$ that

$$\begin{aligned} \|y_n - z_n\| &= \|\alpha_n(u - z_n) + (1 - \alpha_n)(Sz_n - z_n)\| \\ &\leq \alpha_n \|u - z_n\| + (1 - \alpha_n)\|Sz_n - z_n\| \to 0. \end{aligned}$$

Consequently, from (4.2) we conclude that

$$\|x_n - T^n x_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.3}$$

By Step 4 and (4.3), we obtain from Lemma 2.10 that $x_n - Tx_n \to 0$ as $n \to \infty$.

Step 6. We claim that $x_n \to v \in F(S) \cap F(T)$. Indeed, since H is reflexive and $\{x_n\}$ is bounded, we obtain that $\omega_w(\{x_n\})$ is nonempty. Let us show that $\omega_w(\{x_n\})$ is a singleton. Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Since $\|x_n - Tx_n\| \to 0$ by Step 5, it follows from the uniform continuity of T that $\|x_n - T^m x_n\| \to 0$ for all $m \geq 1$. By Proposition 2.12, $v \in \omega_w(\{x_n\}) \subset F(T)$. On the other hand, since $\|x_{n+1} - x_n\| \to 0$, $\|x_{n+1} - y_n\| \to 0$ and $\|y_n - z_n\| \to 0$, we deduce that $\|x_n - z_n\| \to 0$ and hence $z_{n_i} \rightharpoonup v$. Thus, from Lemma 2.11 we have $v \in \omega_w(\{x_n\}) \subset F(S)$. Consequently, $v \in \omega_w(\{x_n\}) \subset F(S) \cap F(T)$.

Since $x_{n+1} = P_{C_n \cap Q_n}(u)$, we obtain that

$$\|u - x_{n+1}\| \leq \|u - P_{F(S)\cap F(T)}(u)\|, \quad \forall n \geq 1.$$

Observe that

$$\|u - x_{n_i}\| \rightharpoonup \|u - v\|.$$

By weak lower semicontinuity of the norm, we have

$$\begin{aligned} \|u - P_{F(S)\cap F(T)}(u)\| &\leq \|u - v\| \\ &\leq \liminf_{i\to\infty} \|u - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|u - x_{n_i}\| \\ &\leq \|u - P_{F(S)\cap F(T)}(u)\|, \end{aligned}$$

which yields

$$\|u - P_{F(S)\cap F(T)}(u)\| = \|u - v\|$$

and

$$\lim_{i\to\infty} \|u - x_{n_i}\| = \|u - P_{F(S)\cap F(T)}(u)\|. \tag{4.4}$$

Hence $v = P_{F(S)\cap F(T)}(u)$ by the uniqueness of the nearest point projection of u onto $F(S) \cap F(T)$. Thus, $\|x_{n_i} - u\| \to \|v - u\|$. This shows that $\|x_{n_i} - u\| \to \|v - u\|$, that is, $x_{n_i} \to v$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, it follows that $\omega_w(\{x_n\}) = \{v\}$ and hence from Lemma 2.5 we have $x_n \rightharpoonup v$. It is easy to see as (4.4) that $\|x_n - u\| \to \|v - u\|$. Therefore, $x_n \to v$. □

From Theorem 4.1, we derive the following results which appeared recently in the literature.

Corollary 4.2 ([26, Theorem 4.1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence in C generated by the following CQ algorithm:*

$$\begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \left\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \vartheta_n\right\}, \\ Q_n = \left\{z \in C : \langle x_n - z, u - x_n\rangle \geq 0\right\}, \\ x_{n+1} = P_{C_n\cap Q_n}(u), \ \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\vartheta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\left\{\|x_n - z\|^2 : z \in F(T)\right\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Proof. In Theorem 4.1, put $S \equiv I$ the identity mapping, and $\alpha_n = 0$ for all $n \geq 1$. Then, $F(S) \cap F(T) = F(T)$ and $y_n = z_n$. In this case, (4.1) reduces to (4.5). Thus, from Theorem 4.1 we obtain the desired conclusion. □

Corollary 4.3 ([17, Theorem 2.2])**.** *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ in $[1, \infty)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in C by the following algorithm:*

$$
\left\{
\begin{array}{l}
u = x_1 \in C \text{ chosen arbitrary}, \\
y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\
C_n = \left\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \vartheta_n\right\}, \\
Q_n = \left\{z \in C : \langle x_n - z, u - x_n \rangle \geq 0\right\}, \\
x_{n+1} = P_{C_n \cap Q_n}(u), \ \forall n \geq 1,
\end{array}
\right. \tag{4.6}
$$

where $\vartheta_n = (k_n^2 - 1)\mathrm{diam}(C)^2$ for all $n \geq 1$, where $\mathrm{diam}(C)$ denotes the diameter of C. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Corollary 4.4 ([24, Theorem 3.4])**.** *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in C by the following algorithm:*

$$
\left\{
\begin{array}{l}
u = x_1 \in C \text{ chosen arbitrary}, \\
y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\
C_n = \left\{z \in C : \|y_n - z\| \leq \|x_n - z\|\right\}, \\
Q_n = \left\{z \in C : \langle x_n - z, u - x_n \rangle \geq 0\right\}, \\
x_{n+1} = P_{C_n \cap Q_n}(u), \ \forall n \geq 1.
\end{array}
\right. \tag{4.7}
$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

5. Nearly asymptotically strict pseudocontractive mappings in intermediate sense

Motivated and inspired by the concept of asymptotically κ-strict pseudocontractive mappings in intermediate sense, we introduce the concept of nearly asymptotically strict pseudocontractive mappings in intermediate sense in the setting of Banach spaces, which are not necessarily Lipschitzian.

Definition 5.1. Let C be a nonempty bounded subset of a uniformly convex Banach space X. A mapping $T : C \to C$ is called a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$
\begin{aligned}
\limsup_{n\to\infty} \sup_{x,y\in C} \big(&\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\
&- \kappa \min\{\|x - T^n x - (y - T^n y)\|^2, g(\|x - T^n x - (y - T^n y)\|)\}\big) \leq 0,
\end{aligned} \tag{5.1}
$$

where $g : [0, 2r] \to \mathbb{R}$ is defined as in Lemma 2.1 and $r = \sup\{\|x\| : x \in C\}$.

Remark 5.2. Let C be a nonempty bounded subset of a Hilbert space H. If we take $g(t) = t^2$, $\forall t \in [0, \infty)$, then each asymptotically κ-strict pseudocontractive mapping $T: C \to C$ in intermediate sense with sequence $\{\gamma_n\}$ must be a nearly asymptotically κ-strict pseudocontractive mappings in intermediate sense with sequence $\{\gamma_n\}$. As pointed out in [26], if $T: C \to C$ is an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ then T is not necessarily uniformly L-Lipschitzian. Therefore, there is no doubt that in a uniformly convex Banach space, every nearly asymptotically κ-strict pseudocontractive mapping $T: C \to C$ in intermediate sense with sequence $\{\gamma_n\}$ is not necessarily uniformly L-Lipschitzian. Actually, this can be seen from Lemma 5.4.

We study some properties and convergence of some iteration processes for the class of nearly asymptotically κ-strict pseudocontractive mappings in the intermediate sense. Rest of the paper we, assume that

$$\begin{aligned}\theta_n := \max\{0, \sup_{x,y\in C} (&\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 \\ &-\kappa \min\{\|x - T^n x - (y - T^n y)\|^2, g(\|x - T^n x - (y - T^n y)\|)\})\}.\end{aligned}$$

Then $\theta_n \geq 0$ ($\forall n \geq 1$), $\theta_n \to 0$ ($n \to \infty$), and (5.1) reduces to the following relation

$$\begin{aligned}\|T^n x - T^n y\|^2 &\leq (1+\gamma_n)\|x-y\|^2 \\ &+\kappa \min\left\{\|x - T^n x - (y - T^n y)\|^2, g(\|x - T^n x - (y - T^n y)\|)\right\} + \theta_n\end{aligned} \tag{5.2}$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 5.3. *Let C be a nonempty bounded subset of a uniformly convex Banach space X and $T: C \to C$ be a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-\kappa}\left(\kappa\|x-y\| + \sqrt{(1+(1-\kappa)\gamma_n)\|x-y\|^2 + (1-\kappa)\theta_n}\right)$$

for all $x, y \in C$ and $n \geq 1$.

Proof. For all $x, y \in C$, we have

$$\begin{aligned}&\|T^n x - T^n y\|^2 \\ &\leq (1+\gamma_n)\|x-y\|^2 \\ &\quad +\kappa \min\left\{\|x - T^n x - (y - T^n y)\|^2, g(\|x - T^n x - (y - T^n y)\|)\right\} + \theta_n \\ &\leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2 + \theta_n \\ &\leq (1+\gamma_n)\|x-y\|^2 + \kappa(\|x-y\| + \|T^n x - T^n y\|)^2 + \theta_n \\ &\leq (1+\kappa+\gamma_n)\|x-y\|^2 + \kappa\left(2\|x-y\|\|T^n x - T^n y\| + \|T^n x - T^n y\|^2\right) + \theta_n.\end{aligned}$$

It gives us that

$$(1-\kappa)\|T^n x - T^n y\|^2 - 2\kappa\|x-y\|\|T^n x - T^n y\| - (1+\kappa+\gamma_n)\|x-y\|^2 - \theta_n \leq 0,$$

which is a quadratic inequality in $\|T^n x - T^n y\|$. Hence, the conclusion follows. □

Lemma 5.4. *Let C be a nonempty bounded subset of a uniformly convex Banach space X and $T: C \to C$ be a uniformly continuous nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Proof. Since T is a nearly asymptotically κ-strict pseudocontractive mapping in the intermediate sense, we obtain from Lemma 5.3 that

$$\begin{aligned}&\|T^{n+1}x_n - T^{n+1}x_{n+1}\|\\ &\le \tfrac{1}{1-\kappa}\left(\kappa\|x_n - x_{n+1}\| + \sqrt{(1+(1-\kappa)\gamma_{n+1})\|x_n - x_{n+1}\|^2 + (1-\kappa)\theta_{n+1}}\right).\end{aligned}$$

Since $\|x_n - x_{n+1}\| \to 0$, we have $\|T^{n+1}x_n - T^{n+1}x_{n+1}\| \to 0$. Observe that

$$\begin{aligned}&\|x_n - Tx_n\|\\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\|\\ &\quad+\|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|.\end{aligned} \tag{5.3}$$

By the uniform continuity of T, we have

$$\|Tx_n - T^{n+1}x_n\| \to 0 \quad \text{as } n \to \infty. \tag{5.4}$$

Since $\|x_n - T^n x_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$, it follows from (5.3) and (5.4) that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. □

6. Weak convergence of modified Mann iteration process

We extend demiclosedness principle (Proposition 2.12) for nearly asymptotically κ-strict pseudocontractive mappings in intermediate sense in the setting of Banach spaces.

Proposition 6.1 (Demiclosedness principle)**.** *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X, which has a weakly continuous duality map J. Let $T : C \to C$ be a continuous nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Then $I-T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I-T)x = 0$.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - T^m x_n\| = 0. \tag{6.1}$$

By Lemma 5.3, we have

$$\begin{aligned}\|T^m x_n - T^m x\| &\le \tfrac{1}{1-\kappa}\left(\kappa\|x_n - x\| + \sqrt{(1+(1-\kappa)\gamma_m)\|x_n - x\|^2 + (1-\kappa)\theta_m}\right)\\ &\le K'\end{aligned}$$

for all $m, n \ge 1$ and some constant $K' > 0$. Since X has a weakly continuous duality map J, there exists a gauge $\varphi(t) = t$ for which the duality map J is single-valued and weak-to-weak* sequentially continuous. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau = \frac{1}{2}t^2, \quad \forall t \ge 0.$$

Then, $J(x) = \partial(\frac{1}{2}\|x\|^2)$ for all $x \in X$. Hence, from $x_n \rightharpoonup x$ and Lemma 2.2 (ii) we deduce that for all $y \in X$,

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|y - x\|^2. \tag{6.2}$$

Define

$$\psi(y) := \limsup_{n\to\infty} \|x_n - y\|^2, \quad \forall y \in X.$$

Then, (6.2) can be rewritten as

$$\psi(y) = \psi(x) + \|x - y\|^2, \quad \forall y \in X. \tag{6.3}$$

Since T is a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$, by relation (5.2), we have

$$\begin{aligned}
\psi(T^m x) &= \limsup_{n\to\infty} \|x_n - T^m x\|^2 \\
&\le \limsup_{n\to\infty} (\|x_n - T^m x_n\| + \|T^m x_n - T^m x\|)^2 \\
&= \limsup_{n\to\infty} \Big(\|x_n - T^m x_n\|^2 + \|T^m x_n - T^m x\|^2 \\
&\quad +2\|x_n - T^m x_n\|\|T^m x_n - T^m x\|\Big) \\
&\le \limsup_{n\to\infty} \|T^m x_n - T^m x\|^2 \\
&\quad + \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K') \\
&\le \limsup_{n\to\infty} \big((1+\gamma_m)\|x_n - x\|^2 + \kappa\|x_n - T^m x_n - (x - T^m x)\|^2 + \theta_m\big) \\
&\quad + \limsup_{n\to\infty} \big(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'\big) \\
&\le \psi(x) + \kappa \limsup_{n\to\infty} \|x_n - T^m x_n - (x - T^m x)\|^2 + \psi(x)\gamma_m + \theta_m \\
&\quad + \limsup_{n\to\infty} \big(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'\big), \quad \forall m \ge 1.
\end{aligned}$$

By (6.3), we have

$$\begin{aligned}
\psi(x) + \|x - T^m x\|^2 &= \psi(T^m x) \\
&\le \psi(x) + \kappa \limsup_{n\to\infty} \|x_n - T^m x_n - (x - T^m x)\|^2 + \psi(x)\gamma_m \\
&\quad +\theta_m + \limsup_{n\to\infty} \big(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x - T^m x\|^2 &\le \kappa \limsup_{n\to\infty} \|x_n - T^m x_n - (x - T^m x)\|^2 + \psi(x)\gamma_m + \theta_m \\
&\quad + \limsup_{n\to\infty} \big(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'\big).
\end{aligned} \tag{6.4}$$

Since $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, it follows from (6.4) that

$$\limsup_{m\to\infty} \|x - T^m x\|^2 \le \kappa \limsup_{m\to\infty} \|x - T^m x\|^2.$$

It means that $T^m x \to x$ as $m \to \infty$. Therefore, the continuity of T implies that $(I - T)x = 0$. □

We now prove the weak convergence of (1.1) for nearly asymptotically κ-strict pseudocontractive mappings in intermediate sense.

Theorem 6.2. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X, which has a weakly continuous duality map J. Let $T : C \to C$ be a uniformly continuous nearly asymptotically κ-strict pseudocontractive* *mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n \theta_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following modified Mann iteration process:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1. \tag{6.5}$$

Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Proof. Let p be an element in $F(T)$. Using relation (2.2), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 \\ &\quad + \alpha_n\|T^n x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - T^n x_n\|) \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[(1 + \gamma_n)\|x_n - p\|^2 \\ &\quad + \kappa \min\{\|x_n - T^n x_n\|^2, g(\|x_n - T^n x_n\|)\} + \theta_n] \\ &\quad - \alpha_n(1 - \alpha_n)g(\|x_n - T^n x_n\|) \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[(1 + \gamma_n)\|x_n - p\|^2 \\ &\quad + \kappa g(\|x_n - T^n x_n\|) + \theta_n] \\ &\quad - \alpha_n(1 - \alpha_n)g(\|x_n - T^n x_n\|) \\ &\leq (1 + \gamma_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n - \kappa)g(\|x_n - T^n x_n\|) + \alpha_n\theta_n \\ &\leq (1 + \gamma_n)\|x_n - p\|^2 - \delta^2 g(\|x_n - T^n x_n\|) + \alpha_n\theta_n \\ &\leq (1 + \gamma_n)\|x_n - p\|^2 + \alpha_n\theta_n \end{aligned} \tag{6.6}$$

for all $n \geq 1$. By Lemma 2.3, last inequality in (6.6) and the assumptions that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n\theta_n < \infty$, we deduce that

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists.} \tag{6.7}$$

Suppose $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r > 0$. It is easy to see from the second last inequality in (6.6) that

$$\delta^2 g(\|x_n - T^n x_n\|) \leq (1 + \gamma_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\theta_n,$$

which implies that $\lim_{n\to\infty} g(\|x_n - T^n x_n\|) = 0$. Since $g : [0, 2r] \to \mathbb{R}$ is a strictly increasing, continuous and convex function such that $g(0) = 0$, we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0.$$

Observe that

$$\|x_{n+1} - x_n\| = \alpha_n\|x_n - T^n x_n\| \leq (1 - \kappa - \delta)\|x_n - T^n x_n\| \to 0 \quad \text{as } n \to \infty.$$

Since $\|x_{n+1} - x_n\| \to 0$, $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$, C is bounded, $\{x_n\}$ is a sequence in C and T is uniformly continuous, we obtain from Lemma 5.4 that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in C$. Since T is uniformly continuous and $\|x_n - Tx_n\| \to 0$, we have

$\|x_n - T^m x_n\| \to 0$ for all $m \geq 1$. By Proposition 6.1, we obtain $x \in F(T)$. To complete the proof, it is sufficient to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, x. Suppose there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $z \neq x$. As in the case of x, we must have $z \in F(T)$. It follows from (6.7) that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - z\|$ exist. Since X has a weakly continuous duality map J, by utilizing Lemma 2.2 (ii) we conclude that X satisfies the Opial condition. Thus, we have

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{n_k} - x\| < \lim_{k\to\infty} \|x_{n_k} - z\| = \lim_{n\to\infty} \|x_n - z\|,$$

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{j\to\infty} \|x_{n_j} - z\| < \lim_{j\to\infty} \|x_{n_j} - x\| = \lim_{n\to\infty} \|x_n - x\|,$$

which leads to a contradiction. So we must have $x = z$. This shows that $\omega_w(\{x_n\})$ is a singleton. Therefore, $\{x_n\}$ converges weakly to x by Lemma 2.5. □

Remark 6.3. It is well known that every real Hilbert space H is uniformly convex and its normalized duality mapping $J = I$ is weakly continuous. When C is a nonempty closed convex bounded subset of H, $T : C \to C$ is a nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ if and only if $T : C \to C$ is an asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$. Thus, Theorem 6.2 is more general than the results studied in [18, 21, 27, 26] to certain extent.

As a consequence of Theorem 6.2, we derive the following corollaries.

Corollary 6.4. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X, which has a weakly continuous duality map J. Let $T : C \to C$ be a uniformly continuous nearly asymptotically κ-strict pseudocontractive mapping in intermediate sense with $F(T) \neq \emptyset$ (in this case, $\gamma_n = 0$, $\forall n \geq 1$). Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n \theta_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by* (6.5). *Then $\{x_n\}$ converges weakly to an element of $F(T)$.*

Corollary 6.5. [26, Theorem 3.4] *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by* (6.5). *Then $\{x_n\}$ converges weakly to an element of $F(T)$.*

Corollary 6.6 ([18, Theorem 3.1]). *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by* (6.5). *Then $\{x_n\}$ converges weakly to an element of $F(T)$.*

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