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STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR FAMILIES OF RELATIVELY NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In 2008, Takahashi, Takeuchi and Kubota [10] proved a strong convergence theorem by the hybrid method for a family of nonexpansive mappings which generalized Nakajo and Takahashi's theorems [6]. Furthermore, they obtained another strong convergence theorem for the family of nonexpansive mappings by a hybrid method which is different from Nakajo and Takahashi. In this paper, we extend Takahashi, Takeuchi and Kubota's results for a single relatively nonexpansive mapping or a family of relatively nonexpansive mappings in a Hilbert space. Using these results, we obtain some new strong convergence theorems in a Hilbert space.

1. Introduction and preliminaries

Let H be a real Hilbert space. We denote by $x_n \rightharpoonup x$ the weak convergence and by $x_n \to x$ the strong convergence.

Let C be a nonempty closed convex subset of a Hilbert space H. Then, for each $x \in H$, there exists a unique nearest point in C denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such P_C is called the metric projection of H onto C. We know that for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $\langle x - z, z - u\rangle \geq 0$ for all $u \in C$. We also know that H has the Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$; see [9] for more details.

Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - Tu\| \leq \|x - y\|$ for all $x \in C$ and $u \in F(T)$. We know from Itoh and Takahashi [2] that if $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex. A point p in C is said to be an asymptotic fixed point of T [7] if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive [4] if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and $\|u - Tx\| \leq \|u - x\|$ for all $x \in C$ and $u \in F(T)$

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Following Nakajo, Shimoji and Takahashi [5], we can give the following definition: Let $\{T_n\}$ and $\mathcal{T}$ be two families of mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, $\{T_n\}$ is said to satisfy the NST-condition (I) with $\mathcal{T}$ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$$

implies that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ for all $T \in \mathcal{T}$. In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the NST-condition with T. Takahashi, Takeuchi and Kubota [10] proved the following strong convergence theorem by using a hybrid method called the shrinking projection method.

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Takahashi, Takeuchi and Kubota[10] also proved the following theorem.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$. For $u_1 = P_C x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|u_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - u_n, u_n - z\rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Using Theorem 1.1 Takahashi, Takeuchi and Kubota [10] also obtained the following new results:

Theorem 1.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)Tu_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)Tu_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x_0$.

Recently, Kohsaka and Takahashi [3] introduced a new nonlinear operator which is deduced from a family of nonexpansive mapping in a Hilbert space. A mapping S from C into itself is said to be nonspreading [3] if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. Iemoto and Takahashi [1] proved that a mapping $S : C \to C$ is nonspreading if and only if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle$$

for all $x, y \in C$. We know from [1] and [3] that if T is nonexpansive or nonspreading with $F(T) \neq \emptyset$, then T is relatively nonexpansive.

In this paper, we consider the NST-condition for a family of relatively nonexpansive mappings in a Hilbert space and then extend Takahashi, Takeuchi and Kubota's results [10] to relatively nonexpansive mappings in a Hilbert space. Using these results, we obtain some new strong convergence theorems.

2. Strong convergence by hybrid methods

In this section, we prove strong convergence theorems by hybrid methods for families of relatively nonexpansive mappings in a Hilbert space.

Lemma 2.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{T}$ be a family of relatively nonexpansive mappings of C into itself such that $F(\mathcal{T})$ is nonempty and let $x_0 \in H$. Let $\{u_n\}$ be a sequence of C satisfying the following conditions:*

(a) *$\|u_n - x_0\| \leq \|u - x_0\|$ for all $u \in F(\mathcal{T})$ and $n \in \mathbb{N}$;*
(b) *for any $T \in \mathcal{T}$, $\|u_n - Tu_n\| \to 0$.*

Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$, where $P_{F(\mathcal{T})}$ is the metric projection of H onto $F(\mathcal{T})$.

Proof. Put $z_0 = P_{F(\mathcal{T})}x_0$. Since $\|u_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \in \mathbb{N}$, $\{u_n\}$ is bounded. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, we have that C is weakly closed and hence $w \in C$. From (b) and the relative nonexpansiveness of T, we have that $w = Tw$ for all $T \in \mathcal{T}$. We also have

$$\|x_0 - z_0\| \leq \|x_0 - w\| \leq \liminf_{i\to\infty} \|x_0 - u_{n_i}\| \leq \limsup_{i\to\infty} \|x_0 - u_{n_i}\| \leq \|z_0 - x_0\|$$

and hence $\|x_0 - z_0\| = \|x_0 - w\|$. From $z_0 = P_{F(\mathcal{T})}x_0$, we have $z_0 = w$. This implies that $\{u_n\}$ converges weakly to z_0. So, we have

$$\|x_0 - z_0\| \leq \liminf_{i\to\infty} \|x_0 - u_n\| \leq \limsup_{i\to\infty} \|x_0 - u_n\| \leq \|z_0 - x_0\|$$

and hence $\lim_{n\to\infty} \|x_0 - u_n\| = \|z_0 - x_0\|$. From $u_n \rightharpoonup z_0$, we also have $x_0 - u_n \rightharpoonup x_0 - z_0$. Since H satisfies the Kadec-Klee property, it follows that $x_0 - u_n \to x_0 - z_0$. So, we have

$$\|u_n - z_0\| = \|u_n - x_0 - (z_0 - x_0)\| \to 0$$

and hence $u_n \to z_0$. □

Using Lemma 2.1, we obtain the following theorem.

Theorem 2.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ be a family of mappings of C into C itself and let $\mathcal{T}$ be a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$. Let $\{u_n\}$ be a sequence of C satisfying the following conditions:*

(a) *$0 \leq \langle x_0 - u_n, u_n - u\rangle$ for all $u \in F(\mathcal{T})$ and $n \in \mathbb{N}$;*
(b) *$0 \leq \langle x_0 - u_n, u_n - u_{n+1}\rangle$ for all $n \in \mathbb{N}$;*
(c) *there exists $M \geq 0$ such that $\|T_n u_n - u_n\| \leq M\|u_n - u_{n+1}\|$ for all $n \in \mathbb{N}$.*

Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Proof. For $u \in F(\mathcal{T})$ and $n \in \mathbb{N}$, from (a) we have

$$\begin{aligned} 0 &\leq \langle x_0 - u_n, u_n - u\rangle \\ &= \langle x_0 - u_n, u_n - x_0 + x_0 - u\rangle \\ &\leq -\|x_0 - u_n\|^2 + \|x_0 - u_n\|\|x_0 - u\| \end{aligned}$$

and hence

$$\|x_0 - u_n\| \leq \|x_0 - u\|. \tag{2.1}$$

We have from (b) that for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - u_n, u_n - u_{n+1}\rangle \\ &= \langle x_0 - u_n, u_n - x_0 + x_0 - u_{n+1}\rangle \\ &\leq -\|x_0 - u_n\|^2 + \|x_0 - u_n\|\|x_0 - u_{n+1}\| \end{aligned}$$

and hence

$$\|x_0 - u_n\| \leq \|x_0 - u_{n+1}\|.$$

Since $\{\|x_0 - u_n\|\}$ is bounded from (2.1), $\lim_{n\to\infty} \|u_n - x_0\|$ exists. Next, we show that $\|u_n - u_{n+1}\| \to 0$. In fact, from (b) we have

$$\begin{aligned} \|u_n - u_{n+1}\|^2 &= \|u_n - x_0 + x_0 - u_{n+1}\|^2 \\ &= \|u_n - x_0\|^2 + \|x_0 - u_{n+1}\|^2 + 2\langle u_n - x_0, x_0 - u_{n+1}\rangle \\ &= \|u_n - x_0\|^2 + \|x_0 - u_{n+1}\|^2 + 2\langle u_n - x_0, x_0 - u_n + u_n - u_{n+1}\rangle \\ &= -\|u_n - x_0\|^2 + \|x_0 - u_{n+1}\|^2 - 2\langle x_0 - u_n, u_n - u_{n+1}\rangle \\ &\leq \|x_0 - u_{n+1}\|^2 - \|x_0 - u_n\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - x_0\|$ exists, we have that $\|u_n - u_{n+1}\| \to 0$. From (c), we have that $\|T_n u_n - u_n\| \to 0$. Since $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$, we have that

$$\|T u_n - u_n\| \to 0 \tag{2.2}$$

for all $T \in \mathcal{T}$. From (2.1), (2.2) and Lemma 2.1, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})} x_0$. □

Using Theorem 2.2, we obtain a generalization of Takahashi, Takeuchi and Kubota's result [10].

Theorem 2.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself and let $\mathcal{T}$ be a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})} x_0$.

Proof. We first show by induction that $F(\mathcal{T}) \subset C_n$ for all $n \in \mathbb{N}$. $F(\mathcal{T}) \subset C_1$ is obvious. Suppose that $F(\mathcal{T}) \subset C_k$ for some $k \in \mathbb{N}$. Then, we have, for any $u \in F(\mathcal{T}) \subset C_k$,

$$\begin{aligned} \|y_k - u\| &= \|\alpha_k u_k + (1 - \alpha_k) T_k u_k - u\| \\ &\leq \alpha_k \|u_k - u\| + (1 - \alpha_k)\|T_k u_k - u\| \\ &\leq \alpha_k \|u_k - u\| + (1 - \alpha_k)\|u_k - u\| \\ &= \|u_k - u\|. \end{aligned}$$

and hence $u \in C_{k+1}$. This implies that

$$F(\mathcal{T}) \subset C_n \text{ for all } n \in \mathbb{N}.$$

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, from [6] we know that $\|y_k - z\| \leq \|u_k - z\|$ is equivalent to

$$\|y_k - u_k\|^2 + 2\langle y_k - u_k, u_k - z\rangle \leq 0.$$

So, C_{k+1} is closed and convex. Then for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\{u_n\}$ is well-defined. From $u_n = P_{C_n}x_0$, we have

$$\langle x_0 - u_n, u_n - y\rangle \geq 0 \quad \text{for all } y \in C_n.$$

Using $F(\mathcal{T}) \subset C_n$, we also have

$$\langle x_0 - u_n, u_n - u\rangle \geq 0 \quad \text{for all } u \in F(\mathcal{T}) \text{ and } n \in \mathbb{N}. \tag{2.3}$$

For $u \in F(\mathcal{T})$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - u_n, u_n - u\rangle \\ &= \langle x_0 - u_n, u_n - x_0 + x_0 - u\rangle \\ &= -\|x_0 - u_n\|^2 + \|x_0 - u_n\|\|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - u_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(\mathcal{T}) \text{ and } n \in \mathbb{N}.$$

From $u_n = P_{C_n}x_0$ and $u_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - u_n, u_n - u_{n+1}\rangle \geq 0. \tag{2.4}$$

From (2.4), we have, for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - u_n, u_n - u_{n+1}\rangle \\ &= \langle x_0 - u_n, u_n - x_0 + x_0 - u_{n+1}\rangle \\ &= -\|x_0 - u_n\|^2 + \|x_0 - u_n\|\|x_0 - u_{n+1}\| \end{aligned}$$

and hence

$$\|x_0 - u_n\| \leq \|x_0 - u_{n+1}\|.$$

Since $\{\|u_n - x_0\|\}$ is bounded, $\lim_{n\to\infty}\|u_n - x_0\|$ exists. Next, we show that $\|u_n - u_{n+1}\| \to 0$. In fact, from (2.4) we have

$$\begin{aligned} \|u_n - u_{n+1}\|^2 &= \|u_n - x_0 + x_0 - u_{n+1}\|^2 \\ &= \|u_n - x_0\|^2 + 2\langle u_n - x_0, x_0 - u_{n+1}\rangle + \|x_0 - u_{n+1}\|^2 \\ &= \|u_n - x_0\|^2 + 2\langle u_n - x_0, x_0 - u_n + u_n - u_{n+1}\rangle + \|x_0 - u_{n+1}\|^2 \\ &= -\|u_n - x_0\|^2 + 2\langle u_n - x_0, u_n - u_{n+1}\rangle + \|x_0 - u_{n+1}\|^2 \\ &\leq -\|u_n - x_0\|^2 + \|x_0 - u_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - x_0\|$ exists, we have that $\|u_n - u_{n+1}\| \to 0$. On the other hand, $u_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - u_{n+1}\| \leq \|u_n - u_{n+1}\|. \tag{2.5}$$

Further, we have

$$\begin{aligned} \|y_n - u_n\| &= \|\alpha_n u_n + (1-\alpha_n)T_n u_n - u_n\| \\ &= (1-\alpha_n)\|T_n u_n - u_n\|. \end{aligned}$$

From (2.5), we have

$$\begin{aligned}\|T_n u_n - u_n\| &= \frac{1}{1-\alpha_n}\|y_n - u_n\| \\ &\leq \frac{1}{1-a}\|y_n - u_n\| \\ &= \frac{1}{1-a}\|y_n - u_{n+1} + u_{n+1} - u_n\| \\ &\leq \frac{2}{1-a}\|u_n - u_{n+1}\|.\end{aligned}$$

Hence, we have

$$\|T_n u_n - u_n\| \leq \frac{2}{1-a}\|u_n - u_{n+1}\|. \tag{2.6}$$

From (2.3), (2.4), (2.6) and Theorem 2.2, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$. □

Using Theorem 2.2, we obtain a generalization of Nakajo and Takahashi's result [6]

Theorem 2.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself and let $\mathcal{T}$ be a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with $\mathcal{T}$. For $u_1 = P_C x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|u_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - u_n, u_n - z\rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Proof. As in the proof of [6], we have that C_n and Q_n are closed and convex for all $n \in \mathbb{N}$. Let $u \in F(\mathcal{T})$. Then, we have, for $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n u_n + (1-\alpha_n)T_n u_n - u\| \\ &\leq \alpha_n\|u_n - u\| + (1-\alpha_n)\|T_n u_n - u\| \\ &\leq \alpha_n\|u_n - u\| + (1-\alpha_n)\|u_n - u\| \\ &= \|u_n - u\|\end{aligned}$$

and hence $u \in C_n$. So, we have that $F(\mathcal{T}) \subset C_n$ for all $n \in \mathbb{N}$. By induction, we show that $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $u_1 = P_C x_0$, we have

$$\langle x_0 - u_1, u_1 - y\rangle \geq 0 \ \text{ for all } y \in C$$

and hence $Q_1 = C$. So, we have $F(\mathcal{T}) \subset Q_1$. Then, $F(\mathcal{T}) \subset C_1 \cap Q_1$. Suppose that $F(\mathcal{T}) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $u_{k+1} = P_{C_k \cap Q_k} x_0$, we have

$$\langle x_0 - u_{k+1}, u_{k+1} - y\rangle \geq 0 \ \text{ for all } y \in C_k \cap Q_k.$$

Since $F(\mathcal{T}) \subset C_k \cap Q_k$, we have

$$\langle x_0 - u_{k+1}, u_{k+1} - u\rangle \geq 0 \quad \text{for all } u \in F(\mathcal{T}).$$

So, we have $F(\mathcal{T}) \subset Q_{k+1}$ and hence $F(\mathcal{T}) \subset C_{k+1} \cap Q_{k+1}$. This implies that $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{u_n\}$ is well-defined. Since $F(\mathcal{T}) \subset Q_n$ for all $n \in \mathbb{N}$, we have

$$\langle x_0 - u_n, u_n - u\rangle \geq 0 \quad \text{for all } u \in F(\mathcal{T}) \text{ and } n \in \mathbb{N}. \tag{2.7}$$

From $u_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$ we also have

$$\langle x_0 - u_n, u_n - u_{n+1}\rangle \geq 0. \tag{2.8}$$

From $u_{n+1} = P_{C_n \cap Q_n} x_0 \in C_n$ we also have

$$\|y_n - u_{n+1}\| \leq \|u_n - u_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Further, we have

$$\|y_n - u_n\| = \|\alpha_n u_n + (1-\alpha_n)T_n u_n - u_n\| = (1-\alpha_n)\|T_n u_n - u_n\|.$$

As in the proof of Theorem 2.3, we also have

$$\|T_n u_n - u_n\| = \frac{1}{1-\alpha_n}\|y_n - u_n\| \leq \frac{2}{1-\alpha}\|u_n - u_{n+1}\|. \tag{2.10}$$

From (2.7), (2.8), (2.10) and Theorem 2.2, $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$. □

3. Deduced results

In this section, using Theorems 2.3 and 2.4, we obtain some strong convergence theorems in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a relatively nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Proof. Define $T_n = T$ for all $n \in \mathbb{N}$. It is obvious that a family $\{T_n\}$ of quasi-nonexpansive mappings of C into itself satisfies the NST-condition (I) with T. So, we obtain the desired result by using Theorem 2.3 □

Let S be a nonspreading mapping with $F(S) \neq \emptyset$ and let T be a nonexpansive mapping with $F(T) \neq \emptyset$. Then S and T are relatively nonexpansive mappings. Using Theorem 3.1, we obtain the following two corollaries.

Corollary 3.2 ([10]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself such that $F(S) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Su_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)}x_0$.

Motivated by [8], we prove the following lemma.

Lemma 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a relatively nonexpansive mapping of C into itself. Let $\{\alpha_n\}$ be a sequence of real numbers with $0 \leq \alpha_n \leq b < 1$. For $n \in \mathbb{N}$, define a mapping T_n of C into itself by*

$$T_n x = \alpha_n x + (1-\alpha_n)Tx, \ \forall x \in C.$$

Then, $\{T_n\}$ is a family of quasi-nonexpansive mappings of C into itself and it satisfies the NST-condition (I) with T.

Proof. Let $u \in F(T) \neq \emptyset$. Then $u \in F(T_n)$ for all $n \in \mathbb{N}$.
We have

$$\|T_n x - u\| = \|(1-\alpha_n)x + \alpha_n Tx - u\| \leq (1-\alpha_n)\|x - u\| + \alpha_n\|Tx - u\| \leq \|x - u\|$$

for all $x \in C$ and $u \in F(T)$. Then, $\{T_n\}$ is a family of quasi-nonexpansive mappings of C into itself. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|T_n z_n - z_n\| = 0$. Then, we have

$$\|T_n z_n - z_n\| = \|\alpha_n z_n + (1-\alpha_n)Tz_n - z_n\| = (1-\alpha_n)\|Tz_n - z_n\| \geq (1-b)\|Tz_n - z_n\|.$$

So we get that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. Hence $\{T_n\}$ satisfies the NST-condition (I) with T. □

Using Lemma 3.4, we obtain the following theorem.

Theorem 3.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a relatively nonexpansive mapping of C into itself and let*

$x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)Tu_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ and $0 \le \beta_n \le b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Proof. Define $T_n = T$ for all $n \in \mathbb{N}$. By Lemma 3.4, we know that $\{T_n\}$ satisfies the NST-condition (I) with T. So, we obtain the desired result by using Theorem 2.3 □

Using Theorem 3.5, we obtain the following two corollaries.

Corollary 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)Tu_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ and $0 \le \beta_n \le b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself such that $F(S) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)Su_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ and $0 \le \beta_n \le b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)}x_0$.

Motivated by [5], we prove the following lemma.

Lemma 3.8. *Let C be a nonempty closed convex subset of a Hilbert space and let S and T be relatively nonexpansive mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers with $0 < a \le \alpha_n \le b < 1$. For $n \in \mathbb{N}$, define a mapping T_n of C into itself by*

$$T_n x = \alpha_n Sx + (1-\alpha_n)Tx \quad \text{for all } x \in C.$$

Then, $\{T_n\}$ is a family of quasi-nonexpansive mappings of C into itself and it satisfies the NST-conditon (I) with $\mathcal{T} = \{S, T\}$.

Proof. It is clearly that $F(S) \cap F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Conversely, take $n \in \mathbb{N}$ and $z \in F(S) \cap F(T)$. Then, we have that for any $v \in F(T_n)$,

$$\begin{aligned} \|v - z\|^2 &= \|T_n v - z\|^2 = \|\alpha_n Sv + (1-\alpha_n)Tv - z\|^2 \\ &= \alpha_n \|Sv - z\|^2 + (1-\alpha_n)\|Tv - z\|^2 - \alpha_n(1-\alpha_n)\|Sv - Tv\|^2 \\ &\le \alpha_n \|v - z\|^2 + (1-\alpha_n)\|v - z\|^2 - \alpha_n(1-\alpha_n)\|Sv - Tv\|^2 \\ &= \|v - z\|^2 - \alpha_n(1-\alpha_n)\|Sv - Tv\|^2 \end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n)\|Sv - Tv\|^2 \le 0.$$

Since $\alpha_n(1-\alpha_n) \neq 0$, we have $Sv = Tv$. Since

$$\|v - Sv\| \le \|v - T_n v\| + \|T_n v - Sv\| = \|v - v\| + \|Sv - Sv\| = 0,$$

we have $v = Sv$.
Similarly, we have $v = Tv$. Then, we have $F(T_n) \subset F(S) \cap F(T)$. This implies

$$F(S) \cap F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$ and let $z \in F(S) \cap F(T)$. Then, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|x_n - z\|^2 &\le \{\|x_n - T_n x_n\| + \|T_n x_n - z\|\}^2 \le \|x_n - Tx_n\| \cdot M + \|T_n x_n - z\|^2 \\ &= \|x_n - T_n x_n\| \cdot M + \alpha_n \|Sx_n - z\|^2 \\ &\quad + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ &\le \|x_n - T_n x_n\| \cdot M + \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - Tx_n\|^2 \end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n)\|Sx_n - Tx_n\|^2 \le \|x_n - T_n x_n\| \cdot M,$$

where $M = \sup_{n\in\mathbb{N}}\{\|x_n - T_n x_n\| + 2\|x_n - z\|\}$. So, we get $\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0$. Since

$$\|x_n - Sx_n\| \le \|x_n - T_n x_n\| + \|T_n x_n - Sx_n\| = \|x_n - T_n x_n\| + (1-\alpha_n)\|Sx_n - Tx_n\|$$

for every $n \in \mathbb{N}$, we obtain $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ Similarly, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We have that $\{T_n\}$ satisfies the NST-condition (I) for $\{S, T\}$. We also have that

$$\begin{aligned} \|T_n x - u\| &= \|\alpha_n Sx + (1-\alpha_n)Tx - u\| \\ &\le \alpha_n \|Sx - u\| + (1-\alpha_n)\|Tx - u\| \\ &\le \alpha_n \|Sx - u\| + (1-\alpha_n)\|Tx - u\| \\ &= \|x - u\| \end{aligned}$$

for all $x \in C$ and $u \in F(T_n)$. So, $\{T_n\}$ is a family of quasi-nonexpansive mappings of C into itself. □

Using Lemma 3.8, we prove the following theorem.

Theorem 3.9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S and T be relatively nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)T u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x_0$.

Proof. Define $T_n x = \beta_n S x + (1-\beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. Then, we obtain the desired result by Theorem 2.3 and Lemma 3.8. □

Using Theorem 3.9, we have the following three corollaries.

Corollary 3.10. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)T u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x_0$.

Corollary 3.11 ([10])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)T u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x_0$.

Corollary 3.12. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S and T be nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)T u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ *and* $0 < b \leq \beta_n \leq c < 1$ *for all* $n \in \mathbb{N}$. *Then,* $\{u_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap F(T)}x_0$.

References

[1] S. Iemoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading in a Hilbert space,* Nonlinear Anal., to appear.

[2] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings,* Pacific. J. Math., **79** (1978), 493–508.

[3] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mapping related to maximal monotone operators in Banach soaces,* Arch. Math. (Basel), **91** (2008), 166–177.

[4] S.y. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in Banach spaces,* J. Approximation Theory, **134** (2009), 257-266.

[5] K. Nakajo, K Shimoji and W. Takahashi, *Strong convergence to common fixed points of nonexpansive mappings in Banach spaces,* J. Nonlinear Convex Anal. **8** (2007), 11–34.

[6] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups,* J. Math. Anal. Appl. **279** (2003), 372–379.

[7] S. Reich, *A weak convergence theorem for alternateing method with Bregman distance,* in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type A.G. Kartsatos (Ed.), Marcel Dekker, Newyork, 1996, pp. 313–318.

[8] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mapping in Banach spaces,* Nonlinear Anal. **70** (2009), 719–734.

[9] W. Takahashi, *Introduction to Nonlinear and Convex Analysis,* Yokohama Publishers, Yokohama, 2009.

[10] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **341** (2008), 276–286.

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