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AN INVARIANCE OF DOMAIN THEOREM FOR DEMICONTINUOUS OPERATORS

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Abstract. We give an invariance of domain theorem for locally strictly monotone, demicontinuous operators of class (S) defined on an open subset of a real reflexive Banach space into its dual space. The method of approach is to use Browder's degree theory for demicontinuous operators of class $(S)_+$.

1. Introduction

It is known in [7,8] that the celebrated invariance of domain theorem for compact vector fields in Banach spaces is closely related to the problem of finding solutions for nonlinear differential equations. The method of approach is to use degree theory or homotopy theory. In this point of view, this problem has been investigated for a large class of (countably) condensing operators; see [2,3,4,5]

We are now interested in studying this problem for demicontinuous operators of class (S) which are defined on an open subset Ω of a reflexive Banach space X into its dual space X^*. Browder [1] introduced a degree theory for demicontinuous operators of class $(S)_+$. Applying the degree, some invariance of domain theorems for these operators are obtained in [6]. In this paper, we give an invariance of domain theorem for locally strictly monotone operators which states as follows:

If $T : \Omega \to X^*$ is a locally strictly monotone, demicontinuous operator of class (S), then $T(\Omega)$ is open in X^*.

This theorem extends a result of [6] in a similar method, with the aid of Browder's degree theory. As a consequence, we get a surjectivity result when T is uniformly monotone on the whole space X.

Let X be a real reflexive Banach space with dual space X^*. We use the notation $\langle f, x\rangle = f(x)$ for $f \in X^*$ and $x \in X$ and $\to$ ($\rightharpoonup$) for usual (weak) convergence, respectively. For a subset B of X, the closure and the boundary of B in X are denoted by $\overline{B}$ and ∂B, respectively.

Let $T : D(T) \subseteq X \to X^*$ be an operator and $B \subseteq D(T)$. Then T is said to be

(1) *demicontinuous* on $D(T)$ if for any sequence (x_n) in $D(T)$ and $x \in D(T)$,

$$x_n \to x \text{ as } n \to \infty \quad \text{implies} \quad Tx_n \rightharpoonup Tx \text{ as } n \to \infty.$$

(2) *of class (S) on B* if for any sequence (x_n) in B and $x \in X$,

$$x_n \rightharpoonup x \text{ as } n \to \infty \text{ and } \lim_{n\to\infty} \langle Tx_n, x_n - x\rangle = 0 \text{ imply } x_n \to x \text{ as } n \to \infty.$$

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(3) *of class* $(S)_+$ *on* B if for any sequence (x_n) in B and $x \in X$,

$$x_n \rightharpoonup x \text{ as } n \to \infty \text{ and } \limsup_{n\to\infty} \langle Tx_n, x_n - x\rangle \le 0 \ \text{ imply } \ x_n \to x \text{ as } n \to \infty.$$

(4) *strictly monotone on* B if

$$\langle Tx - Ty, x - y\rangle > 0 \qquad \text{for all } \ x, y \in B \text{ with } x \neq y.$$

(5) *strongly* ϕ*-expansive on* B if

$$\langle Tx - Ty, x - y\rangle \ge \phi(\|x - y\|) \qquad \text{for all } \ x, y \in B,$$

where $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous in a neighborhood of 0 and $\phi(0) = 0$.

(6) *uniformly monotone on* $D(T)$ if

$$\langle Tx - Ty, x - y\rangle \ge \varphi(\|x - y\|)\|x - y\| \qquad \text{for all } x, y \in D(T),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous, $\varphi(0) = 0$, and $\varphi(t) \to \infty$ as $t \to \infty$.

An operator $T : D(T) \subseteq X \to X^*$ is said to be *locally strongly* ϕ*-expansive* if T is strongly ϕ-expansive on a neighborhood of any point in $D(T)$.

2. Main result

Using a degree theory for demicontinuous operators of class $(S)_+$ in [1], we give our main result for locally strictly monotone operators.

The following statement on the duality map is taken from Proposition 8 of [1].

Lemma 2.1. *Let* X *be a real reflexive Banach space which is normed so that* X *and its dual* X^* *are locally uniformly convex. Then there is a unique homeomorphism* $J : X \to X^*$ *given by the conditions that*

$$\langle Jx, x\rangle = \|x\|^2 \quad \text{and} \quad \|Jx\| = \|x\| \quad \text{for each } x \in X.$$

The duality map J *is strictly monotone and of class* $(S)_+$.

Our key tool is the following degree for demicontinuous operators of class $(S)_+$ given in [1, Theorem 4].

Lemma 2.2. *Let* X *be a real reflexive Banach space and let* $\mathcal{F}$ *be the family of operators* $f : \overline{B} \to X^*$*, where* B *is a bounded open subset of* X *and* f *is a demicontinuous operator of class* $(S)_+$*. Then there exists one and only one degree function d on* $\mathcal{F}$ *which is invariant under the class of affine homotopies in* $\mathcal{F}$ *and is normalized by the duality map* J *in the sense of Lemma 2.1.*

Theorem 2.3. *Let* Ω *be an open subset of a real reflexive Banach space* X*, where* X *is normed so that* X *and* X^* *are both locally uniformly convex. Suppose that* $T : \Omega \to X^*$ *is demicontinuous and that for each* $x \in \Omega$ *there exits a closed ball* $\overline{B}(x, r) \subset \Omega$ *on which* T *is strictly monotone and of class* (S)*. Then* $T(\Omega)$ *is open in* X^**. Here* $B(x, r)$ *denotes the open ball in* X *of radius* $r > 0$ *and centered at* x*.*

Proof. Let $y_0 \in T(\Omega)$ be arbitrary but fixed, that is, $y_0 = Tx_0$ for some $x_0 \in \Omega$. Let $\overline{B}(x_0, r)$ be a closed ball contained in Ω such that T is strictly monotone and of class (S) on $\overline{B}(x_0, r)$. Since T is injective on $\overline{B}(x_0, r)$, we have

$$y_0 \notin T(\partial B(x_0, r)). \tag{2.1}$$

We first claim that $T(\partial B(x_0, r))$ is closed in X^*. For any sequence (x_n) in $\partial B(x_0, r)$ with $Tx_n \to y$ as $n \to \infty$, since (x_n) is a bounded sequence in the reflexive Banach space X, there is a subsequence (x_{n_k}) of (x_n) which converges weakly to some $x \in \overline{B}(x_0, r)$. Since T is of class (S) on $\overline{B}(x_0, r)$, it follows from

$$\lim_{k\to\infty} \langle Tx_{n_k}, x_{n_k} - x\rangle = \lim_{k\to\infty} \langle Tx_{n_k} - y, x_{n_k} - x\rangle = 0$$

that (x_{n_k}) converges to x and this limit x indeed belongs to $\partial B(x_0, r)$. Since T is demicontinuous, (Tx_{n_k}) converges weakly to Tx and by the uniqueness of weak limit we have $y = Tx \in T(\partial B(x_0, r))$. Thus, $T(\partial B(x_0, r))$ is closed in X^*. From (2.1), we can choose a real number $\rho > 0$ such that

$$\overline{B}(y_0, \rho) \cap T(\partial B(x_0, r)) = \emptyset. \tag{2.2}$$

Since T is monotone and of class (S) on $\overline{B}(x_0, r)$, it is easily verified that T is of class $(S)_+$ on $\overline{B}(x_0, r)$. Now we will apply degree theory for demicontinuous operators of class $(S)_+$. We first consider a homotopy $H : [0, 1] \times \overline{B}(x_0, r) \to X^*$ defined by

$$H(t, x) := tTx + (1 - t)J(x - x_0), \quad \alpha_1(t) = ty_0.$$

Then $\alpha_1(t) \neq H(t, x)$ for all $(t, x) \in [0, 1] \times \partial B(x_0, r)$. In fact, if $\alpha_1(t) = H(t, x)$ for some $(t, x) \in [0, 1] \times \partial B(x_0, r)$, then we have $t(Tx_0 - Tx) = (1 - t)J(x - x_0)$ and hence by Lemma 2.1

$$t\langle Tx_0 - Tx, x - x_0\rangle = (1 - t)\|x - x_0\|^2.$$

In case $t = 0$, this is a contradiction to $x \neq x_0$; in case $0 < t \leq 1$, we have $\langle Tx - Tx_0, x - x_0\rangle = (1 - t)/(-t)\, \|x - x_0\|^2 \leq 0$, in contradiction to the fact that T is strictly monotone on $\overline{B}(x_0, r)$. By the invariance property of the degree d under homotopy stated in Lemma 2.2, we obtain

$$d\,(T, B(x_0, r), y_0) = d\,(J(\cdot - x_0), B(x_0, r), 0). \tag{2.3}$$

Next, we consider a homotopy $G : [0, 1] \times \overline{B}(x_0, r) \to X^*$ defined by

$$G(t, x) := tJ(x - x_0) + (1 - t)Jx, \quad \alpha_2(t) = (1 - t)Jx_0.$$

Since J is strictly monotone by Lemma 2.1, a similar argument shows as above that $\alpha_2(t) \notin G(t, \partial B(x_0, r))$ for all $t \in [0, 1]$. As the degree d is normalized by the duality map J and $x_0 \in B(x_0, r)$, we have

$$d\,(J(\cdot - x_0), B(x_0, r), 0) = d\,(J, B(x_0, r), Jx_0) = 1. \tag{2.4}$$

For each $y \in B(y_0, \rho)$, we observe a path $\alpha_3 : [0, 1] \to X^*$ given by $\alpha_3(t) = ty + (1 - t)y_0$. From $\alpha_3(t) \notin T(\partial B(x_0, r))$ for all $t \in [0, 1]$ by (2.2), it follows that

$$d\,(T, B(x_0, r), y) = d\,(T, B(x_0, r), y_0). \tag{2.5}$$

For all $y \in B(y_0, \rho)$, we obtain in view of (2.3), (2.4), and (2.5) that

$$d\,(T, B(x_0, r), y) = 1$$

and hence $y \in T(B(x_0, r))$. Thus, $B(y_0, \rho) \subseteq T(B(x_0, r)) \subset T(\Omega)$. We conclude that $T(\Omega)$ is open in X^*. □

As a consequence of Theorem 2.3, we obtain Theorem 1 of [6] for locally strongly ϕ-expansive operators.

Corollary 2.4. *Let X and Ω be as in Theorem 2.3. If $T : \Omega \to X^*$ is demicontinuous and locally strongly ϕ-expansive, then $T(\Omega)$ is open in X^*.*

Proof. Note that if T is strongly ϕ-expansive on a closed ball contained in Ω, then T is strictly monotone and of class (S) on the ball. The conclusion follows from Theorem 2.3. □

Moreover, we can show that T is a bijection when T is uniformly monotone on the whole space.

Corollary 2.5. *Let X be as in Theorem 2.3. If $T : X \to X^*$ is demicontinuous and uniformly monotone, then T is a bijection on X and the inverse operator T^{-1} is continuous on X^*.*

Proof. In view of uniform monotonicity, the following relation holds:

$$\langle Tx - Ty, x - y\rangle \geq \varphi(\|x - y\|)\|x - y\| \qquad \text{for all } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Then T is locally strongly ϕ-expansive with $\phi(t) = \varphi(t)t$. Applying Corollary 2.4 with $\Omega = X$, $T(X)$ is open in X^*. Moreover, $T(X)$ is closed in X^*. Indeed, let (x_n) be any sequence in X such that $Tx_n \to y$ as $n \to \infty$. Since (Tx_n) is a Cauchy sequence in X^* and since φ^{-1} is strictly increasing and continuous at 0, it follows from the relation $\|Tx_m - Tx_n\| \geq \varphi(\|x_m - x_n\|)$ for all $m, n \in \mathbb{N}$ that (x_n) is a Cauchy sequence in the Banach space X and converges to some $x \in X$. Since T is demicontinuous, we obtain that $Tx_n \rightharpoonup Tx$ as $n \to \infty$ and so $y = Tx \in T(X)$. Since $T(X)$ is thus closed and open in X^*, we have $T(X) = X^*$. Hence T is bijective. To show that T^{-1} is continuous on X^*, we use the following inequality

$$\|T^{-1}w - T^{-1}z\| \leq \varphi^{-1}(\|w - z\|) \qquad \text{for all } w, z \in X^*$$

with the continuity of φ^{-1}. This completes the proof. □

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