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KKM-FAN PRINCIPLE IN $\aleph_0$-SPACES AND SOME APPLICATIONS

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Abstract. In this paper we first introduce the topological structure of $\aleph_0$-spaces which is a generalization of hyperconvex metric spaces. We then establish an associated KKM finite intersection lemmas. As applications we give a $\aleph_0$-space version of Fan's best approximation theorem for set-valued mappings and some fixed point theorems.

1. Introduction and preliminaries

Let us start by briefly introducing some needed notations and terminology. Throughout this paper, for each integer $n \geq 0$ we denote by $\mathcal{F}_n$ the set of all nonempty subset of $\langle n \rangle := \{0, 1, \ldots, n\}$. For each $J \in \mathcal{F}_n$, we denote by $\Delta_J := conv\{e_i : i \in J\}$ the convex hull of $\{e_i : i \in J\}$, and $\Delta_n = \Delta_{\langle n \rangle}$ where $e_0, e_1, \ldots, e_n$ are the unit vectors of $\mathbb{R}^{n+1}$. If $card(J)$ denotes the cardinality of J, the set $\Delta_n^k = \underset{card(J)\leq k}{\cup} \Delta_J$ is called the k-skeleton of Δ_n.

Recall that $\{M_i \subset \Delta_n : i \in \langle n \rangle\}$ is a **KKM family** if for each $J \in \mathcal{F}_n$ $\Delta_J \subset \cup_{i\in J} M_i$.

The KKM (Knaster-Kuratowski-Mazurkiewicz) or three polish's lemma is one of the most interesting results in nonlinear functional analysis. This lemma [16] states that each KKM family of closed subsets of Δ_n has a nonempty intersection. The KKM lemma is in fact equivalent to several fundamental results as Sperner's lemma [20], Brouwer's fixed point theorem [5], Ky Fan's minimax inequality [7], and others see [23] for more details. The dual form of the KKM lemma says that the KKM lemma holds true when the word **closed** is replaced by **open**. In the last decades, the KKM lemma and Ky Fan's minimax inequality have been improved by relaxing the closedness, the convexity, the compactness condition or even more by extending the framework to more general nonlinear structures. Our object in this paper is to use hyperconvex structure in metric spaces and H-structure in topological spaces in order to extend some recent intersection and minimax inequality results as in Khamsi [12] and Kirk, Sims and Yuan [14].

To present interconnection between hyperconvex structure and H-structure let us recall the definitions.

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Definition 1.1 (Aronszjn and Panitchpakdi [2])**.** A metric space (X, d) is said to be hyperconvex if for any class $\{x_\alpha : \alpha \in \mathcal{A}\}$ in X and $\{r_\alpha : \alpha \in \mathcal{A}\}$ in $\mathbb{R}_+$, one has

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \;\; \forall \alpha, \beta \in \mathcal{A} \implies \bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha) \neq \emptyset$$

where $B(x_\alpha, r_\alpha)$ is the closed ball centered at x_α with radius r_α.

For any nonempty bounded subset A of X, its convex hull $co(A)$ is defined by

$$co(A) := \cap\{B : B \text{ is a closed ball containing } A\}.$$

Remark that $A = co(A)$ iff A is an intersection of balls. In this case we will say A is an admissible subset of X.

Note that every hyperconvex metric space is complete. Also, it is quite easy to see that an admissible subset of a hyperconvex metric space is hyperconvex.

The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi in [2], where it is shown that a metric space is hyperconvex if and only if it is injective with respect to the nonexpansive mappings. Since every metric space has an injective hull [11], it follows that every metric space is isometric with a subspace of a (minimal) hyperconvex superspace. Also it is known that the space $\mathcal{C}(E)$ of all continuous real functions on a Stonian space E (extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathcal{C}(E)$ for some Stonian space E. Classical and concrete examples of hyperconvex spaces include the well-known spaces $(\mathbb{R}^n, \|\cdot\|_\infty)$, l^∞ and L^∞.

A topological space X is said to be contractible if the identity of X is homotopic to a constant mapping.

Definition 1.2 (Horvath [9], Bardaro and Ceppiletti [3])**.** Let X be a topological space, $\mathcal{F}(X)$ the set of all finite nonempty subset of X, and $\Gamma : \mathcal{F}(X) \to 2^X$ a set-valued mapping (2^X denotes the set of all nonempty subsets of X), then (X, Γ) is a H-structure (or H-space) if :

(a) $\Gamma(A)$ is nonempty and contractible for every $A \in \mathcal{F}(X)$; and
(b) $\Gamma(A) \subset \Gamma(B)$ if $A \subset B$ and $A, B \in \mathcal{F}(X)$.

Recently, Horvath [9] obtained some generalizations of KKM and Ky Fan's geometric lemma by replacing convexity assumption with the H-structure. More details concerning these definitions and properties can be found in the recent Yuan's book [23].

The starting point of this paper is to introduce a generalized hyperconvex structure on topological space and propose the topological structure of $\aleph_0$-space. Based on the useful KKM-Lemma [16] and its recent dual form [21, 13, 17, 23] we discuss some properties of $\aleph_0$-space in topological spaces by employing Horvath's approch [9, 18] of H-spaces and hyperconvex structures [8, 12]. Afterword, we present some KKM-finite intersection properties of set-valued mappings. These results improve and unify some corresponding results in the literature.

Definition 1.3. Let (X, d_1) and (Y, d_2) be two metric spaces, and $T : X \to Y$ a continuous mapping. The continuity modulus of T is an increasing mapping

$\delta : [0, +\infty[\to [0, +\infty]$ such that $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ and

$$\sup_{\substack{x,y \in X \\ d_1(x,y) \leq \epsilon}} d_2(T(x), T(y)) \leq \delta(\varepsilon).$$

If moreover δ is subadditive, i.e. $\delta(\varepsilon_1 + \varepsilon_2) \leq \delta(\varepsilon_1) + \delta(\varepsilon_2)$, $\forall \varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we say that T admits a subadditive modulus of continuity.

If a mapping T has some modulus of continuity, it is uniformly continuous.

Conversely, for each uniformly continuous mapping T there exists a minimal modulus of continuity δ_T defined as follows:

$$\delta_T(\varepsilon) = \sup\{d_2(T(x), T(y)) : x, y \in X, d_1(x, y) \leq \varepsilon\}.$$

Every other modulus of continuity δ of T satisfies : $\delta_T(\varepsilon) \leq \delta(\varepsilon)$ for all $\varepsilon > 0$.

For a continuous mapping T, see [2, Theorem 1], in order that there exists a subadditive modulus of continuity majorating a given modulus of continuity δ, it is necessary and sufficient that $\limsup_{\varepsilon \to +\infty} \frac{\delta(\varepsilon)}{\varepsilon} < +\infty$.

Definition 1.4 ([2, 6])**.** Let $m \in \mathbb{N}$ and (X, d) a metric space. (X, d) is said to be m-hyperconvex if for any class $\{x_\alpha : \alpha \in \mathcal{A}\}$ in X and $\{r_\alpha : \alpha \in \mathcal{A}\}$ in $\mathbb{R}_+$ with $card(\mathcal{A}) < m$, one has

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \;\; \forall \alpha, \beta \in \mathcal{A} \Longrightarrow \bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha) \neq \emptyset$$

If X is m-hyperconvex for every $m \in \mathbb{N}$, we say that X is $\aleph_0$-hyperconvex.

Remark that if cardinality of $\mathcal{A}$ is not fixed, as introduced in Definition 1.1, we only say X to be hyperconvex.

It is clear that hyperconvexity is stronger than m-hyperconvexity, for each m. The notion of m-hyperconvexity is also stronger than m'-hyperconvexity if $m' < m$, and the inclusion is strict.

Take, for example, the Euclidean plane $\mathbb{R}^2$ with the natural metric. It is easy to see that $\mathbb{R}^2$ is a 3-hyperconvex space but not 4-hyperconvex. Thus $\mathbb{R}^2$ is neither $\aleph_0$-hyperconvex nor hyperconvex.

Given a metric space X and $A \subset M$. We will denote by $h(A)$ the $\aleph_0$-hyperconvex hull of A in M, i.e. $h(A)$ is a minimal element, relatively to order of inclusion, of the set of $\aleph_0$-hyperconvex subsets of M.

Let Y be an other metric space and $T : Y \to X$ a compact mapping, i.e. T is continuous and $T(A)$ is relatively compact in X for each bounded subset A of Y.

Fix $y_0 \in Y$, and denote, for each $n \geq 2$, $B(y_0, n)$ the closed ball centered at y_0 with radius n. Then by induction we define

$$H_n = h\left(h(B_n) \cup H_{n-1}\right) \text{ where } B_n := T(B(y_0, n)) \text{ and } H_1 = h(B(y_0, 1)).$$

Following Espinola and Lopez [6], we have the following characterization of a complete $\aleph_0$-hyperconvex metric space.

Theorem 1.5. *Let (X, d) be a metric space, then the following assersions are equivalent:*

(1) *X is $\aleph_0$-hyperconvex and complete;*

(2) *if* $Y \cup \{z\}$ *is a metric space which contains metrically* Y, $n \in \mathbb{N}$ *for which* $n-1 < d(y_0, z) < n$, *and* $T : Y \to X$ *a compact uniformly continuous mapping with a subadditive modulus of continuity* δ_T, *then there exists* $\overline{T}$: $Y \cup \{z\} \to X$ *an extension of* T *such that* $\overline{T}(z) \in H_{n+1}$ *and* δ_T *is a modulus of continuity for* $\overline{T}$;

(3) *if* Y *is a metric space,* $T : Y \to X$ *a compact uniformly continuous mapping with a subadditive modulus of continuity* δ_T, *and* Z *is a metric space which contains metrically* Y, *then there exists* $\overline{T} : Z \to X$ *an extension of* T *such that*

$$\delta_{\overline{T}}(\varepsilon) = \max\{(1+\eta)\,\varepsilon\delta_T(1), (1+\eta)\,\delta_T(\varepsilon)\}$$

is a modulus of continuity for $\overline{T}$.

Definition 1.6. Let H be a nonempty set, M be a topological space and $R : \mathcal{F}(H) \to 2^M$. The triplet (H, M, R) is said to be a $\aleph_0$-space if

(1) $\forall A \in \mathcal{F}(H)$, $R(A)$ is a nonempty complete $\aleph_0$-hyperconvex space;
(2) $\forall A, B \in \mathcal{F}(H)$, $A \subset B$ implies $R(A) \subset R(B)$ metrically.

Remark 1.7. This definition extend the similar one in [12] since if M is a hyperconvex space and seting $R(A) = co(A)$ we have (M, M, R) is a $\aleph_0$-space.

Remark 1.8. If $H = \langle n \rangle$ for some $n \in \mathbb{N} \setminus \{0\}$ we say that (M, R) is a $(n, \aleph_0)$-space.

Remark 1.9. Let us remark that $R(J)$ may be a metric space for a nonmetrisable topological space M. As an example, one can consider $R(J)$ as a closed ball in a normed space M with a separable dual space M' endowed with the weak topology $\sigma(M, M')$. Then, for $\sigma(M, M')$, M is nonmetrisable but $R(J)$ is metrisable. This says that Definition 1.6 and Remark 1.8 extend the definition 3 in [12].

Remark 1.10. Consider in $(\mathbb{R}^3, \|.\|_2)$ the triangle $\triangle = conv\{e_0, e_1, e_2\}$. Then $\triangle$ is convex and thus contractible, but not $\aleph_0$-hyperconvex. We conclude that contractibility and $\aleph_0$-hyperconvexity are different notions.

Proposition 1.11. *The product of finite* $\aleph_0$*-hyperconvex spaces (respectively,* $\aleph_0$*-spaces) is a* $\aleph_0$*-hyperconvex space (respectively,* $\aleph_0$*-space).*

Proof. Consider $X_1, X_2, \ldots, X_i, \ldots, X_n$ a finite family of $\aleph_0$-hyperconvex spaces, and $X := \prod_i X_i$. Let $\left(x_\alpha = (x_\alpha^1, \ldots, x_\alpha^n)\right)_{\alpha \in \mathcal{A}}$ be a net in X and $(r_\alpha)_{\alpha \in \mathcal{A}}$ a net of positive real numbers such that $d(x_\alpha, x_\beta) := \max_{1 \le i \le n} d_i(x_\alpha^i, x_\beta^i) \le r_\alpha + r_\beta$.

Then for each $1 \le i \le n$ one has $d(x_\alpha^i, x_\beta^i) \le r_\alpha + r_\beta$, and since X_i are $\aleph_0$-hyperconvex spaces, we obtain $\bigcap_{\alpha \in \mathcal{A}} B_{X_i}(x_{\alpha,i}, r_\alpha) \neq \emptyset$. This implies that $\bigcap_{\alpha \in \mathcal{A}} B_X(x_\alpha, r_\alpha)$, since equal to $\prod_i \bigcap_{\alpha \in \mathcal{A}} B_{X_i}(x_{\alpha,i}, r_\alpha)$, is nonempty; and thus X is $\aleph_0$-hyperconvex space.

Let $(H, X_1, R_1), \ldots, (H, X_n, R_n)$ be $\aleph_0$-spaces, and define the mapping $R : \mathcal{F}(H) \to 2^X$ by $R(A) := \prod_i R_i(P_i(A))$, where P_i is the projection on the space X_i. One can verify that (H, X, R) is a $\aleph_0$-space. □

Proposition 1.12. *Let* M *be a compact* $\aleph_0$*-hyperconvex metric space, then every nonempty admissible subset is* $\aleph_0$*-hyperconvex.*

Proof. Let B be a nonempty admissible subset of M. Since an admissible subset is an intersection of balls, set $B := \underset{i\in I}{\cap} B(y_i, t_i)$, where I index the family of all balls containing B and the families $(y_i)_{i\in I} \subset M$ and $(t_i)_{i\in I} \subset]0, +\infty[$. Consider $(x_\alpha) \subset B$ and $r_\alpha > 0$ such that $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$. For $D := \underset{\alpha}{\cap} B(x_\alpha, r_\alpha)$ let us show that $D \cap B \neq \emptyset$.

Let $K_i := B(y_i, t_i) \cap D$ and fixe $J \in \mathcal{F}(I)$. Let us consider $(y_i, t_i)_{i\in J}$, (x_α, r_α) and check that this family satisfies the condition of Definition 1.3 what affirms that

$$\underset{i\in J}{\cap} B(y_i, t_i) \cap \Big(\underset{\alpha}{\cap} B(x_\alpha, r_\alpha)\Big) = \underset{i\in J}{\cap} B(y_i, t_i) \cap D \neq \emptyset \text{ implies } \underset{i\in J}{\cap} K_i \neq \emptyset.$$

Consequently $\underset{i\in I}{\cap} K_i = D \cap B \neq \emptyset$ from which B is $\aleph_0$-hyperconvexe. □

2. Continuous simplicial selection

In order to give the main result of this section we first prove the following technical extension lemma.

Lemma 2.1. *Let M be a topological space and $g : \Delta_n^k \longrightarrow M$ be a mapping with a subadditive modulus of continuity δ_g. Consider $f : \Delta_n^{k+1} \longrightarrow M$ a continuous mapping such that $f = g$ on Δ_n^k (i.e. f is an extension of g to Δ_n^{k+1}), and suppose that for each $J \in \mathcal{F}_n$, with card $(J) \le k+1$, the modulus of continuity δ_f^J of f on Δ_J is subadditive. Then f has a subadditive modulus of continuity δ_f on Δ_n^{k+1}.*

Proof. Fix $\varepsilon > 0$, sufficiently small, and consider $\Delta_J, \Delta_{J'}$ two faces of Δ_n^{k+1} with $\Delta_J \cap \Delta_{J'} \neq \emptyset$. Note that the case $\Delta_J \cap \Delta_{J'} = \emptyset$ is omited since, in this case, the distance between Δ_J and $\Delta_{J'}$ may be larger than ε.

Denote by $\Delta_I := \Delta_J \cap \Delta_{J'}$ which is a subset of Δ_n^k, and let $x \in \Delta_J$ and $y \in \Delta_{J'}$ such that $d(x, y) \le \varepsilon$, where d is the Euclidean distance. One has

$$x = \sum_{i\in J} \alpha_i e_i = \sum_{i\in J_1} \alpha_i e_i + \sum_{i\in I} \alpha_i e_i \text{ and } y = \sum_{i\in J'} \beta_i e_i = \sum_{i\in J_1'} \beta_i e_i + \sum_{i\in I} \beta_i e_i$$

where

$$\alpha_i, \beta_i \ge 0, \sum_{i\in J} \alpha_i = \sum_{i\in J'} \beta_i = 1 \text{ and } J_1 = J\backslash I,\ J_1' = J'\backslash I.$$

Fix $i_1 \in I$ and consider $x_1 := \sum_{i\in I} \alpha_i' e_i$ and $y_1 := \sum_{i\in I} \beta_i' e_i$ where $\alpha_i' = \alpha_i$, $\beta_i' = \beta_i$ if $i \neq i_1$ and $\alpha_{i_1}' = \alpha_{i_1} + \sum_{i\in J_1} \alpha_i$, $\beta_{i_1}' = \beta_{i_1} + \sum_{i\in J_1'} \beta_i$.

Clearly, $x_1, y_1 \in \Delta_I$, $d(x, x_1) = \left(\sum_{i\in J_1} \alpha_i^2 + (\sum_{i\in J_1} \alpha_i)^2 \right)^{1/2}$.

According, since *card* $(J_1) \le n$, to Hölder inequality we obtain

$$\sum_{i\in J_1} \alpha_i \le \sqrt{n} \left(\sum_{i\in J_1} \alpha_i^2 \right)^{1/2}$$

and then

$$d(x,x_1) \le \left((n+1)\sum_{i\in J_1}\alpha_i^2 \right)^{1/2} \le \sqrt{n+1}d(x,y).$$

Since

$$d(x,y) = \left(\sum_{i\in I}(\alpha_i-\beta_i)^2 + \sum_{i\in J_1}\alpha_i^2 + \sum_{i\in J_1'}\beta_i^2 \right)^{1/2} \le \varepsilon,$$

we deduce

$$d(x,x_1) \le \sqrt{n+1}\varepsilon.$$

Similarly, by seting $c := \sqrt{n+1}$, we have

$$d(y,y_1) = \left(\sum_{i\in J_1'}\beta_i^2 + \left(\sum_{i\in J_1'}\beta_i \right)^2 \right)^{1/2} \le \sqrt{n+1}\varepsilon \le c\varepsilon.$$

Thus

$$d(x_1,y_1) \le d(x,x_1) + d(x,y) + d(y,y_1) \le (2c+1)\varepsilon.$$

Since $x \in \Delta_J, y \in \Delta_{J'}$ and $x_1, y_1 \in \Delta_I$, if we denote the restriction of f on J by f_J, it follows that

$$\begin{aligned} d(f(x),f(y)) &= d(f_J(x),f_{J'}(y)) \\ &\le d(f_J(x),f_J(x_1)) + d(f_J(x_1),f_{J'}(y_1)) + d(f_{J'}(y_1),f_{J'}(y)) \\ &\le d(f_J(x),f_J(x_1)) + d(f_I(x_1),f_I(y_1)) + d(f_{J'}(y_1),f_{J'}(y)) \\ &\le d(f_J(x),f_J(x_1)) + d(g(x_1),g(y_1)) + d(f_{J'}(y_1),f_{J'}(y)) \\ &\le \delta_{f_J}(c\varepsilon) + \delta_g((2c+1)\varepsilon) + \delta_{f_{J'}}(c\varepsilon). \end{aligned}$$

We deduce that

$$\sup_{\{x\in\Delta_J, y\in\Delta_{J'}\,:\,d(x,y)\le\varepsilon\}} d(f(x),f(y)) \le \delta_f(\varepsilon) := \delta_f^0(\varepsilon) + \frac{\zeta}{\rho}\varepsilon$$

where $\delta_f^0(\varepsilon) := \max_{J,J'\in\mathcal{F}_n} \left(\delta_{f_J}(c\varepsilon) + \delta_g(c\varepsilon) + \delta_{f_{J'}}(c\varepsilon)\right)$, $\rho := \inf_{\{J,J':\Delta_J\cap\Delta_{J'}=\emptyset\}} d(\Delta_J,\Delta_{J'})$ and $\zeta := \sup_{x,y\in\Delta_n^{k+1}} d(f(x),f(y))$.

We confirm then that δ_f is a modulus of continuity of f which is subadditive since δ_f^0 is so. □

Theorem 2.2. *Let M be a topological space and $R : \mathcal{F}_n \to 2^M$ be a set-valued mapping. Suppose (M,R) to be a $(n,\aleph_0)$-space, then there exists a continuous mapping $f : \Delta_n \to M$ such that for each $J \in \mathcal{F}_n$ one has $f(\Delta_J) \subset R(J)$ and which modulus of continuity is subadditive.*

Proof. For each $i \in \langle n\rangle$ we choose some $x_i \in R(\{i\})$ and define $f^0 : \Delta_n^0 \to M$ by $f^0(e_i) = x_i$. In this way we obtain a uniformly continuous f^0 with a subadditive modulus of continuity δ_{f^0}.

Suppose, by induction, we have constructed a continuous mapping $f^k : \Delta_n^k \to M$ on the k-skeleton of Δ_n which modulus of continuity δ_{f^k} is subadditive, and $f^k(\Delta_J) \subset R(J)$ if $card(J) \leq k+1$, (i.e. $\dim(\Delta_J) \leq k$).

Consider Δ_J a $k+1$-dimensional face, and for each $i \in J$ set $J_i = J \setminus \{i\}$. Then $\partial\Delta_J = \underset{i\in J}{\cup}\Delta_{J_i}$ is a subset of a k-skeleton of Δ_n; thus

$$f^k(\partial\Delta_J) \subset \underset{i\in J}{\cup} f^k(\Delta_{J_i}) \subset R(J)$$

Recall that $R(J)$ is a complete $\aleph_0$-hyperconvex space, then by Theorem 1.5 f^k can be extended to a continuous mapping $f_J^{k+1} : \Delta_J \to R(J)$ with a subadditive modulus of continuity $\delta_{f_J^{k+1}}$.

Let us define f^{k+1} on Δ_n^{k+1} by $f^{k+1}/\Delta_J := f_J^{k+1}$ for each J, and claim that f^{k+1} is well defined. Indeed, let Δ_J and $\Delta_{J'}$ be two different faces of Δ_n which dimension is $k+1$ and $\Delta_J \cap \Delta_{J'} \neq \emptyset$; then $f_J^{k+1} = f_{J'}^{k+1} = f^k$ on $\Delta_J \cap \Delta_{J'}$ since $\Delta_J \cap \Delta_{J'} \in \Delta_n^k$. Thus f^{k+1} is well defined and continuous, and from Lemma 2.1 its modulus of continuity is subadditive. □

3. Generalized KKM finite intersection property

Proposition 3.1. *Let (M,R) be a $(n,\aleph_0)$-space and $M_0,\ldots,M_n$ be a family of closed (or open) subsets of M, and suppose that $R(J) \subset \underset{i\in J}{\cup} M_i$ for each $J \in \mathcal{F}_n$. Then $\underset{i\in\langle n\rangle}{\cap} M_i \neq \emptyset$.*

Proof. Using Theorem 2.2, there exists a continuous mapping $f : \Delta_n \to M$ with a subadditive modulus of continuity and such that for each $J \in \mathcal{F}_n$ one has $f(\Delta_J) \subset R(J)$. Thus the family $\{f^{-1}(M_i) : i \in \langle n\rangle\}$ satisfies conditions of KKM-Lemma or its dual respectively provided that M_i are supposed closed or open respectively, i.e. $\Delta_J \subset \underset{i\in J}{\cup} f^{-1}(M_i)$. Consequently $\underset{i\in\langle n\rangle}{\cap} f^{-1}(M_i) \neq \emptyset$ and hence $\underset{i\in\langle n\rangle}{\cap} M_i \neq \emptyset$. □

Corollary 3.2. *Let M be a complete $\aleph_0$-hyperconvex space, $\{M_i : i \in \langle n\rangle\}$ a recovering family of closed (or open) subsets of M, and $\{F_i : i \in \langle n\rangle\}$ a family of subsets of M. Suppose that*

(i) *for all $i \in \langle n\rangle$, $F_i \cap M_i = \emptyset$;*
(ii) *for all $J \subset \langle n\rangle$ with $card(J) \leq n$ one has $\underset{i\in J}{\cap} F_i$ is a nonempty complete $\aleph_0$-hyperconvex space.*

Then $\underset{i\in\langle n\rangle}{\cap} M_i \neq \emptyset$.

Proof. Introducing the mapping R defined by $R(J) := \underset{i\notin J}{\cap} F_i$ for each J strictly included in $\langle n\rangle$ and $R(\langle n\rangle) = M$, then Proposition 3.1 implies the desired result. □

Remark that if we suppose $F_i = M_i$ for all $i \in \langle n\rangle$, the disappointed condition (ii) is automatically satisfied since M_i are supposed to be closed.

Proposition 3.3. *Let M be a complete $\aleph_0$-hyperconvex space, $\{M_i : i \in \langle n\rangle\}$ a recovering family of closed subsets of M, and $\{F_i : i \in \langle n\rangle\}$ a family of subsets of M. Suppose that*

(i) *for all* $i \in \langle n\rangle$, $F_i \subset M_i$;
(ii) *for all* $J \subset \langle n\rangle$ *with* $card(J) \leq n$ *one has* $\underset{i\in J}{\cap} F_i$ *is a nonempty* $\aleph_0$*-hyperconvex space.*

Then $\underset{i\in\langle n\rangle}{\cap} M_i \neq \emptyset$.

For the proof of this proposition we need the following lemma for which we present a proof since we found it anywhere in the consulted bibliography.

Lemma 3.4. *Let* $g : \Delta_n \to \Delta_n$ *be a continuous mapping such that* $g(\Delta_J) \subset \Delta_J$ *for every* $J \in \mathcal{F}_n$. *Then* g *is surjective.*

Proof. Let us note that $g(\Delta_n) \subset \Delta_n$ is true. To prove the converse inclusion fix $y \in \Delta_n$ and consider for all $i \in \langle n\rangle$ the closed subset

$$E_i(y) := \{x \in \Delta_n \ : \ \langle y - g(x), e_i - g(x)\rangle \leq 0\}.$$

Then the family $\{E_i(y) : i \in \langle n\rangle\}$ satisfies conditions of KKM-Lemma, i.e., for every finite subset $J \subset \langle n\rangle$ there holds $\Delta_J \subset \underset{i\in J}{\cap} E_i(y)$. Indeed, suppose to the contrary that $z \in \Delta_J$, but $z \notin E_i(y)$ for all $i \in J$. Then for some $\delta > 0$ one has

$$\langle y - g(z), e_i - g(z)\rangle \geq \delta > 0 \text{ for all } i \in J.$$

Using $z \in \Delta_J$ and the assumption of lemma we conclude that $g(z) \in \Delta_J$, that is $g(z) = \sum_{i\in J} g_i e_i$, $g_i \geq 0$ and $\sum_{i\in J} g_i = 1$. Hence

$$0 = \langle y - g(z), \sum_{i\in J} g_i e_i - g(z)\rangle = \sum_{i\in J} g_i \langle y - g(z), e_i - g(z)\rangle \geq \sum_{i\in J} g_i \delta = \delta$$

a contradiction with $\delta > 0$. It follows from the KKM Lemma that there exists $x \in \Delta_n$ such that $x \in \underset{i\in\langle n\rangle}{\cap} E_i(y)$, that is

$$\langle y - g(x), e_i - g(x)\rangle \leq 0 \text{ for all } i \in \langle n\rangle.$$

Now $y \in \Delta_n$ then $y = \sum\limits_{i\in\langle n\rangle} y_i e_i$ for some $y_i \geq 0$ such that $\sum\limits_{i\in\langle n\rangle} y_i = 1$, and hence

$$\|y - g(x)\|^2 = \langle y - g(x), y - g(x)\rangle = \sum_{i\in\langle n\rangle} y_i \langle y - g(x), e_i - g(x)\rangle \leq 0.$$

It follows then that there exists some $x \in \Delta_n$ such that $g(x) = y$, and since y was arbitrary in Δ_n the claimed result follows. □

Proof of Proposition. Consider the mappings R defined by $R(J) := \underset{i\notin J}{\cap} F_i$ if J is different from $\langle n\rangle$ and $R(\langle n\rangle) = M$, then using assumption (ii) (M, R) is a $(n, \aleph_0)$-space. By Theorem 2.2 there is a continuous function $f : \Delta_n \to M$ for which $f(\Delta_J) \subset R(J)$ for each $J \in \mathcal{F}_n$.
Suppose that $\underset{i\in\langle n\rangle}{\cap} M_i = \emptyset$, and consider the continuous mapping g defined on Δ_n by

$$g(x) := \frac{1}{\sum\limits_{i\in\langle n\rangle} d(f(x), M_i)} \sum_{i\in\langle n\rangle} d(f(x), M_i) e_i.$$

Fix $x \in \Delta_J$, one has $f(x) \in \bigcap_{i \notin J} F_i$ and then $f(x) \in \bigcap_{i \notin J} M_i$. We obtain $d(f(x), M_i) = 0$ for each $i \notin J$, and thus $g(x) \in \Delta_J$. We conclude that $g(\Delta_J) \subset \Delta_J$. Using Lemma 3.4, it follows that g is surjective.

Let $x_0 \in \Delta_n$ for which $g(x_0) \notin \partial\Delta_n$, then $f(x_0) \notin M_i$ for each $i \in \langle n \rangle$. Consequently, $f(x_0)$ is outside the set $\bigcup_{i \in \langle n \rangle} M_i$, which contradicts $\{M_i : i \in \langle n \rangle\}$ is a recovering family of M. □

Remark 3.5. Suppose $M = \Delta_n$ and $F_i = co(\{e_j : j \neq i\})$ for $i \in \langle n \rangle$, then Proposition 3.3 is reduced to Alexandroff and Pasynkoff's theorem [1]. Also by setting $M = \Delta_n$ and $F_i = M \setminus M_i$ which are closed subsets, we obtain the Klee and Berge's intersection theorem, see [4, 15].

Remark 3.6. Combining the proof of Lemma 3.4 and Proposition 3.3 we can see that Lemma 3.4 is equivalent to KKM lemma.

Proposition 3.7. *Let Z be a topologiacal space and $\{T_i : i \in \langle n \rangle\}$ a family of nonempty subsets of Z. Suppose that for each $J \subset \langle n \rangle$ the space $\bigcup_{i \in J} T_i$ is complete and $\aleph_0$-hyperconvex, then $\bigcap_{i \in \langle n \rangle} T_i \neq \emptyset$.*

Proof. Let us consider $R(J) := \bigcup_{i \in J} T_i$ for each $J \in \mathcal{F}_n$, and using Theorem 2.2 there exists a continuous mapping $f : \Delta_n \to Z$ satisfying $f(\Delta_J) \subset R(J)$ for every $J \in \mathcal{F}_n$.

Let us denote by d the metric defined on $\bigcup_{i \in \langle n \rangle} T_i$. Suppose that $\bigcap_{i \in \langle n \rangle} T_i = \emptyset$, and define the continuous mapping $F : \Delta_n \to \Delta_n$ by

$$F(x) := \frac{1}{\sum_{i \in \langle n \rangle} d(f(x), T_i)} \sum_{i \in \langle n \rangle} d(f(x), T_i) e_i.$$

By Brouwer's Fixed Point Theorem there exists $x_0 \in \Delta_n$ such that $F(x_0) = x_0$. Let $J_0 := \{i \in \langle n \rangle : d(f(x_0), T_i) \neq 0\}$, then $x_0 \in \Delta_{J_0}$ and $f(x_0) \notin T_i$ for each $i \in J_0$. Hence $f(x_0) \in f(\Delta_{J_0}) \subset \Delta_{J_0} \subset \bigcup_{i \in J_0} T_i$, thereby contradicting $f(x_0) \notin T_i$ for $i \in J_0$. □

Theorem 3.8. *Consider for $i \in \langle n \rangle$ $(X_i; R_i)$ a $\aleph_0$-space and $B_i \subset X := \prod_{j \in \langle n \rangle} X_j$. For $x, y \in X$, let us denote by x_i the i^{th} component of x, $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \prod_{j \neq i} X_j$, and $[y_i, x^i] = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \in X$. Suppose that for $i \in \langle n \rangle$ and each $x \in X$*

(1) *X_i is compact ;*
(2) *$B^i(x) := \{y \in X : [y_i, x^i] \in B_i\}$ is nonempty and open ;*
(3) *$B_i(x) := \{y \in X : [x_i, y^i] \in B_i\}$ is nonempty ;*
(4) *if $A \in \mathcal{F}(B_i(x))$ then $R(A) \subset B_i(x)$.*

Then $\bigcap_{i \in \langle n \rangle} B_i \neq \emptyset$.

Proof. Consider $G : X \to 2^X$ defined by $G(x) := X \setminus \bigcap_{i\in\langle n\rangle} B^i(x)$. For each $y \in X$ there exists $x \in X$ such that $[y_i, x^i] \in B_i$, which means $y \in \bigcap_{i\in\langle n\rangle} B^i(x)$. Thus

$$\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \left(X \setminus \bigcap_{i\in\langle n\rangle} B^i(x) \right) = X \setminus \bigcup_{x\in X} \bigcap_{i\in\langle n\rangle} B^i(x) = \emptyset.$$

One can easily justify by contradiction that G is $\aleph_0$-KKM mapping; therefore, there exists a finite subset A of X and $z \in R(A)$ such that $z \notin \bigcup_{x\in A} G(x)$. We deduce that for each $x \in A$ and each $i \in \langle n\rangle$, $x \in B_i(z)$, and using the assumption 4 we have $R(A) \subset B_i(z)$ which implies $z \in B_i(z)$. Thus $z \in B_i$ for each $i \in \langle n\rangle$, and then the conclusion is true. □

Remark 3.9. Theorem 3.8 improves Theorem 5.2 in [14] by considering a $\aleph_0$-hyperconvex space.

To prove the main fixed point theorem of this section we need the following continuous selection theorem which includes Tarafdar's similar result, see [22].

Theorem 3.10. *Let M be a compact topological space, H a topological space, (H, H, R) a $\aleph_0$-space, and $F : M \to 2^H$. Suppose that:*

(i) *for each $x \in M$ and $A \in \mathcal{F}(F(x))$, one has $R(A) \subset F(x)$;*
(ii) *$M = \bigcup_{y\in H} int(F^{-1}(y))$, where int is the topological interior in M.*

Then there exists a continuous selection f of F which can be expressed as the composite of two continuous mappings $g : \triangle_n \to H$ and $h : M \to \triangle_n$.

If moreover $M = H$, F admits a fixed point.

Proof. Using condition (ii) and compactness of M, there exists $y_1, \dots, y_n$ such that

$$M = \bigcup_{i\in\langle n\rangle} int(F^{-1}(y_i)).$$

Let $(\psi_i)_i$ be an associated partition of unity, and consider $h : M \to \triangle_n$ defined by $h(x) = \sum_{i\in\langle n\rangle} \psi_i(x) e_i$, and $R_n : \mathcal{F}_n \to 2^M$ defined by $R_n(J) := R(\{y_i \ : \ i \in J\})$ if $J \in \mathcal{F}_n$.

Using Theorem 2.2 one can find a continuous mapping $g : \triangle_n \to H$ for which $g(\Delta_J) \subset R_n(J)$ for each $J \in \mathcal{F}_n$.

We claim that $g(h(x)) \in F(x)$ for each $x \in M$. Indeed, let us fixe some $x \in M$ and set $J(x) := \{i \in \langle n\rangle \ : \ \psi_i(x) \neq 0\}$. Since $\psi_i(x) \neq 0$ is equivalent to $x \in int(F^{-1}(y_i))$, then $y_i \in F(x)$. Setting $A(x) = \{y_i \ : \ i \in J(x)\}$ and taking into account the assumption (i) we have $R(A(x)) \subset F(x)$, and then

$$g(h(x)) \in g(\triangle_{J(x)}) \subset R_n(J(x)) = R(A(x)) \subset F(x).$$

We confirm then that $f := g \circ h$ is a continuous selection of F.

Consider now the continuous mapping $h \circ g : \triangle_n \to \triangle_n$, then using Brouwer's Fixed Point Theorem one has the existence of $x_0 \in \triangle_n$ such that $h \circ g(x_0) = x_0$. Taking $y_0 = g(x_0) \in M$, then

$$y_0 = g(x_0) = g(h \circ g(x_0)) = g \circ h(y_0) = f(y_0) \in F(y_0),$$

and thus y_0 is a fixed point of F. □

Theorem 3.11. *Let M be a compact $\aleph_0$-hyperconvex space, and $T : M \to 2^M$ with nonempty closed valued in M. Suppose that*

(i) *for each $x \in M, A \in \mathcal{F}(F(x))$ one has $co(A) \subset F(x)$;*
(ii) $M = \underset{y\in H}{\cup} int(T^{-1}(y))$.

Then, there exists $x_0 \in M$ such that $x_0 \in T(x_0)$.

Proof. Let us first remark, by setting $R(A) = co(A)$ and using Proposition 1.12, that (M, M, R) is a $\aleph_0$-space. Hence all assumptions of Theorem 3.10 are satisfied, and then F has a fixed point. □

Remark 3.12. Note that this theorem extends Theorems 2.11.26, 2.11.32, 2.11.41, 2.11.22 in [23] and Theorem 3 in [19].

4. Generalized Ky Fan Lemma

Definition 4.1. Let (H, M, R) be a $\aleph_0$-space. A set-valued mapping $F : H \to 2^M$ is said to be $\aleph_0$-R-KKM if for each $A \in \mathcal{F}(H)$,

$$R(A) \subset \underset{x\in A}{\cup} F(x).$$

Theorem 4.2. *Let (H, M, R) be a $\aleph_0$-space, $Y = R(H)$ and $F : H \to 2^{M\cap Y}$. Suppose that for some $x_0 \in X$*

(1) *$F(x_0)$ is relatively compact in Y;*
(2) *F is $(x_0, \aleph_0)$-R-KKM;*
(3) *for each $A \in \mathcal{F}(H, x_0)$ and $x \in A$ one has $F(x) \cap R(A)$ is closed in $R(A)$;*
(4) *for each $A \in \mathcal{F}(H, x_0)$ one has $cl_M(\underset{x\in A}{\cap} F(x)) \cap R(A) = \underset{x\in A}{\cap} F(x) \cap R(A)$.*

Then $\underset{x\in X}{\cap} F(x) \neq \emptyset$.

Proof. Let us fix some $A \in \mathcal{F}(H, x_0)$ and consider the set-valued mapping T defined for each $x \in A$ by $T_A(x) := F(x) \cap R(A)$. We confirm that T_A is $\aleph_0$-R-KKM from A to $R(A)$; then using Proposition 3.1 and the assumptions 2 - 3 we conclude that $\underset{x\in A}{\cap} T_A(x) \neq \emptyset$. This implies that the set-valued mapping $V : \mathcal{F}(H, x_0) \to 2^M$ which values are $V(A) := cl_Y(\underset{x\in A}{\cap} F(x)) \cap R(A)$ is well defined. We have for each $A \in \mathcal{F}(H, x_0)$, $V(A)$ is a subset of the compact subset $cl_Y F(x_0)$ of Y. Since the family $\{V(A) : A \in \mathcal{F}(X, x_0)\}$ satisfies the finite intersection property, we conclude that $\underset{A\in\mathcal{F}(X,x_0)}{\cap} V(A) \neq \emptyset$, let $\bar{x}$ be one point.

In this way we obtain for each $x \in X$, by setting $A(x) := \{x, x_0\}$ which is an element of $\mathcal{F}(H, x_0)$, $\bar{x} \in V(A(x)) = cl_Y(\underset{x\in A(x)}{\cap} F(x)) \cap R(A) = \underset{x\in A(x)}{\cap} F(x) \cap R(A)$; the last equality follows from the assumption 4. Thus $\bar{x} \in F(x)$, which completes the proof. □

Remark 4.3. Note that as a special case we find Theorem 4 in [12].

Theorem 4.4. *Let X be a compact topological space, Y a complete metric space and $f : X \times Y \to \mathbb{R}$. Suppose that*

(1) *for each $x \in X$ the mapping $y \to f(x, y)$ is lower semicontinuous;*

(2) *for each* $A \in \mathcal{F}(X)$, $\{y \in Y : \inf_{x \in A} f(x,y) \leq 0\}$ *is* $\aleph_0$*-hyperconvex and closed.*

Then there exists $y_0 \in X$ *such that* $f(x, y_0) \leq 0$ *for every* $x \in X$.

Proof. It suffices to consider $R(A) := \{y \in Y : \inf_{x \in A} f(x,y) \leq 0\}$ and $F(x) := \{y \in Y : f(x,y) \leq 0\}$. Then F is a $\aleph_0$-R-KKM, and all assumptions of Theorem 4.2 are satisfied. □

Remark 4.5. Note that Condition 2 is equivalent to

2'. If, for each $A \in \mathcal{F}(X)$, $(y_\alpha) \subset Y$, $(r_\alpha) \subset \mathbb{R}^+$ such that for all α, β $d(y_\alpha, y_\beta) \leq r_\alpha + r_\beta$ there exists some $x \in A$ such that for each α $f(x, y_\alpha) \leq 0$, then we have $d(z, y_\alpha) \leq r_\alpha$ for all α implies the existence of $x \in A$ such that $f(x,z) \leq 0$.

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