Journal of Nonlinear and Convex Analysis
Volume 11, Number 2, 2010, 243–266

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONVERGENCE OF MULTIVALUED PRAMARTS

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Abstract. We state various convergence results for multivalued pramarts in a separable Banach space and its dual. Some new structure results for pramarts are also discussed. The conditional expectation for a special class closed convex valued integrable multifunction is also proven.

1. Introduction

The almost sure convergence of vector valued pramarts in Banach spaces were studied by several authors. There is a rich bibliograpy on this subject, see the book by Egghe[16] and the references therein. In this paper we present various convergence results for convex weakly compact valued pramarts in Banach spaces by introducing some new tools based upon the multivalued biting Dunford-Pettis theorem developed in [3, 9] which allow to extend classical a.s. norm convergence results for vector-valued pramarts to the a.s. convergence with respect to the linear topology [2] for convex weakly compact valued pramarts. Main results are given in section 4-5-6 where several convergence results for multivalued pramarts are presented in both the primal space and the dual space. The paper is organized as follows. In section 2 we summarize some basic properties of Mosco convergence and linear topology and measurable multifunctions. In section 3 we state and summarize for references the multivalued conditional expectation for closed convex valued integrable multifunctions and the multivalued biting Dunford-Pettis theorem. Section 4 is devoted to the convergence with respect to the linear topology for convex weakly compact valued integrably bounded pramarts in the case when the underlying Banach space is separable and have the RNP and its strong dual is separable. In section 5 we present the w^* Kuratowski convergence for convex weak star compact valued pramarts in the weak* dual of a separable Banach space, here the dual space is nolonger strongly separable. In section 6 further structure results for pramarts in both the primal space and the dual space are discussed.

2. Notations and preliminaries

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_n)_{n\in\mathbf{N}}$ is an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\in\mathbf{N}}\mathcal{F}_n$. E is a separable Banach space and E^* is its topological dual. Let $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) be the closed unit ball of E (resp. E^*) and 2^E the collection of all subsets of E. Let $c(E)$ (resp. $cc(E)$) (resp. $cwk(E)$) (resp. $\mathcal{R}wk(E)$) be the set of nonempty closed (resp. closed convex) (resp. convex weak compact) (resp.

2010 *Mathematics Subject Classification.* 28B20, 60G42, 46A17, 54A20.

Key words and phrases. Conditional expectation, multivalued biting lemma, multivalued Dunford-Pettis theorem, tightness, Mosco convergence, linear topology pramarts, subpramarts.

ball-weakly compact closed convex) subsets of E, here a closed convex subset in E is ball-weakly compact if its intersection with any closed ball in E is weakly compact. For $A \in cc(E)$, the distance function and the support function associated with A are defined respectively by

$$d(x, A) = \inf\{\|x - y\| \; : y \in A\}, (x \in E)$$

$$\delta^*(x^*, A) = \sup\{\langle x^*, y\rangle \; : y \in A\}, (x^* \in E^*).$$

We also define

$$|A| = \sup\{||x|| \; : x \in A\}.$$

and denote by $\mathcal{H}_E$ the Hausdorff distance defined on the $c(E)$ assciated with the topology of the norm in E. Given a sub-σ-algebra $\mathcal{B}$ in Ω, a multifunction $X : \Omega \to 2^E$ is $\mathcal{B}$-measurable if for every open set U in E the set

$$X^- U := \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{B}$. A function $f : \Omega \to E$ is a $\mathcal{B}$-measurable selection of X if $f(\omega) \in X(\omega)$ for all $\omega \in \Omega$. A Castaing representation of X is a sequence $(f_n)_{n\in\mathbf{N}}$ of $\mathcal{B}$-measurable selections of X such that

$$X(\omega) = cl\{f_n(\omega), n \in \mathbf{N}\} \quad \forall w \in \Omega$$

where the closure is taken with respect to the topology of associated with the norm in E. It is known that a nonempty closed-valued multifunction $X : \Omega \to c(E)$ is $\mathcal{B}$-measurable iff it admits a Castaing representation. If $\mathcal{B}$ is complete, the $\mathcal{B}$-measurability is equivalent to the measurability in the sense of graph, namely the graph of X is a member of $\mathcal{B} \otimes \mathcal{B}(E)$, here $\mathcal{B}(E)$ denotes the Borel tribe on E. A $cc(E)$-valued $\mathcal{B}$-measurable $X : \Omega \to cc(E)$ is integrable if the set $S^1_X(\mathcal{B})$ of all $\mathcal{B}$-measurable and integrable selections of X is nonempty. We denote by $L^1_E(\mathcal{B})$ the space of E-valued $\mathcal{B}$-measurable and Bochner-integrable functions defined on Ω and $\mathcal{L}^1_{cwk(E)}(\mathcal{B})$ the space of all $\mathcal{B}$-measurable multifunctions $X : \Omega \to cwk(E)$, such that $|X| \in L^1_{\mathbf{R}}(\mathcal{B})$. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is bounded (resp. uniformly integrable) if the sequence $(|X_n|)_{n\in\mathbf{N}}$ is bounded (resp. uniformly integrable) in $L^1_{\mathbf{R}}(\mathcal{F})$. A $cc(E)$-valued sequence $(X_n)_{n\in\mathbf{N}}$ Mosco-converges [19] to $X_\infty \in cc(E)$ if

$$X_\infty = s\text{-}liX_n = w\text{-}lsX_n$$

where

$$s\text{-}li\, X_n = \{x \in E : ||x_n - x|| \to 0,\, x_n \in X_n\}$$

and

$$w\text{-}ls\, X_n = \{x \in E : x = w\text{-}\lim_{j\to\infty} x_{n_j},\, x_{n_j} \in X_{n_j}\}$$

and s (resp. w) is the strong (resp. weak) topology in E. If $(X_n)_{n\in\mathbf{N}}$ Mosco-converges to X_∞ in $cc(E)$, we write

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty.$$

A $cc(E)$-valued sequence $(X_n)_{n\in\mathbf{N}}$ converges to $X_\infty \in cc(E)$ with respect to the linear topology τ_L [2] if

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad \forall x^* \in E^*.$$

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad \forall x \in E.$$

Beer showed that the τ_L-topology is stronger than the Mosco-topology. We refer to [11] for the theory of Measurable Multifunctions and Convex Analysis, and to [16, 20] for basic theory of martingales and adapted sequences.

3. Multivalued conditional expectation and Multivalued Dunford-Pettis theorem

A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ of $cc(E)$-valued multifunctions is adapted if each X_n is $\mathcal{F}_n$-measurable. Given a sub-σ-algebra, $\mathcal{B}$ of $\mathcal{F}$ and an integrable $\mathcal{F}$-measurable $cc(E)$-valued multifunction $X : \Omega \rightrightarrows E$, Hiai and Umegaki [18] showed the existence of a $\mathcal{B}$-measurable $cc(E)$-valued integrable mutifunction, denoted by $E^{\mathcal{B}}X$ such that

$$\mathcal{S}^1_{E^{\mathcal{B}}X}(\mathcal{B}) = cl\{E^{\mathcal{B}} f : f \in \mathcal{S}^1_X(\mathcal{F})\}$$

the closure being taken in $L^1_E(\Omega, \mathcal{A}, P)$; $E^{\mathcal{B}}X$ is the multivalued conditional expectation of X relative to $\mathcal{B}$. If $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$, and the strong dual E^*_b is separable, then $E^{\mathcal{B}}X \in \mathcal{L}^1_{cwk(E)}(\mathcal{B})$ with $\mathcal{S}^1_{E^{\mathcal{B}}X}(\mathcal{B}) = \{E^{\mathcal{B}} f : f \in \mathcal{S}^1_X(\mathcal{F})\}$. This result was stated by the first author in ([3], Theorem 3). A unified approach for general conditional expectation of $cc(E)$-valued integrable multifunctions is given in [21] allowing to recover both the $cc(E)$-valued conditional expectation of $cc(E)$-valued integrable multifunctions in the sense of [18] and the $cwk(E)$-valued conditional expectation of $cwk(E)$-valued integrably bounded multifunctions given in [3]. For more information on multivalued conditional expectation and related subjects we refer to [1, 6, 11, 18, 21]. In the context of this paper we present a specific version of conditional expectation that we summarize below.

Proposition 3.1. *Assume that E^*_b is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and an integrable $\mathcal{F}$-measurable $cc(E)$-valued multifunction $X : \Omega \rightrightarrows E$. Assume further there is a $\mathcal{F}$-measurable ball-weakly compact $cc(E)$-valued multifunction $K : \Omega \rightrightarrows E$ such that $X(\omega) \subset K(\omega)$ for all $\omega \in \Omega$. Then there is a unique (for the equality a.s.) $\mathcal{B}$-measurable $cc(E)$-valued multifunction Y satisfying the property*

$$(*) \qquad \forall v \in L^\infty_{E^*}(\mathcal{B}), \int_\Omega \delta^*(v(\omega), Y(\omega))P(d\omega) = \int_\Omega \delta^*(v(\omega), X(\omega))P(d\omega).$$

$E^{\mathcal{B}}X := Y$ is the conditional expectation of X.

Proof. The proof is a careful adaptation of the one of Theorem VIII.35 in [11]. For technical reason we will assume that $\mathcal{B}$ is complete. Let u_0 be an integrable selection of X. For every $n \geq 1$, let

$$X_n(\omega) = X(\omega) \cap (u_0(\omega) + n\overline{B}_E) \quad \forall n \in \mathbf{N} \quad \forall \omega \in \Omega.$$

As $X(\omega) \subset K(\omega)$ for all $\omega \in \Omega$, we get

$$X_n(\omega) = X(\omega) \cap (u_0(\omega) + n\overline{B}_E) \subset K(\omega) \cap (u_0(\omega) + n\overline{B}_E) \quad \forall n \in \mathbf{N} \quad \forall \omega \in \Omega.$$

As $K(\omega)$ is ball-weakly compact, it is immediate that $X_n \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$. so that, by virtue of ([3] or ([21], Remarks of Theorem 3), the conditional expectation $E^{\mathcal{B}}X_n \in$

$\mathcal{L}^1_{cwk(E)}(\mathcal{B})$. It follows that

$$(**) \qquad \int_\Omega \delta^*(v(\omega), E^{\mathcal{B}} X_n(\omega)) P(d\omega) = \int_\Omega \delta^*(v(\omega), X_n(\omega)) P(d\omega).$$

$\forall n \in \mathbf{N}, \forall v \in L^\infty_{E^*}(\mathcal{B})$. Now let

$$Y(\omega) = cl(\cup_{n\in\mathbf{N}} E^{\mathcal{B}} X_n(\omega)) \quad \forall \omega \in \Omega.$$

Then Y is $\mathcal{B}$-measurable and convex. Now the required property (*) follows from (**) and the monotone convergence theorem. Indeed

$$\forall n \in \mathbf{N}, \forall v \in L^\infty_{E^*}(\mathcal{B}), \langle u_0, v\rangle \le \delta^*(v, X_n) \uparrow \delta^*(v, X)$$
$$\langle v, E^{\mathcal{B}} u_0\rangle \le \delta^*(v, E^{\mathcal{B}} X_n) \uparrow \delta^*(v, Y).$$

Now the uniqueness follows exactly as in the proof of Theorem VIII.35 via the measurable projection theorem ([11], Theorem III.32). □

New existence results of conditional expectation for convex weakly compact valued multifunctions and its applications to martingales are available in [1, 6].

For the convenience of the reader recall and summarize a tightness condition and a compactness result in the space $\mathcal{L}^1_{cwk(E)}(\Omega, \mathcal{F}, P)$. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E}(\mathcal{F})$ is *cwk*(E)*-tight* if, for every $\varepsilon > 0$, there is a $cwk(E)$-valued $\mathcal{F}$-measurable multifunction $\Gamma_\varepsilon : \Omega \to E$ such that

$$\sup_{n\in\mathbf{N}} P(\Omega \setminus \{\omega \in \Omega : X_n(\omega) \subset \Gamma_\varepsilon(\omega)\}) \le \varepsilon.$$

The following is a multivalued biting-Dunford-Pettis (biting-compactness for short) theorem in the space $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. See ([5], Theorem 6.1).

Theorem 3.2. *Suppose that E is a separable Banach space, $(X_n)_{n\in\mathbf{N}}$ is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying one of the following conditions:*
(a) $(X_n)_{n\in\mathbf{N}}$ *is cwk(E)-tight.*
(b) E^*_b *is separable, E has the RNP and for each $A \in \mathcal{F}$, $\cup_{n=1}^\infty \int_A X_n\, dP$ is relatively weakly compact in E.*

Then there exist an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ such that $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(X'_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that, for each $p \in \mathbf{N}$ and for each $v \in L^\infty_{E^*}(A_p \cap \mathcal{F})$, the following holds:

$$\lim_{n\to\infty} \int_{A_p} \delta^*(v, X'_n)\, dP = \int_{A_p} \delta^*(v, X_\infty)\, dP.$$

We finish this section by mentionning a useful result.

Proposition 3.3. *Suppose that E is a separable Banach space, $(X_n)_{n\in\mathbf{N}}$ is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following condition: There is a $\mathcal{F}$-measurable $\mathcal{R}wk(E)$-valued multifunction $K : \Omega \rightrightarrows E$ such that $X_n(\omega) \subset K(\omega)$ for all $n \in N$ and for all $\omega \in \Omega$. Then $(X_n)_{n\in\mathbf{N}}$ is cwk(E)-tight.*

Proof. See e.g. ([7], Proposition 3.3 (i)). □

We will show in next section the convergence problem for multivalued pramarts.

4. Convergence of pramarts in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$

From now we will assume in the majority of this section that the strong dual E^*_b is *separable* in order to ensure the weak compactness of the conditional expectation for multifunctions in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and also the validity of the multivalued Dunford-Pettis theorem in this space (see Theorem 3.2). This assumption can be removed in some particular cases. We provide in this section the a.s. τ_L-convergence for bounded pramarts in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Let us recall and summarize some definitions.

Definition 4.1. A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is an adapted sequence if each X_n is $\mathcal{F}_n$-measurable. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon]) \leq \varepsilon$$

where T denotes the set of bounded stopping times.

It is clear that if $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, then, for each $x^* \in \overline{B}_{E^*}$, the adapted sequence $(\delta^*(x^*, X_n), \mathcal{F}_n)_{n\in\mathbf{N}}$ is a real-valued pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ because

$$|\delta^*(x^*, X_\sigma) - \delta^*(x^*, E^{\mathcal{F}_\sigma} X_\tau)| \leq h(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau).$$

It is clear that this definition covers the notion of vector-valued pramarts in $L^1_E(\mathcal{F})$. Indeed an adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_E(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma > \sigma_\varepsilon \Rightarrow P([\,|X_\sigma - E^{\mathcal{F}_\sigma} X_\tau| > \varepsilon]) \leq \varepsilon$$

We also need the classical notion of subpramarts.

Definition 4.2. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_{\mathbf{R}}(\mathcal{F})$ is a subpramart, if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([(X_\sigma - E^{\mathcal{F}_\sigma} X_\tau)^+ > \varepsilon]) \leq \varepsilon$$

We will need some technical lemmas.

Lemma 4.3. *Assume that E^*_b is separable and $D^*_1 := (e^*_m)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\sup_{m\in\mathbf{N}} [\delta^*(e^*_m, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)^+] \leq \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) \quad a.s.$$

for all $\sigma, \tau \in T, \tau \geq \sigma$.

Proof. For each $m \in \mathbf{N}$, we have

$$\delta^*(e^*_m, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)^+ \leq [\delta^*(e^*_m, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)]^+ \quad a.s.$$

Indeed, if $\delta^*(e^*_m, X_\sigma) > 0$, then we have

$$\begin{aligned} \delta^*(e^*_m, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)^+ &= \delta^*(e^*_m, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)^+ \\ &\leq \delta^*(e^*_m, X_\sigma) - [E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)]^+ \\ &\leq [\delta^*(e^*_m, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)]^+ \quad a.s. \end{aligned}$$

If $\delta^*(e_m^*, X_\sigma) \le 0$, then we have

$$\begin{aligned}\delta^*(e_m^*, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+ &= 0 - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+ \\ &\le 0 \le [\delta^*(e_m^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)]^+.\end{aligned}$$

So we get the estimate

$$\begin{aligned}\delta^*(e_m^*, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+ &\le |\delta^*(e_m^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)| \\ &\le \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) \quad a.s.\end{aligned}$$

Finally by taking the supremum on $m \in \mathbf{N}$ in the preceding estimate, we get the required inequality

$$\sup_{m\in\mathbf{N}} [\delta^*(e_m^*, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+] \le \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) \quad a.s.$$

Alternatively we may apply the techniques developed in Choukairi ([12, 13], Theorem 3.1). For each $m, n \in \mathbf{N}$, let us set

$$\varphi_{m,n}(\omega) := \delta^*(e_m^*, X_n(\omega))$$

Let σ, $\tau \in T, \tau \ge \sigma$ and let us set

$$\varphi_{m,\tau}(\omega) := \delta^*(e_m^*, X_\tau(\omega))$$

$$\varphi_{m,\sigma}(\omega) := \delta^*(e_m^*, X_\sigma(\omega))$$

From Jensen inequality we have

$$|E^{\mathcal{F}_\sigma}\varphi_{m,\tau}(\omega)| \le E^{\mathcal{F}_\sigma}|\varphi_{m,\tau}|(\omega) \quad a.s.$$

Then for a.s. $\omega \in \Omega$ we have that

$$\begin{aligned}\delta^*(e_m^*, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+ &= \varphi_{m,\sigma}^+ - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}^+) \\ &= \frac{1}{2}[\varphi_{m,\sigma} + |\varphi_{m,\sigma}| - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) - E^{\mathcal{F}_\sigma}(|\varphi_{m,\tau}|)] \\ &\le \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma}| - |E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &\le \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &= [\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})]^+ \\ &= [\delta^*(e_m, X_\sigma) - \delta^*(e_m, E^{\mathcal{F}_\sigma}X_\tau)]^+ \\ &\le \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau).\end{aligned}$$

□

Lemma 4.4. *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$. Let $x \in E$ and let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\tau))^+ \le \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau)$$

a.s. for all $m \in \mathbf{N}$, σ, $\tau \in T, \tau \ge \sigma$.

Proof. We can apply the techniques developed in Lemma 4.3. For simplicity for each $m, n \in \mathbf{N}$, let us set

$$\varphi_{m,n,x}(\omega) := \langle e_m^*, x\rangle - \delta^*(e_m^*, X_n(\omega))$$

Let σ, $\tau \in T, \tau \geq \sigma$ and let us set

$$\varphi_{m,\tau,x}(\omega) := \langle e_m^*, x\rangle - \delta^*(e_m^*, X_\tau(\omega))$$

$$\varphi_{m,\sigma,x}(\omega) := \langle e_m^*, x\rangle - \delta^*(e_m^*, X_\sigma(\omega))$$

From Jensen inequality we have

$$|E^{\mathcal{F}_\sigma}\varphi_{m,\tau,x}(\omega)| \leq E^{\mathcal{F}_\sigma}|\varphi_{m,\tau,x}|(\omega) \quad a.s.$$

Then for a.s. $\omega \in \Omega$ we have that

$$\begin{aligned}
\varphi^+_{m,\sigma,x} - E^{\mathcal{F}_\sigma}(\varphi^+_{m,\tau,x}) &= \frac{1}{2}[\varphi_{m,\sigma,x} + |\varphi_{m,\sigma,x}| - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x}) - E^{\mathcal{F}_\sigma}(|\varphi_{m,\tau,x}|)] \\
&\leq \frac{1}{2}[\varphi_{m,\sigma,x} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x}) + |\varphi_{m,\sigma,x}| - |E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x})|] \\
&\leq \frac{1}{2}[\varphi_{m,\sigma,x} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x}) + |\varphi_{m,\sigma,x} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x})|] \\
&= [\varphi_{m,\sigma,x} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau,x})]^+ \\
&= [\delta^*(e_m, X_\sigma) - \delta^*(e_m, E^{\mathcal{F}_\sigma}X_\tau)]^+ \\
&\leq \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau)
\end{aligned}$$

thereby proving the required inequality. □

Remarks. 1) Lemma 4.3-4.4 show that the sequence $((\delta^*(e_m^*, X_n)^+_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ and $(([\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ are uniform sequence of positive subpramarts in the terminology of Egghe ([16], definition VIII.1.14).
2) Lemma 4.3-4.4 hold true if we remplace $D_1^* =: (e_m^*)_{m\in\mathbf{N}}$ by $E_1^* =: (f_m^*)_{m\in\mathbf{N}}$ where $(f_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology $\tau(E^*, E)$.

There is a useful application of Lemma 4.3.

Lemma 4.5. *Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a martingale in $L^1_{\mathbf{R}}(\mathcal{F})$. Then the following holds:*

$$(Y_\sigma - X_\sigma)^+ - E^{\mathcal{F}_\sigma}(Y_\tau - X_\tau)^+ \leq |X_\sigma - E^{\mathcal{F}_\sigma}X_\tau|$$

a.s. for all σ, $\tau \in T, \tau \geq \sigma$.

Proof. Since $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a martingale in $L^1_{\mathbf{R}}(\mathcal{F})$ we have

$$Y_\sigma = E^{\mathcal{F}_\sigma}Y_\tau \quad \forall\, \sigma, \tau \in T, \quad \tau \geq \sigma.$$

Hence the result follows by applying Lemma 4.3 to the pramart $(Z_n)_{n\in\mathbf{N}} = (Y_n - X_n)_{n\in\mathbf{N}}$ □

The following result has some importance in the pramart convergence.

Lemma 4.6. *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $L_E^1(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a martingale in $L_E^1(\mathcal{F})$. Then the following holds:*

$$\sup_{m\in\mathbf{N}} [\langle e_m^*, Y_\sigma - X_\sigma\rangle^+ - E^{\mathcal{F}_\sigma}\langle e_m^*, Y_\tau - X_\tau\rangle^+] \leq ||X_\sigma - E^{\mathcal{F}_\sigma} X_\tau||_E \quad a.s.$$

for all σ, $\tau \in T, \tau \geq \sigma$.

Proof. Since $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a martingale in $L_E^1(\mathcal{F})$ we have

$$Y_\sigma = E^{\mathcal{F}_\sigma} Y_\tau \quad \forall\, \sigma, \tau \in T, \quad \tau \geq \sigma.$$

In particular we have

$$\langle e_m^*, Y_\sigma\rangle = \langle e_m^*, E^{\mathcal{F}_\sigma} Y_\tau\rangle = E^{\mathcal{F}_\sigma}\langle e_m^*, Y_\tau\rangle \quad \forall m \in \mathbf{N}, \forall\, \sigma, \tau \in T, \tau \geq \sigma.$$

Now we may apply Lemma 4.5 to the pramarts

$$(\langle e_m^*, Y_n - X_n\rangle)_{n\in\mathbf{N}} = (\langle e_m^*, Y_n\rangle - \langle e_m^*, X_n\rangle)_{n\in\mathbf{N}}.$$

This yields for a.s. $\omega \in \Omega$

$$\begin{aligned}\langle e_m^*, Y_\sigma - X_\sigma\rangle^+ - E^{\mathcal{F}_\sigma}\langle e_m^*, Y_\tau - X_\tau\rangle^+ &\leq |\langle e_m^*, X_\sigma - E^{\mathcal{F}_\sigma} X_\tau\rangle| \\ &\leq ||X_\sigma - E^{\mathcal{F}_\sigma} X_\tau||_E.\end{aligned}$$

By taking the supremum on $m \in \mathbf{N}$ in the preceding estimate, we get the result. □

Theorem 4.7. *Assume that E_b^* is separable and E have the RNP. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the weak compactness condition: For each $A \in \mathcal{F}$, $\cup_{n=1}^\infty \int_A X_n dP$ is relatively $\sigma(E, E^*)$-compact. Then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that*

$$\lim_{n\to\infty} |X_n(\omega)| = |X_\infty(\omega)| \quad a.s.$$

$$\lim_{n\to\infty} \delta^*(x^*, X_n(\omega)) = \delta^*(x^*, X_\infty(\omega)) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}.$$

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

Proof. The proof is divided in two steps.
Step 1 **Claim:** There exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}.$$

We will use the biting-compactness method developed in ([5], Theorem 2.7). Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, that is,

$$\sup_{n\in\mathbf{N}} \int_\Omega |X_n| dP = \sup_{n\in\mathbf{N}} \int_\Omega \sup_{x^*\in\overline{B}_{E^*}} |\delta^*(x^*, X_n)| dP < \infty$$

for each $x^* \in \overline{B}_{E^*}$, the L^1-bounded pramart $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function $m_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. Now applying Theorem 3.2 to the bounded sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ with $\lim_{p\to\infty} P(A_p) =$

1, a subsequence $(X'_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that, for each $p \in \mathbf{N}$, and each $v \in L^\infty_{E^*}(A_p \cap \mathcal{F})$,

$$\lim_{n\to\infty} \int_{A_p} \delta^*(v, X'_n)dP = \int_{A_p} \delta^*(v, X_\infty)dP. \tag{4.1}$$

So by identifying the limits, we get

$$\lim_{n\to\infty} \int_A \delta^*(x^*, X'_n)dP = \int_A \delta^*(x^*, X_\infty)dP = \int_A m_{x^*}dP \tag{4.2}$$

for each $p \in \mathbf{N}$, each $A \in A_p \cap \mathcal{F}$ and each $x^* \in \overline{B}_{E^*}$. Consequently, there is a negligible set $N_{p,x^*} \in A_p \cap \mathcal{F}$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = m_{x^*}(\omega) = \delta^*(x^*, X_\infty) \quad \forall\omega \notin N_{p,x^*}$$

Let $D^*_1 = (e^*_m)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the topology of the dual norm of E^*. Set

$$N_p = \bigcup_{x^*\in D^*_1} N_{p,x^*}$$

Then N_p is negligible in $A_p \cap \mathcal{F}$ and we have

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X_\infty) \quad \forall m \in \mathbf{N}, \quad \forall\omega \in A_p \setminus N_p. \tag{4.3}$$

By Lemma 4.3 $((\delta^*(e^*_m, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded subpramarts [16]. Further by (4.3) we have

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n)^+ = \delta^*(e^*_m, X_\infty)^+ \quad \forall m \in \mathbf{N}, \quad \forall\omega \in A_p \setminus N_p. \tag{4.4}$$

Then $N := \cup_{p\in\mathbf{N}} N_p$ is negligible. Applying Lemma VIII.1.15 in [16] to $((\delta^*(e^*_m, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ yields

$$\begin{aligned}\lim_{n\to\infty} |X_n| = \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \delta^*(e^*_m, X_n)^+ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \delta^*(e^*_m, X_n)^+ \\ &= \sup_{m\in\mathbf{N}} \delta^*(e^*_m, X_\infty)^+ = |X_\infty|\end{aligned}$$

for all $\omega \in \Omega \setminus N$. So we get

$$\sup_{n\in\mathbf{N}} |X_n| < \infty \quad \forall\omega \in \Omega \setminus N. \tag{4.5}$$

Now by (4.5) it is not difficult to check that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad \forall\omega \in \Omega \setminus N, \quad \forall x^* \in \overline{B}_{E^*} \tag{4.6}$$

because for each $\omega \in \Omega \setminus N$, $x^* \in \overline{B}_{E^*}$ and $e^* \in D^*_1$ we have the estimate

$$\begin{aligned}|\delta^*(x^*, X_n) - \delta^*(e^*, X_\infty)| &\le ||x^* - e^*|| \sup_{n\in\mathbf{N}} |X_n| \\ &+|\delta^*(e^*, X_n) - \delta^*(e^*, X_\infty)| + ||x^* - e^*||\, |X_\infty|.\end{aligned}$$

Step 2 **Claim:** $\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega))$ *a.s.* $\forall x \in E$. Let $x \in E$. By Lemma 4.4 and the scalar convergence obtained in the first step we see that

$(([\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded subpramarts [16] with

$$\lim_{n\to\infty}[\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^+ = [\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\infty)]^+ \quad a.s.$$

From Lemma VIII.1.15 in [16] we deduce that

$$\begin{aligned}\lim_{n\to\infty} d(x, X_n(\omega)) &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}}\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n(\omega))\\ &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}}[\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n(\omega))]^+\\ &= \sup_{m\in\mathbf{N}}\lim_{n\to\infty}[\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n(\omega))]^+\\ &= \sup_{m\in\mathbf{N}}[\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\infty(\omega))]^+\\ &= \sup_{m\in\mathbf{N}}[\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\infty(\omega))] = d(x, X_\infty(\omega)) \quad a.s.\end{aligned}$$

Hence we get

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega))$$

a.s. for all $x \in E$ by equicontinuity of the distance function and the separability of E. □

When the pramarts $(X_n)_{n\in\mathbf{N}}$ in Theorem 4.7 are single-valued, namely $X_n \in L^1_E(\Omega, \mathcal{F}, P)$, Theorem 4.7 is reduced to

Corollary 4.8. *Assume that E_b^* is separable and E have the RNP. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L^1_E(\mathcal{F})$ satisfying the weak compactness condition: For each $A \in \mathcal{F}$, $\cup_{n=1}^\infty \int_A X_n dP$ is relatively $\sigma(E, E^*)$-compact. Then $(X_n)_{n\in\mathbf{N}}$ norm converges a.s. to an element $X_\infty \in L^1_E(\mathcal{F})$.*

There is a variant of Theorem 4.7.

Theorem 4.9. *Assume that E_b^* is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the condition: There is a $\mathcal{R}wk(E)$-valued $\mathcal{F}$-measurable multifunction $K : \Omega \rightrightarrows E$ such that $X_n(\omega) \subset K(\omega)$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$. Then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that*

$$\lim_{n\to\infty} |X_n(\omega)| = |X_\infty(\omega)| \quad a.s.$$

$$\lim_{n\to\infty} \delta^*(x^*, X_n(\omega)) = \delta^*(x^*, X_\infty(\omega) \quad a.s \quad \forall x^* \in \overline{B}_{E^*}.$$

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

Proof. The proof is divided in two steps.
Step 1 **Claim:** There exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}.$$

We will use the biting-compactness method developed in the proof of Theorem 4.7. However this need a careful look. Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, that is,

$$\sup_{n\in\mathbf{N}}\int_\Omega |X_n| dP = \sup_{n\in\mathbf{N}}\int_\Omega \sup_{x^*\in\overline{B}_{E^*}} |\delta^*(x^*, X_n)| dP < \infty$$

for each $x^* \in \overline{B}_{E^*}$, the L^1-bounded pramart $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function $m_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. Now applying Theorem 3.2 to the bounded sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ with $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(X'_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that, for each $p \in \mathbf{N}$, and each $v \in L^\infty_{E^*}(A_p \cap \mathcal{F})$,

$$\lim_{n\to\infty} \int_{A_p} \delta^*(v, X'_n)dP = \int_{A_p} \delta^*(v, X_\infty)dP \tag{4.7}$$

So by identifying the limits, we get

$$\lim_{n\to\infty} \int_A \delta^*(x^*, X'_n)dP = \int_A \delta^*(x^*, X_\infty)dP = \int_A m_{x^*}dP \tag{4.8}$$

for each $p \in \mathbf{N}$, each $A \in A_p \cap \mathcal{F}$ and each $x^* \in \overline{B}_{E^*}$. Consequently, there is a negligible set $N_{p,x^*} \in A_p \cap \mathcal{F}$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = m_{x^*}(\omega) = \delta^*(x^*, X_\infty) \quad \forall \omega \notin N_{p,x^*}$$

Let $E_1^* = (f_m^*)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology $\tau(E^*, E)$. Set

$$N_p = \bigcup_{x^* \in E_1^*} N_{p,x^*}$$

Then N_p is negligible in $A_p \cap \mathcal{F}$ and we have

$$\lim_{n\to\infty} \delta^*(f_m^*, X_n) = \delta^*(f_m^*, X_\infty) \quad \forall m \in \mathbf{N}, \quad \forall \omega \in A_p \setminus N_p. \tag{4.9}$$

By Lemma 4.3 $((\delta^*(f_m^*, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded pramarts. Further by (4.9) we have

$$\lim_{n\to\infty} \delta^*(f_m^*, X_n)^+ = \delta^*(f_m^*, X_\infty)^+ \quad \forall m \in \mathbf{N}, \quad \forall \omega \in A_p \setminus N_p. \tag{4.10}$$

Then $N := \cup_{p\in\mathbf{N}} N_p$ is negligible. Now applying Lemma VIII.1.15 to

$$((\delta^*(f_m^*, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

yields

$$\begin{aligned} \lim_{n\to\infty} |X_n| = \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \delta^*(f_m^*, X_n)^+ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \delta^*(f_m^*, X_n)^+ \\ &= \sup_{m\in\mathbf{N}} \delta^*(f_m^*, X_\infty)^+ = |X_\infty| \end{aligned}$$

for all $\omega \in \Omega \setminus N$. Therefore we get

$$r(\omega) := \sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty \quad \forall \omega \in \Omega \setminus N. \tag{4.11}$$

Let us set

$$\Gamma(\omega) = K(\omega) \cap r(\omega)\overline{B}_E, \quad \forall \omega \in \Omega \setminus N.$$

Then $\Gamma(\omega)$ is convex weakly compact because $K(\omega)$ is weakly ball-compact for each $\omega \in \Omega$. Now we show that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$$

for all $x^* \in \overline{B}_{E^*}$ and for all $\omega \in \Omega \setminus N$. We will use an argument in ([4], Lemma 3.2). Let $x^* \in \overline{B}_{E^*}$ and $f^*_m \in E^*_1$. We have the estimate

$$\begin{aligned}|\delta^*(x^*, X_n) - \delta^*(x^*, X_\infty)| \quad \leq \quad & \max\{\delta^*(x^* - f^*_m, X_n), \delta^*(f^*_m - x^*, X_n)\}\\ & +|\delta^*(f^*_m, X_n) - \delta^*(f^*_m, X_\infty)|\\ & +\max\{\delta^*(x^* - f^*_m, X_\infty), \delta^*(f^*_m - x^*, X_\infty)\}.\end{aligned}$$

Let $\omega \in \Omega \setminus N$ be fixed and $\varepsilon > 0$. Since $\Gamma(\omega)$ is weakly compact, there is $f^*_m \in M^*_1$ such that

$$\max\{\delta^*(x^* - f^*_m, \Gamma(\omega)), \delta^*(f^*_m - x^*, \Gamma(\omega))\} \leq \varepsilon.$$

Since $X_n(\omega) \subset \Gamma(\omega)$ for all $n \in \mathbf{N} \cup \{\infty\}$, it follows that

$$|\delta^*(x^*, X_n(\omega)) - \delta^*(x^*, X_\infty(\omega))| \leq |\delta^*(f^*_m, X_n(\omega)) - \delta^*(f^*_m, X_\infty(\omega))| + 2\varepsilon.$$

So we deduce that

$$\lim_{n\to\infty} \delta^*(x^*, X_n(\omega)) = \delta^*(x^*, X_\infty(\omega)).$$

Step 2 **Claim:** $\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E$. Let $x \in E$. By Lemma 4.4 and the scalar convergence obtained in the first step we conclude that $(([\langle f^*_m, x\rangle - \delta^*(f^*_m, X_n)]^+_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded subpramarts with

$$\lim_{n\to\infty} [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_n)]^+ = [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_\infty)]^+ \quad a.s.$$

From Lemma VIII.1.15 we deduce that

$$\begin{aligned}\lim_{n\to\infty} d(x, X_n(\omega)) &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \langle f^*_m, x\rangle - \delta^*(f^*_m, X_n(\omega))\\ &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_n(\omega))]^+\\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_n(\omega))]^+\\ &= \sup_{m\in\mathbf{N}} [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_\infty(\omega))]^+\\ &= \sup_{m\in\mathbf{N}} [\langle f^*_m, x\rangle - \delta^*(f^*_m, X_\infty(\omega))] = d(x, X_\infty(\omega)) \quad a.s.\end{aligned}$$

Hence we get

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega))$$

a.s. for all $x \in E$ by equicontinuity of the distance function and the separability of E. □

When the pramarts $(X_n)_{n\in\mathbf{N}}$ in Theorem 4.9 are single-valued, namely $X_n \in L^1_E(\Omega, \mathcal{F}, P)$, Theorem 4.9 is reduced to

Corollary 4.10. *Assume that E is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L^1_E(\mathcal{F})$ satisfying the condition: There is a $\mathcal{R}wk(E)$-valued $\mathcal{F}$-measurable multifunction $K : \Omega \rightrightarrows E$ such that $X_n(\omega) \in K(\omega)$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$. Then $(X_n)_{n\in\mathbf{N}}$ norm converges a.s. to an element $X_\infty \in L^1_E(\mathcal{F})$.*

5. Pramarts in $L^1_{E^*}[E](\mathcal{F})$ and $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)_{n\in\mathbf{N}}$ an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\geq 1}^{\infty}\mathcal{F}_n$. Let E be a separable Banach space, $D = (x_p)_{p\in\mathbf{N}}$ is a dense sequence in E, E^* is the topological dual of E, $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) is the closed unit ball of E (resp. E^*). We denote by E^*_s (resp. E^*_b) (resp. $E^*_{m^*}$) the topological dual E^* endowed with the topology $\sigma(E^*, E)$ of pointwise convergence, alias w^* topology (resp. the topology associated with the dual norm $||.||_{E^*_b}$) (resp. the topology $m^* = \sigma(E^*, H)$, where H is the linear space of E generated by D, that is the Hausdorff locally convex topology defined by the sequence of semi-norms

$$p_k(x^*) = \max\{|\langle x^*, x_p\rangle| : p \leq k\},\ \ x^* \in E^*, k \geq 1.$$

Recall that the topology m^* is metrizable, for instance, by the metric

$$d_{E^*_{m^*}}(x^*_1, x^*_2) := \sum_{p=1}^{p=\infty} \frac{1}{2^p}|\langle x_p, x^*_1\rangle - \langle x_p, x^*_2\rangle|, \quad x^*_1, x^*_2 \in E^*.$$

We assume from now that $d_{E^*_{m^*}}$ is held fixed. Further, we have $m^* \subset w^* \subset s^*$. When E is infinite dimensional these inclusions are strict. On the other hand, the restrictions of m^* and w^* to any bounded subset of E^* coincide and the Borel tribe $\mathcal{B}(E^*_s)$ and $\mathcal{B}(E^*_{m^*})$ associated with E^*_s and $E^*_{m^*}$ are equal. Noting that E^* is the countable union of closed balls, we deduce that the space E^*_s is Suslin, as well as the metrizable topological space $E^*_{m^*}$. A $2^{E^*_s}$-valued multifunction (alias mapping for short) $X : \Omega \rightrightarrows E^*_s$ is $\mathcal{F}$-measurable if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Given a $\mathcal{F}$-measurable mapping $X : \Omega \rightrightarrows E^*_s$ and a Borel set $G \in \mathcal{B}(E^*_s)$, the set

$$X^- G = \{\omega \in \Omega : X(\omega) \cap G \neq \emptyset\}$$

is $\mathcal{F}$-measurable, that is $X^- G \in \mathcal{F}$. In view of the completeness hypothesis on the probability space, this is a consequence of the Projection Theorem (see e.g. Theorem III.23 of [11]) and of the equality

$$X^- G = \mathrm{proj}_\Omega \{Gr(X) \cap (\Omega \times G)\}.$$

Further if $u : \Omega \to E^*_s$ is a scalarly $\mathcal{F}$-measurable mapping, that is, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable, then the function $f : (\omega, x^*) \mapsto ||x^* - u(\omega)||_{E^*_b}$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable, and for every fixed $\omega \in \Omega$, $f(\omega, .)$ is lower semicontinuous on E^*_s, shortly, f is a normal integrand, indeed, we have

$$||x^* - u(\omega)||_{E^*_b} = \sup_{j\in\mathbf{N}} \langle e_j, x^* - u(\omega)\rangle$$

here $D_1 = (e_j)_{j\geq 1}$ is a dense sequence in the closed unit ball of E. As each function $(\omega, x^*) \mapsto \langle e_j, x^* - u(\omega)\rangle$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable and continuous on E^*_s for each $\omega \in \Omega$, it follows that f is a normal integrand. Consequently, the graph of u belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Besides these facts, let us mention that the function distance $d_{E^*_b}(x^*, y^*) = ||x^* - y^*||_{E^*_b}$ is lower semicontinuous on $E^*_s \times E^*_s$, being the supremum of w^*-continuous functions. If X is a $\mathcal{F}$-measurable mapping, the distance function $\omega \mapsto d_{E^*_b}(x^*, X(\omega))$ is $\mathcal{F}$-measurable, by using the lower semicontinuity of the function $d_{E^*_b}(x^*, .)$ on E^*_s and measurable projection theorem ([11], Theorem III.23) and recalling that E^*_s is a Suslin space. A mapping $u : \Omega \Rightarrow E^*_s$ is said to be scalarly integrable, if, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable and integrable. We denote by $L^1_{E^*}[E](\mathcal{F})$ the subspace of all $\mathcal{F}$-measurable mappings u such that the function $|u| : \omega \mapsto ||u(\omega)||_{E^*_b}$ is integrable. The measurability of $|u|$ follows easily from the above considerations. Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of $\mathcal{F}$-measurable w^*-closed convex mappings, the sequential weak* upper limit $w^*\text{-}ls\, X_n$ of $(X_n)_{n\in\mathbf{N}}$ is defined by

$$w^*\text{-}ls\, X_n = \{x^* \in E^* : x^* = \sigma(E^*, E)\text{-}\lim_{j\to\infty} x^*_j;\ x^*_j \in X_{n_j}\}.$$

Similarly the sequential weak* lower limit $w^*\text{-}li\, X_n$ of $(X_n)_{n\in\mathbf{N}}$ is defined by

$$w^*\text{-}li\, X_n = \{x^* \in E^* : x^* = \sigma(E^*, E)\text{-}\lim_{n\to\infty} x^*_n;\ x^*_n \in X_n\}.$$

The sequence $(X_n)_{n\in\mathbf{N}}$ weak star Kuratowski (w^*K for short) converges to a $\mathcal{F}$-measurable w^*-closed convex valued mapping $X_\infty : \Omega \Rightarrow E^*_s$ if the following holds

$$w^*\text{-}ls\, X_n \subset X_\infty \subset w^*\text{-}li\, X_n \quad a.s.$$

Shortly

$$w^*\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

By $cwk(E^*_s)$ we denote the set of all nonempty convex $\sigma(E^*, E)$-compact subsets of E^*_s. A $cwk(E^*_s)$-valued mapping $X : \Omega \Rightarrow E^*_s$ is scalarly $\mathcal{F}$-measurable if the function $\omega \to \delta^*(x, X(\omega))$ is $\mathcal{F}$-measurable for every $x \in E$. Let us recall that any scalarly $\mathcal{F}$-measurable $cwk(E^*_s)$-valued mapping is $\mathcal{F}$-measurable. Indeed, let $(e_k)_{k\in\mathbf{N}}$ be a sequence in E which separates the points of E^*, then we have $x \in X(\omega)$ iff, $\langle e_k, x\rangle \le \delta^*(e_k, X(\omega))$ for all $k \in \mathbf{N}$. By $\mathcal{L}^1_{cwk(E^*_s)}(\Omega, \mathcal{F}, P)$ (shortly $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$) we denote the of all scalarly integrable $cwk(E)$-valued multifunctions X such that the function $|X| : \omega \to |X(\omega)|$ is integrable, here $|X(\omega)| := \sup_{y^*\in X(\omega)} ||y^*||_{E^*_b}$, by the above consideration, it is easy to see that $|X|$ is $\mathcal{F}$-measurable. Let $\mathcal{H}^*_{E^*_b}$ be the Hausdorff distance asscociated with the dual norm $||.||_{E^*_b}$ on bounded closed convex subsets in E^*, and X, Y be two convex weak* compact valued measurable mapping, then $\mathcal{H}^*_{E^*_b}(X, Y) = \sup_{j\in\mathbf{N}}[\delta^*(e_j, X) - \delta^*(e_j, Y)]$ is measurable. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is *bounded* (resp. *uniformly integrable*) if $(|X_n|)_{n\in\mathbf{N}}$ is bounded (resp. uniformly integrable) in $\mathcal{L}^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$. We refer to [14] for the weak star convergence of closed bounded convex sets in a dual space.

For the convenience of the reader we recall and summarize the existence and uniqueness of the conditional expectation in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. See ([21], Theorem 3).

Theorem 5.1. *Given $\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and a sub σ-algebra $\mathcal{B}$ of $\mathcal{F}$, there exists a unique (for equality a.s.) mapping $\Sigma := E^{\mathcal{B}}\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{B})$, that is the conditional expectation of Γ with respect to $\mathcal{B}$, which enjoys the following properties:*

a) $\int_\Omega \delta^*(v, \Sigma)dP = \int_\Omega \delta^*(v, \Gamma)dP$ *for all* $v \in L^\infty_E(\mathcal{B})$.

b) $\Sigma \subset E^{\mathcal{B}}|\Gamma|\overline{B}_{E^*}$ *a.s.*

c) $\mathcal{S}^1_\Sigma(\mathcal{B})$ *is sequentially* $\sigma(L^1_{E^*}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ *compact (here* $\mathcal{S}^1_\Sigma(\mathcal{B})$ *denotes the set of all* $L^1_{E^*}[E](\mathcal{B})$ *selections of* Σ*) and satisfies the inclusion*

$$E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F}) \subset \mathcal{S}^1_\Sigma(\mathcal{B}).$$

d) *Furthermore one has*

$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$

for all $v \in L^\infty_E(\mathcal{B})$.
e) $E^{\mathcal{B}}$ *is increasing:* $\Gamma_1 \subset \Gamma_2$ *a.s. implies* $E^{\mathcal{B}}\Gamma_1 \subset E^{\mathcal{B}}\Gamma_2$ *a.s.*

For more information for the conditional expectation of multifunctions, we refer to [6, 18, 21].

Definition 5.2. A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is an adapted sequence if each X_n is $\mathcal{F}_n$-measurable. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\mathcal{H}_{E^*_b}(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) > \varepsilon]) \leq \varepsilon$$

where T denotes the set of bounded stopping times and the conditional expectation is defined by Theorem 5.1.

Although the dual E^* is nolonger separable, we can develop our convergence results for pramarts in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ because the above conditional expectation are ensured by Theorem 5.1 and the measurability of $\mathcal{H}_{E^*_b}(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau)$ is ensured by the formula given above.

We begin with a simple result dealing with pramarts in $L^1_{E^*}[E](\mathcal{F})$ that is even new.

Theorem 5.3. *Let* $(X_n)_{n\in\mathbf{N}}$ *be a bounded pramart in* $L^1_{E^*}[E](\mathcal{F})$. *Then there is* $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ *such that*

$$\lim_{n\to\infty} |X_n(\omega)| = |X_\infty(\omega)| \quad a.s.$$

$$\lim_{n\to\infty} \langle x, X_n(\omega)\rangle = \langle x, X_\infty(\omega)\rangle \quad a.s. \quad \forall x \in \overline{B}_E$$

Proof. We will use a simple variant of the biting-compactness method developed in [8]. Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $L^1_{E^*}[E](\mathcal{F})$, there are subsequences $(X_{n_k})_{k\in\mathbf{N}}$, $(Y_{n_k})_{k\in\mathbf{N}}$, $(Z_{n_k})_{k\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ such that $(Y_{n_k})_{k\in\mathbf{N}}$ is uniformly integrable and $\lim_{k\to\infty} |Z_{n_k}| = 0$ a.s. and such that

$$X_{n_k} = Y_{n_k} + Z_{n_k} \quad \forall k \in \mathbf{N} \tag{5.1}$$

By virtue of Theorem 6.5.9 in [8] we may assume there is $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ such that

$$\lim_{k\to\infty} \int_\Omega \langle u, Y_{n_k}\rangle dP = \int_\Omega \langle u, X_\infty\rangle dP \tag{5.2}$$

for all $u \in L^\infty_E(\mathcal{F})$. Let $(e_m)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_E$. Now by combining (5.1)-(5.2) and the pramart a.s. convergence of each $(\langle x, X_n\rangle)$ $(x \in E)$, it is not difficult to see that

$$\lim_{n\to\infty} \langle e_m, X_n\rangle = \langle e_m, X_\infty\rangle \quad a.s. \tag{5.3}$$

By Lemma 4.3 and (5.3) $((\langle e_m, X_n\rangle^+)_{m\in\mathbf{N}})_{n\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded subpramarts in the terminology of [16] with

$$\lim_{n\to\infty} \langle e_m, X_n\rangle^+ = \langle e_m, X_\infty\rangle^+, \quad a.s. \tag{5.4}$$

Applying Lemma VIII.1.15 in [16] to $((\langle e_m, X_n\rangle^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ yields

$$\lim_{n\to\infty}|X_n| = \lim_{n\to\infty}\sup_{m\in\mathbf{N}}\langle e_m, X_n\rangle^+ = \sup_{m\in\mathbf{N}}\lim_{n\to\infty}\langle e_m, X_n\rangle^+$$
$$= \sup_{m\in\mathbf{N}}\langle e_m, X_\infty\rangle^+ = |X_\infty| \quad a.s.$$

Therefore we get

$$\sup_{n\in\mathbf{N}}|X_n(\omega)| < \infty \quad a.s. \tag{5.5}$$

Now by (5.3)-(5.5) it is not difficult to check that

$$\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle, \quad a.s. \quad \forall x\in \overline{B}_E$$

thereby proving the theorem. □

Now we proceed to the w^*K convergence of pramarts in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Recall that the Banach space E is weakly compactly generated (WCG) if there exist a weakly compact subset of E whose linear span is dense in E.

Theorem 5.4. *Assume that E is WCG. Let $(X_n)_{n\in\mathbf{N}}$ be a bounded pramart $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then there is $X_\infty\in\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that*

$$\lim_{n\to\infty}|X_n(\omega)| = |X_\infty(\omega)| \quad a.s.$$
$$w^*K\text{-}\lim_{n\to\infty}X_n(\omega) = X_\infty(\omega) \quad a.s.$$

Proof. We will use again the biting-compactness method. Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, that is,

$$\sup_{n\in\mathbf{N}}\int_\Omega |X_n|dP = \sup_{n\in\mathbf{N}}\int_\Omega \sup_{x\in\overline{B}_E}|\delta^*(x, X_n)|dP < \infty$$

for each $x\in\overline{B}_E$, the L^1-bounded pramart $(\delta^*(x, X_n))_{n\in\mathbf{N}}$ converges a.s. towards to an integrable function $\varphi_x\in L^1_{\mathbf{R}}(\mathcal{F})$. Let $(e_m)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_E$. Applying ([10], Theorem 6.1(4)), involving a special biting convergence in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ gives $X_\infty\in\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that

$$\lim_{n\to\infty}\delta^*(e_m, X_n) = \varphi_{e_m} = \delta^*(e_m, X_\infty) \quad a.s. \tag{5.6}$$

By Lemma 4.3 and (5.6) $((\delta^*(e_m, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is a uniform sequence of positive L^1-bounded subpramarts [16]. with

$$\lim_{n\to\infty}\delta^*(e_m, X_n)^+ = \delta^*(e_m, X_\infty)^+, \quad a.s. \quad \forall m\in\mathbf{N}, \tag{5.7}$$

Applying Lemma VIII.1.15 in [16] to $((\delta^*(e, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ yields

$$\lim_{n\to\infty}|X_n| = \lim_{n\to\infty}\sup_{m\in\mathbf{N}}\delta^*(e_m, X_n)^+ = \sup_{m\in\mathbf{N}}\lim_{n\to\infty}\delta^*(e_m, X_n)^+$$
$$= \sup_{m\in\mathbf{N}}\delta^*(e_m, X_\infty)^+ = |X_\infty| \quad a.s.$$

Therefore we get

$$\sup_{n\in\mathbf{N}}|X_n(\omega)| < \infty \quad a.s. \tag{5.8}$$

Now by (5.6)-(5.8) it is not difficult to check that

$$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty), \quad a.s. \quad \forall x \in \overline{B}_E. \tag{5.9}$$

Since E is WCG, applying (5.8)-(5.9) and Theorem 4.1 in [14] yield the w^*K convergence for $(X_n)_{n\in\mathbf{N}}$ □

We will prove in next section the validity of Theorem 5.4 without assuming that E is WCG.

6. Further structure results for pramarts

In this section we investigate in some structure results for both pramarts and multivalued pramarts. We will use the following lemma.

Lemma 6.1. *Assume that E_b^* is separable and $E_1^* := (f_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a martingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds*

$$(\delta^*(f_m^*, Y_\sigma) - \delta^*(f_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\delta^*(f_m^*, Y_\tau) - \delta^*(f_m^*, X_\tau))^+$$
$$\leq \mathcal{H}_E(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma) \quad a.s. \quad \forall m \in \mathbf{N}.$$

Proof. For simplicity, for each $m \in \mathbf{N}$, $n \in \mathbf{N}$, set $A_n^m = \delta^*(f_m^*, X_n)$ and $B_n^m = \delta^*(f_m^*, Y_n)$. Then $(A_n^m)_{n\in\mathbf{N}}$ is a pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ and $(B_n^m)_{n\in\mathbf{N}}$ is a martingale in $L^1_{\mathbf{R}}(\mathcal{F})$. Applying Lemma 4.5 to $(A_n^m)_{n\in\mathbf{N}}$ and $(B_n^m)_{n\in\mathbf{N}}$ yields for a.s. $\omega \in \Omega$

$$\begin{aligned}(B_\sigma^m - A_\sigma^m)^+ - E^{\mathcal{F}_\sigma}(B_\tau^m - A_\tau^m)^+ &\leq |A_\sigma^m - E^{\mathcal{F}_\sigma} A_\tau^m| \\ &= |\delta^*(f_m^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(f_m^*, X_\tau)| \\ &= |\delta^*(f_m^*, X_\sigma) - \delta^*(f_m^*, E^{\mathcal{F}_\sigma} X_\tau)| \\ &\leq \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau)\end{aligned}$$

where $A_\sigma^m = \delta^*(f_m^*, X_\sigma)$ and $B_\tau^m = \delta^*(f_m^*, Y_\tau)$. It follows that

$$(([\delta^*(f_m^*, Y_n) - \delta^*(f_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a positive uniform sequence of subpramarts.

Alternatively we may repeat the techniques given the proof of Lemma 4.4. For simplicity for each $m, n \in \mathbf{N}$, let us set

$$\varphi_{m,n}(\omega) := \delta^*(e_m^*, Y_n) - \delta^*(e_m^*, X_n)$$

Let σ, $\tau \in T, \tau \geq \sigma$ and let us set

$$\varphi_{m,\sigma} := \delta^*(e_m^*, Y_\sigma) - \delta^*(e_m^*, X_\sigma)$$

$$\varphi_{m,\tau} := \delta^*(e_m^*, Y_\tau) - \delta^*(e_m^*, X_\tau)$$

From Jensen inequality we have

$$|E^{\mathcal{F}_\sigma}\varphi_{m,\tau}| \leq E^{\mathcal{F}_\sigma}|\varphi_{m,\tau}| \quad a.s.$$

Then for a.s. $\omega \in \Omega$ we have that

$$\begin{aligned} \varphi_{m,\sigma}^{+} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}^{+}) &= \frac{1}{2}[\varphi_{m,\sigma} + |\varphi_{m,\sigma}| - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) - E^{\mathcal{F}_\sigma}(|\varphi_{m,\tau}|)] \\ &\leq \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma}| - |E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &\leq \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &= [\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})]^{+} \\ &= [\delta^*(e_m, X_\sigma) - \delta^*(e_m, E^{\mathcal{F}_\sigma} X_\tau)]^{+} \\ &\leq \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) \end{aligned}$$

thereby proving the required inequality. □

Remark. A dual variant of Lemma 6.1 is available: Let $(e_m)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_E$, $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ a pramart in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ a martingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then the following holds

$$\begin{aligned} &(\delta^*(e_m, Y_\sigma) - \delta^*(e_m, X_\sigma))^{+} - E^{\mathcal{F}_\sigma}(\delta^*(e_m, Y_\tau) - \delta^*(e_m, X_\tau))^{+} \\ &\qquad \leq \mathcal{H}_{E^*_b}(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma) \quad a.s. \quad \forall m \in \mathbf{N}. \end{aligned}$$

We begin with a convergence theorem for pramarts in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$.

Theorem 6.2. *Assume that E^*_b is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying: there is $X_\infty \in \mathcal{L}^1_{cwk(E}(\mathcal{F})$ such that*

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}.$$

Then the following hold:

$$\lim_{n\to\infty} \mathcal{H}_E(E^{\mathcal{F}_n} X_\infty, X_n) = 0 \quad a.s.$$

Consequently

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

Proof. Step 1. **Claim 1** M-$\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ a.s. See Theorem 3.1 in [1].
Step 2 **Claim 2** $\lim_{n\to\infty} \mathcal{H}_E(E^{\mathcal{F}_n} X_\infty, X_n)$ a.s.
Let $E^*_1 = (f^*_m)_{m\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ with respect Mackey topology $\tau(E^*, E)$. We have

$$\begin{aligned} \mathcal{H}_E(E^{\mathcal{F}_n} X_\infty, X_n,) &= \sup_{m\in\mathbf{N}} [\delta^*(f^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(f^*_m, X_n)] \\ &= \sup_{m\in\mathbf{N}} [\delta^*(f^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(f^*_m, X_n)]^{+} \end{aligned}$$

As $(\delta^*(f^*_m, X_n) - \delta^*(f^*_m, E^{\mathcal{F}_n} X_\infty))_{n\in\mathbf{N}}$ are real-valued bounded pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0, and from Lemma 6.1

$$(([\delta^*(f^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(f^*_m, X_n)]^{+})_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a uniform sequence of positive subpramarts, applying Lemma VIII.1.15 in [16] yields

$$\begin{aligned}\lim_{n\to\infty}\mathcal{H}(E^{\mathcal{F}_n}X_\infty, X_n) &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}}[\delta^*(f_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(f_m^*, X_n)]\\ &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}}[\delta^*(f_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(f_m^*, X_n)]^+\\ &= \sup_{m\in\mathbf{N}}\lim_{n\to\infty}[\delta^*(f_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(f_m^*, X_n)]^+ = 0\end{aligned}$$

almost surely.

Step 3 By Claim 1-2 and Proposition 3.1 in [1], we conclude that

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Hence Proposition 3.2 in [1] shows that

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x\in E.$$

□

The following decomposition theorem is a combined effort of Theorem 4.7 and Theorem 6.2.

Corollary 6.3. *Assume that E_b^* is separable and E have the RNP. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L_E^1(\mathcal{F})$ satisfying the following condition: For each $A\in\mathcal{F}$, $\cup_{n=1}^\infty \int_A X_n dP$ is relatively $\sigma(E,E^*)$-compact. Then there exist a regular martingale $(Y_n)_{n\in\mathbf{N}}$ in $L_E^1(\mathcal{F})$ and a pramart $(Z_n)_{n\in\mathbf{N}}$ in $L_E^1(\mathcal{F})$ such that*

$$X_n = Y_n + Z_n, \quad \forall n\in\mathbf{N}.$$

$$\lim_{n\to\infty}|Z_n| = 0 \quad a.s.$$

Proof. According to Theorem 4.7 and Theorem 6.2 there is $X_\infty\in L_E^1(\mathcal{F})$ such that

$$\lim_{n\to\infty}|X_n - E^{\mathcal{F}_n}X_\infty| = 0 \quad a.s.$$

Hence the result follows by setting

$$Y_n = E^{\mathcal{F}_n}X_\infty \quad \forall n\in\mathbf{N}.$$

$$Z_n = X_n - Y_n \quad \forall n\in\mathbf{N}.$$

Consequently $(X_n)_{n\in\mathbf{N}}$ converges a.s. in norm to an integrable function in $L_E^1(\mathcal{F})$.

□

The following decomposition result is consequence of Theorem 4.9 and Theorem 6.2.

Corollary 6.4. *Assume that E is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L_E^1(\mathcal{F})$ satisfying the following condition: There is a weakly ball-compact closed convex valued $\mathcal{F}$-measurable multifunction $K:\Omega\rightrightarrows E$ such that $X_n(\omega)\in K(\omega)$ for all $n\in\mathbf{N}$ and for all $\omega\in\Omega$. Then there exist a regular martingale $(Y_n)_{n\in\mathbf{N}}$ in $L_E^1(\mathcal{F})$ and a pramart $(Z_n)_{n\in\mathbf{N}}$ in $L_E^1(\mathcal{F})$ such that*

$$X_n = Y_n + Z_n, \quad \forall n\in\mathbf{N}.$$

$$\lim_{n\to\infty} |Z_n| = 0 \quad a.s.$$

Proof. According to Theorem 4.9 and Theorem 6.2 there is $X_\infty \in L^1_E(\mathcal{F})$ such that

$$\lim_{n\to\infty} |X_n - E^{\mathcal{F}_n} X_\infty| = 0 \quad a.s.$$

Hence the result follows by setting

$$Y_n = E^{\mathcal{F}_n} X_\infty \quad \forall n \in \mathbf{N}.$$

$$Z_n = X_n - Y_n \quad \forall n \in \mathbf{N}.$$

Consequently $(X_n)_{n\in\mathbf{N}}$ converges a.s. in norm to an integrable function in $L^1_E(\mathcal{F})$. □

Now we present the weak star Kuratowski convergence for bounded pramarts in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$.

Theorem 6.5. *Assume that E is separable. Let $(X_n)_{n\in\mathbf{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ satisfying: there is $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that*

$$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s. \quad \forall x \in \overline{B}_E.$$

Then the following hold:

$$\lim_{n\to\infty} |X_n| = |X_\infty| \quad a.s.$$

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Proof. Step 1 Let $(e_j)_{j\in\mathbf{N}}$ be a dense sequence in $\overline{B}_E$. Arguing as in the proof of Theorem 6.2 using the scalar convergence a.s. of $(X_n)_{n\in\mathbf{N}}$ towards X_∞ and the fact that $((\delta^*(e_j, X_n)^+)_{n\in\mathbf{N}})_{j\in\mathbf{N}}$ is a positive uniform sequence of subpramarts and applying again Lemma VIII.1.5 in [16] yields

$$\begin{aligned}\lim_{n\to\infty} |X_n| &= \lim_{n\to\infty} \sup_{j\in\mathbf{N}} \delta^*(e_j, X_n)\\ &= \lim_{n\to\infty} \sup_{j\in\mathbf{N}} \delta^*(e_j, X_n)^+ = \sup_{j\in\mathbf{N}} \lim_{n\to\infty} \delta^*(e_j, X_n)^+\\ &= \sup_{j\in\mathbf{N}} \delta^*(e_j, X_\infty)^+ = \sup_{j\in\mathbf{N}} \delta^*(e_j, X_\infty) = |X_\infty| \quad a.s.\end{aligned}$$

So we get

$$(^*) \qquad \sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty \quad a.s.$$

Step 2 **Claim 1**: $(^{**})$ $w^*K\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty$ a.s. See Theorem 3.1 in [6].
Step 3 **Claim 2**: $(^{***})$ $\lim_{n\to\infty} \mathcal{H}_{E^*_b}(E^{\mathcal{F}_n} X_\infty, X_n) = 0 \quad a.s.$
As $(\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty)_{n\in\mathbf{N}}$ are real-valued bounded pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0, and from the remark of Lemma 6.1

$$(([\delta^*(e_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(e_j, X_n)]^+)_{n\in\mathbf{N}})_{j\in\mathbf{N}}$$

is a uniform sequence of positive subpramarts, applying Lemma VIII.1.15 in [16] yields

$$\begin{aligned}\lim_{n\to\infty}\mathcal{H}_{E_b^*}(E^{\mathcal{F}_n}X_\infty, X_n) &= \lim_{n\to\infty}\sup_{j\in\mathbf{N}}[\delta^*(e_j, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_j, X_n)] \\ &= \lim_{n\to\infty}\sup_{j\in\mathbf{N}}[\delta^*(e_j, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_j, X_n)]^+ \\ &= \sup_{j\in\mathbf{N}}\lim_{n\to\infty}[\delta^*(e_j, E^{\mathcal{F}_n}X_\infty - \delta^*(e_j, X_n)]^+ = 0\end{aligned}$$

almost surely. By (*), (**), (***) and Lemma 3.1 in [6], we conclude that

$$w^*K\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n}X_\infty = X_\infty \quad a.s.$$

□

The following decomposition theorem is consequence of Theorem 5.3 and Theorem 6.5.

Corollary 6.6. *Let $(X_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L^1_{E^*}[E](\mathcal{F})$. Then there exist a regular martingale $(Y_n)_{n\in\mathbf{N}} = (E^{\mathcal{F}_n}X_\infty)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ and a pramart $(Z_n)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ such that*

$$X_n = Y_n + Z_n, \quad \forall n\in\mathbf{N}.$$

$$\lim_{n\to\infty}|Z_n| = 0 \quad a.s.$$

Consequenlty

$$w^*\text{-}\lim_{n\to\infty} X_n = w^*\text{-}\lim_{n\to\infty} Y_n = X_\infty \quad a.s.$$

Proof. According to Theorem 5.3 and Theorem 6.5 there is $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ such that

$$\lim_{n\to\infty}|X_n - E^{\mathcal{F}_n}X_\infty|_{E_b^*} = 0 \quad a.s.$$

Hence the result follows by setting

$$Y_n = E^{\mathcal{F}_n}X_\infty \quad \forall n\in\mathbf{N}.$$

$$Z_n = X_n - Y_n \quad \forall n\in\mathbf{N}.$$

so that

$$w^*\text{-}\lim_{n\to\infty} X_n = w^*\text{-}\lim_{n\to\infty} Y_n = X_\infty \quad a.s.$$

□

We finish this paper by providing some specific properties for pramarts which may be useful in other convergence results and provides some interesting analogies with multivalued martingales. This shed a new light on the study of multivalued pramarts in both the primal space and the dual space.

Lemma 6.7. *Assume that E_b^* is separable. Let $x\in E$ and $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that $\sup_{n\in\mathbf{N}}\int_\Omega d(x, X_n)dP < \infty$. Then $(d(x, X_n))_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart converging a.s. to an integrable function in $L^1_{\mathbf{R}}(\mathcal{F})$.*

Proof. By [15], Lemma 4.3) for any sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and for any $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ we have

$$d(x, E^{\mathcal{B}}X) \leq E^{\mathcal{B}}d(x, X) \quad a.s. \tag{6.1}$$

Since $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) > \varepsilon]) \leq \varepsilon. \tag{6.2}$$

By (6.1) for $\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon$, we have

$$d(x, X_\sigma) - E^{\mathcal{F}_\sigma}d(x, X_\tau) \leq d(x, X_\sigma) - d(x, E^{\mathcal{F}_\sigma}X_\tau) \quad a.s. \tag{6.3}$$

By (6.3) and ([2], Lemma 1.5.1, p. 29) we deduce

$$\begin{aligned}[d(x, X_\sigma) - E^{\mathcal{F}_\sigma}d(x, X_\tau)]^+ &\leq |d(x, X_\sigma) - d(x, E^{\mathcal{F}_\sigma}X_\tau)| \\ \leq \sup_{x\in E} |d(x, X_\sigma) - d(x, E^{\mathcal{F}_\sigma}X_\tau)| &= \mathcal{H}_E(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) \quad a.s.\end{aligned} \tag{6.4}$$

Using (6.4) and (6.2) and the definition 4.2 (of subpramart) it is easy to conclude that $(d(x, X_n))_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart which converges a.s. to an integrable function by virtue of Millet-Sucheston theorem, see ([16], Theorem VIII.1.11). □

Remarks. 1) If E is separable, and if $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a L^1-bounded pramart in $L^1_E(\mathcal{F})$, the techniques of Lemma 6.7 show that $(|X_n|, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart, similarly if $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that $\sup_{n\in\mathbf{N}} \int_\Omega d_{E^*_b}(0, X_n)dP < \infty$, then $(d_{E^*_b}(0, X_n))_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart.
2) In the context of convex weakly-weakly* compact valued pramarts given here, it is easy to see that a convex weakly-weakly* compact valued martingale: $X_n = E^{\mathcal{F}_n}X_{n+1}, \forall n \in \mathbf{N}$ is a convex weakly-weakly* compact valued pramart. Further these pramarts enjoy similar convergence properties like convex weakly-weakly* compact valued sub-martingales: $X_n \subset E^{\mathcal{F}_n}X_{n+1}, \forall n \in \mathbf{N}$, see [1, 6]. Mosco convergence for *unbounded* closed convex supermartingales $(X_n, \mathcal{F}_n)$: $E^{\mathcal{F}_n}X_{n+1} \subset X_n, \forall n \in \mathbf{N}$, is available, see [1, 6, 15] and the references therein, further if $d(0, X_n)_{n\in\mathbf{N}}$ is bounded in L^1, then $d(0, X_n)_{n\in\mathbf{N}}$ is a L^1-bounded submartingale, hence it is a subpramart (compare with Lemma 6.7) above. In despite of these similarities, it seems that it is difficult to find a unified closed convex valued pramart theory generalizing the closed convex valued sub-super martingales. Clearly the class of convex weakly-weakly* compact valued pramarts contains the class of convex weakly-weakly* compact valued *uniform amarts*:

$$\lim_{\sigma\leq\tau;\tau\in T} \int_\Omega h(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau)dP = 0.$$

3) If the pramarts given in Corollary 6.1–6.3 are uniformly integrable, then the decomposition formula given therein is unique. Indeed assume that

$$X_n = Y_n + Z_n = Y'_n + Z'_n$$

where $(Y_n, \mathcal{F}_n)$ and $(Y'_n, \mathcal{F}_n)$ are martingales and $(Z_n, \mathcal{F}_n)$ and $(Z'_n, \mathcal{F}_n)$ and pramarts with

$$\lim_{n\to\infty} |Z_n| = \lim_{n\to\infty} |Z'_n| = 0 \quad a.s.$$

Then for each $A \in \cup_n \mathcal{F}_n$

$$\lim_{n\to\infty} \int_A (Z_n - Z'_n) dP = 0.$$

But $Y'_n - Y_n = Z'_n - Z_n$ form also a martingale so that, for each $m \in \mathbf{N}$ and $A \in \mathcal{F}_m$

$$\lim_{n\to\infty} \int_A (Z_n - Z'_n) dP = \int_A (Z_m - Z'_m) dP = 0.$$

Hence $Z_m = Z'_m$ a.s. and also $Y_m = Y'_m$ a.s. for each $m \in \mathbf{N}$.

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Manuscript received February 22, 2010

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