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U-CONVEXITY OF ψ-DIRECT SUMS OF BANACH SPACES

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ABSTRACT. We shall characterize the U-convexity of the ψ-direct sum $(X_1 \oplus \cdots \oplus X_n)_\psi$ of Banach spaces $X_1, \ldots, X_n$, where ψ is a continuous convex function on $\Delta_n (= \{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geq 0\ (\forall j),\ \sum_{j=1}^{n-1} t_j \leq 1\})$ with some appropriate conditions.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be *absolute* if

$$\|(x_1, x_2, \ldots, x_n)\| = \|(|x_1|, |x_2|, \ldots, |x_n|)\|$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{C}$, and *normalized* if $\|(1, 0, \ldots, 0)\| = \|(0, 1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1$. Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$. Recently, Saito, Kato and Takahashi [12] showed that for any absolute normalized norms on $\mathbb{C}^n$ there corresponds a continuous convex function on $\Delta_n (= \{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geq 0\ (\forall j),\ \sum_{j=1}^{n-1} t_j \leq 1\})$ with some appropriate conditions, as follows. For any $\|\cdot\| \in AN_n$ we define

$$\psi(s_1, \ldots, s_{n-1}) = ||(1 - \sum_{i=1}^{n-1} s_i, s_1, \ldots, s_{n-1})|| \quad ((s_1, \ldots, s_{n-1}) \in \Delta_n). \tag{1.1}$$

Then ψ is a continuous convex function on Δ_n, and satisfies the following conditions:

(A_0) $\quad \psi(0, 0, \ldots, 0) = \psi(1, 0, 0, \ldots, 0) = \psi(0, 1, 0, \ldots, 0) = \cdots = \psi(0, \ldots, 0, 1) = 1,$

(A_1) $\quad \psi(s_1, \ldots, s_{n-1}) \geq (s_1 + \cdots + s_{n-1})\psi\left(\frac{s_1}{s_1 + \cdots + s_{n-1}}, \ldots, \frac{s_{n-1}}{s_1 + \cdots + s_{n-1}}\right),$ if $s_1 + \cdots + s_{n-1} \neq 0,$

(A_2) $\quad \psi(s_1, \ldots, s_{n-1}) \geq (1 - s_1)\psi\left(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_{n-1}}{1 - s_1}\right),$ if $s_1 \neq 1,$

$\vdots \qquad \vdots$

(A_n) $\quad \psi(s_1, \ldots, s_{n-1}) \geq (1 - s_{n-1})\psi\left(\frac{s_1}{1 - s_{n-1}}, \ldots, \frac{s_{n-2}}{1 - s_{n-1}}, 0\right),$ if $s_{n-1} \neq 1.$

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Let Ψ_n be the set of all continuous convex functions ψ on Δ_n satisfying (A_0), (A_1), $\dots$, (A_n). Conversely, for every $\psi \in \Psi_n$, we define

$$\|(x_1, x_2, \dots, x_n)\|_\psi = \begin{cases} (|x_1| + \cdots + |x_n|)\psi\left(\frac{|x_2|}{|x_1|+\cdots+|x_n|}, \dots, \frac{|x_n|}{|x_1|+\cdots+|x_n|}\right) & \text{if } (x_1, \dots, x_n) \neq (0, \dots, 0), \\ 0 & \text{if } (x_1, \dots, x_n) = (0, \dots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_n$ and satisfies (1.1). Hence AN_n and Ψ_n are in a one-to-one correspondence under (1.1).

For any Banach spaces $X_1, X_2, \dots, X_n$, we define the ψ-direct sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ to be their direct sum equipped with the norm

$$\|(x_1, x_2, \dots, x_n)\|_\psi = \|(\|x_1\|, \|x_2\|, \dots, \|x_n\|)\|_\psi$$

for $x_1 \in X_1, \dots, x_n \in X_n$. This extends the notion of the ℓ_p-direct sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_p$ of Banach spaces. In [5, 11, 13], the authors characterized the strict, uniform, and uniform non-squareness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ by means of the associate function ψ. Smoothness and uniform smoothness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ are treated in [3, 9].

Let X be a Banach space. Let X^* be the dual space of X, $S_X = \{x \in X : \|x\| = 1\}$, and for $x \in X$ with $x \neq 0$, $D(X, x) = \{\alpha \in S_{X^*} : \langle \alpha, x \rangle = \|x\|\}$. A Banach space X is called a *u-space* if for any $x, y \in S_X$ with $\|x + y\| = 2$, we have $D(X, x) = D(Y, y)$ (see [3]). A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in S_X$, we have $\|x + y\| \leq 2(1 - \delta)$ whenever $\langle \alpha, y \rangle < 1 - \varepsilon$ for some $\alpha \in D(X, x)$ (see [7]). Gao and Lau [4] showed that if a Banach space X is a U-space, then X has uniform normal structure.

In this paper, we characterize the U-convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$. We first characterize the U-convexity of $(\mathbb{C}^n, \|\cdot\|_\psi)$ by means of ψ. Namely, we show that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is a U-space (resp. a u-space) if and only if ψ is a u-function (see the notation of u-function in §2 and §3). We next prove that $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is a u-space if and only if $X_1, X_2, \dots, X_n$ are u-spaces and ψ is a u-function. Moreover, we show that $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is a U-space if and only if $X_1, X_2, \dots, X_n$ are U-spaces and ψ is a u-function.

Let I be an index set and $\{X_i\}_{i \in I}$ be a family of Banach spaces. We define the Banach space $\ell_\infty(I, X_i)$ by

$$\ell_\infty(I, X_i) = \left\{ \{x_i\} \in \prod_{i \in I} X_i : \|\{x_i\}\| = \sup_{i \in I} \|x_i\| < \infty \right\}.$$

Let $\mathcal{U}$ be an ultrafilter in I and let $N_\mathcal{U} = \{\{x_i\} \in \ell_\infty(I, X_i) : \lim_\mathcal{U} \|x_i\| = 0\}$. The *ultraproduct* of $\{X_i\}$ is the quotient space $\ell_\infty(I, X_i)/N_\mathcal{U}$ equipped with the quotient norm. Note that for each $\{x_i\}_\mathcal{U}$ in the ultraproduct of $\{X_i\}$, we have $\|\{x_i\}_\mathcal{U}\| = \lim_\mathcal{U} \|x_i\|$. In particular, for a Banach space X, the *ultrapower* denoted by $X_\mathcal{U}$ is the ultraproduct of $\{X_i\}$ if $I = \mathbb{N}$ and $X_i = X$ for all $i \in \mathbb{N}$ (for details see [3, 4]). Dhompongsa, Kaewkhao and Saejung [3] showed the following.

Proposition 1.1 ([3]). *Let X be a Banach space and X^* the dual space of X. Then* (i) *If X^* is a u-space, then X is a u-space.*

(ii) *If X is a U-space, then X is a u-space. The converse holds, whenever dim $X < \infty$.*
(iii) *X is a U-space if and only if $X_{\mathcal{U}}$ is a u-space.*

Proposition 1.2 ([3]). *Let $X_1, X_2, \ldots, X_n$ be Banach spaces and $\psi \in \Psi_n$. Then $((X_1 \oplus \cdots \oplus X_n)_\psi)_{\mathcal{U}}$ is isometric to $((X_1)_{\mathcal{U}} \oplus \cdots \oplus (X_n)_{\mathcal{U}})_\psi$.*

Let f be a continuous convex function from a convex subset C of a real Banach space X into $\mathbb{R}$. We denote by $\partial f(x)$ the subdifferential of f at $x \in C$; $\partial f(x) = \{a \in X^* : f(y) \geq f(x) + \langle a, y - x\rangle \text{ for } y \in C\}$. For $n \in \mathbb{N}$ with $n \geq 2$, put $I_n = \{0, 1, \ldots, n-1\}$. We also put $p_0 = (0, 0, \ldots, 0) \in \Delta_n$ and

$$p_j = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0) \in \Delta_n,$$

where $j = 1, 2, \ldots, n-1$.

Definition 1.3 ([8]). For each $\psi \in \Psi_n$, we define *the extended function* $\widetilde{\psi}$ of ψ as

$$\widetilde{\psi}(t) = \sup \left\{ \psi(s) + \langle a, t - s\rangle : \begin{array}{l} s = (s_1, s_2, \ldots, s_{n-1}) \in \Delta_n, \\ a \in \partial\psi(s), \\ \psi(s) + \langle a, p_j - s\rangle \geq 0 \text{ for } j \in I_n \end{array} \right\}$$

for all $t \in \mathbb{R}^{n-1}$.

Then $\widetilde{\psi}$ has the following properties:

(1) $\widetilde{\psi}(t) = \psi(t)$ holds for all $t \in \Delta_n$.
(2) $\widetilde{\psi}$ is a convex function on $\mathbb{R}^{n-1}$ with $\widetilde{\psi}(t) < \infty$ for all $t \in \mathbb{R}^{n-1}$.
(3) For every $t = (t_1, t_2, \ldots, t_n) \in \Delta_n$, $a \in \partial\widetilde{\psi}(t)$ if and only if $a \in \psi(t)$ and $\psi(t) + \langle a, p_j - t\rangle \geq 0$ for all $j \in I_n$.
(4) If $n = 2$, then

$$\widetilde{\psi}(t) = \begin{cases} 1 - t & \text{if } t < 0, \\ \psi(t) & \text{if } 0 \leq t \leq 1, \\ t & \text{if } t > 1 \end{cases}$$

and

$$\partial\widetilde{\psi}(t) = \begin{cases} [-1, \psi'_R(0)] & \text{if } t = 0, \\ [\psi'_L(t), \psi'_R(t)] & \text{if } 0 < t < 1, \\ [\psi'_L(1), 1] & \text{if } t = 1, \end{cases}$$

where $\psi'_L(t)$ (resp. $\psi'_R(t)$) is the left (resp. right) derivative of ψ at t (see [8]).

The following was proved by Bonsall and Duncan [2].

Lemma 1.4 ([2]). *For each* $t \in [0,1]$, *put* $x(t) = \frac{1}{\psi(t)}(1-t,t)$. *Then*

$$D\big(\mathbb{C}^2, x(t)\big) = \begin{cases} \left\{ \begin{pmatrix} 1 \\ c(1+a) \end{pmatrix} : a \in \partial\widetilde{\psi}(0), |c| = 1 \right\}, & \textit{if } t = 0, \\ \left\{ \begin{pmatrix} \psi(t) - at \\ \psi(t) + a(1-t) \end{pmatrix} : a \in \partial\widetilde{\psi}(t) \right\}, & \textit{if } 0 < t < 1, \\ \left\{ \begin{pmatrix} c(1-a) \\ 1 \end{pmatrix} : a \in \partial\widetilde{\psi}(1), |c| = 1 \right\}, & \textit{if } t = 1 \end{cases}$$

holds.

In [8], Mitani, Saito and Suzuki gave the n-dimensional version of Lemma 1.4.

Lemma 1.5 ([8], p. 106). *Let* $\psi \in \Psi_n$. *For every* $t = (t_1, t_2, \dots, t_{n-1}) \in \Delta_n$, *we put*

$$t_0 = 1 - \sum_{i=1}^{n-1} t_i \quad \text{and} \quad x(t) = \frac{1}{\psi(t)}(t_0, t_1, \dots, t_{n-1}) \in \mathbb{C}^n.$$

Then

$$D\big(\mathbb{C}^n, x(t)\big) = \left\{ \begin{pmatrix} e^{i\theta_0}\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ e^{i\theta_1}\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ \vdots \\ e^{i\theta_{n-1}}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\widetilde{\psi}(t), \\ \theta_j \in [0, 2\pi) \\ \quad \textit{for } j \in I_n \textit{ with } t_j = 0, \\ \theta_j = 0 \\ \quad \textit{for } j \in I_n \textit{ with } t_j > 0 \end{array} \right\}.$$

Moreover, Mitani, Oshiro and Saito [9] gave the following.

Lemma 1.6 ([9], p.154). *Let* $X_1, X_2, \dots, X_n$ *be Banach spaces and* $\psi \in \Psi_n$. *For every* $x = (x_1, x_2, \dots, x_n) \in S_{(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi}$,

$$D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) = \left\{ \begin{pmatrix} a_1 f_1 \\ a_2 f_2 \\ \vdots \\ a_n f_n \end{pmatrix} : \begin{array}{l} (a_1, \dots, a_n) \in D\big(\mathbb{C}^n, (\|x_1\|, \|x_2\|, \dots, \|x_n\|)\big), \\ \quad f_i \in D(X_i, x_i) \textit{ for } i \textit{ with } x_i \neq 0, \\ \quad f_i \in S_{X_i^*} \textit{ for } i \textit{ with } x_i = 0 \end{array} \right\}.$$

2. U-convexity of absolute norms on $\mathbb{C}^2$

Definition 2.1. A function $\psi \in \Psi_2$ is said to be a *u-function* if, for every s and t with $0 \le s < t \le 1$, $\widetilde{\psi}$ is differentiable at s and t, whenever ψ is affine on $[s,t]$.

Our aim in this section is to prove that $\big(\mathbb{C}^2, \|\cdot\|_\psi\big)$ is a u-space if and only if ψ is a u-function.

Remark 2.2. Our definition of u-function is different from that of u-function of Dhompongsa, Kaewkhao and Saejung [3], where a function $\psi \in \Psi_2$ is said to be a *u-function in the sense of Dhompongsa, Kaewkhao and Saejung (in short, DKS)* if for all interval $[s,t]\subset(0,1)$, ψ is differentiable at s and t whenever ψ is affine on $[s,t]$. In [3], they proved that for any Banach spaces X and Y and any $\psi\in\Psi_2$, $(X\oplus Y)_\psi$ is a u-space (resp. U-space) if and only if X and Y are u-spaces (resp. U-spaces) and ψ is a u-function in the sense of DKS. In particular, for any $\psi\in\Psi_2$, $(\mathbb{C}^2,\|\cdot\|_\psi)$ is a u-space if and only if ψ is a u-function in the sense of DKS. However, we can construct a counter-example of this result. We put

$$\psi_0(t)=\begin{cases}-\frac{1}{2}t+1, & \text{if } 0\le t\le\frac{1}{3},\\ \frac{3}{2}t^2-\frac{3}{2}t+\frac{7}{6}, & \text{if } \frac{1}{3}<t\le\frac{2}{3},\\ \frac{1}{2}t+\frac{1}{2}, & \text{if } \frac{2}{3}<t\le 1.\end{cases}$$

Then it is obvious that $\psi_0\in\Psi_2$ and ψ_0 is differentiable on $(0,1)$. Hence ψ_0 is a u-function in the sense of DKS. However, $(\mathbb{C}^2,\|\cdot\|_{\psi_0})$ is not a u-space. Indeed, we consider the two points $x(0)$ and $x(\frac{1}{3})$ in $\mathbb{C}^2$. Clearly, $\|x(0)\|_{\psi_0}=\|x(\frac{1}{3})\|_{\psi_0}=1$ and $\|x(0)+x(\frac{1}{3})\|_{\psi_0}=2$. Note that $\partial\widetilde{\psi_0}(0)=[-1,-\frac{1}{2}]$ and $\partial\widetilde{\psi_0}(\frac{1}{3})=\{-\frac{1}{2}\}$. Then, by Lemma 1.4, we have

$$D\big(\mathbb{C}^2,x(0)\big)=\left\{\begin{pmatrix}1\\ c(1+a)\end{pmatrix}: a\in[-1,-\tfrac{1}{2}],|c|=1\right\}$$

and

$$D\big(\mathbb{C}^2,x(\tfrac{1}{3})\big)=\left\{\begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix}\right\}.$$

Hence $D\big(\mathbb{C}^2,x(0)\big)\neq D\big(\mathbb{C}^2,x(\frac{1}{3})\big)$. Thus $\big(\mathbb{C}^2,\|\cdot\|_{\psi_0}\big)$ is not a u-space.

Smoothness of the points 0 and 1 for ψ is important to characterize the U-convexity of $\big(\mathbb{C}^2,\|\cdot\|_\psi\big)$. Let us present the correct version.

Theorem 2.3. *Let $\psi\in\Psi_2$. Then the following are equivalent:*
(i) $\big(\mathbb{C}^2,\|\cdot\|_\psi\big)$ *is a u-space.*
(ii) *For every s and t with $0\le s<t\le 1$, we have $\partial\widetilde{\psi}(s)=\partial\widetilde{\psi}(t)$, whenever ψ is affine on $[s,t]$.*
(iii) *ψ is a u-function.*

Proof. (i)⇒(ii): Assume that $\big(\mathbb{C}^2,\|\cdot\|_\psi\big)$ is a u-space. Fix s and t with $0\le s<t\le 1$. There is no case when $s=0$ and $t=1$. Let ψ be affine on $[s,t]$. As the proof of Theorem 14 in [3], we have $\|x(s)\|_\psi=\|x(t)\|_\psi=1$ and $\|x(s)+x(t)\|_\psi=2$. Hence it follows from the assumption that $D\big(\mathbb{C}^2,x(s)\big)=D\big(\mathbb{C}^2,x(t)\big)$. We show $\partial\widetilde{\psi}(s)=\partial\widetilde{\psi}(t)$. Take any $a\in\partial\widetilde{\psi}(s)$. If $0<s<t<1$, then from Lemma 1.4,

$$f:=\begin{pmatrix}\psi(s)-as\\ \psi(s)+a(1-s)\end{pmatrix}\in D\big(\mathbb{C}^2,x(s)\big).$$

By $f\in D\big(\mathbb{C}^2,x(t)\big)$ and Lemma 1.4, there exists $b\in\partial\widetilde{\psi}(t)$ satisfying

$$f=\begin{pmatrix}\psi(t)-bt\\ \psi(t)+b(1-t)\end{pmatrix}.$$

So $\psi(s)-as=\psi(t)-bt$ and $\psi(s)+a(1-s)=\psi(t)+b(1-t)$. These imply $a=b$. Hence we have $a\in\partial\widetilde{\psi}(t)$. If $0<s<t=1$, then $f\in D\big(\mathbb{C}^2,x(s)\big)$. By $f\in D\big(\mathbb{C}^2,x(1)\big)$ and Lemma 1.4, there exist c with $|c|=1$ and $b\in\partial\widetilde{\psi}(1)$ such that

$$f=\begin{pmatrix} c(1-b) \\ 1 \end{pmatrix}.$$

So $\psi(s)-as=c(1-b)$ and $\psi(s)+a(1-s)=1$. Note that $\psi(s)-as=1-b$ because $\psi(s)-as\geq 0$ and $1-b\geq 0$. Hence $a=b$, that is, $a\in\partial\widetilde{\psi}(1)$. In other cases, we similarly have $a\in\partial\widetilde{\psi}(t)$. Hence $\partial\widetilde{\psi}(s)\subset\partial\widetilde{\psi}(t)$. Similarly, $\partial\widetilde{\psi}(s)\supset\partial\widetilde{\psi}(t)$. Thus we have (i)⇒(ii).

(ii)⇒(i): Assume that the assertion (ii) holds. We show that $\big(\mathbb{C}^2,\|\cdot\|_\psi\big)$ is a u-space. Take any $x=(x_0,x_1)$ and $y=(y_0,y_1)\in S_{(\mathbb{C}^2,\|\cdot\|_\psi)}$ with $\|x+y\|_\psi=2$. Put

$$s=\frac{|x_1|}{|x_0|+|x_1|}\quad\text{and}\quad t=\frac{|y_1|}{|y_0|+|y_1|}.$$

Without loss of generality, we may assume that $s\leq t$. As the proof of Theorem 14 in [3], we have

$$\begin{aligned}2&\leq\|(|x_0|+|y_0|,|x_1|+|y_1|)\|_\psi\\&=(|x_0|+|y_0|+|x_1|+|y_1|)\psi\big((1-\lambda)s+\lambda t\big)\\&\leq(|x_0|+|y_0|+|x_1|+|y_1|)\{(1-\lambda)\psi(s)+\lambda\psi(t)\}\\&=\|x\|_\psi+\|y\|_\psi=2,\end{aligned}$$

where $\lambda=(|y_0|+|y_1|)/(|x_0|+|y_0|+|x_1|+|y_1|)$. So ψ is affine on $[s,t]$. Hence it follows from the assumption that $\partial\widetilde{\psi}(s)=\partial\widetilde{\psi}(t)$.

We first show $D\big(\mathbb{C}^2,x(s)\big)=D\big(\mathbb{C}^2,x(t)\big)$. We consider the case when $0<s<t<1$. Take any $f\in D\big(\mathbb{C}^2,x(s)\big)$. By Lemma 1.4, there exists $a\in\partial\widetilde{\psi}(s)$ satisfying

$$f=\begin{pmatrix} \psi(s)-as \\ \psi(s)+a(1-s) \end{pmatrix}.$$

From $a\in\partial\widetilde{\psi}(s)$ and $a\in\partial\widetilde{\psi}(t)$, we have $\psi(s)-\psi(t)=a(s-t)$, which implies $\psi(s)-as=\psi(t)-at$ and $\psi(s)+a(1-s)=\psi(t)+a(1-t)$. Hence it follows from $a\in\partial\widetilde{\psi}(t)$ and Lemma 1.4 that

$$f=\begin{pmatrix} \psi(t)-at \\ \psi(t)+a(1-t) \end{pmatrix}\in D(\mathbb{C}^2,x(t)\big).$$

We next consider the case when $0=s<t<1$. Take any $f\in D(\mathbb{C}^2,x(0)\big)$. Then there exist $a\in\partial\widetilde{\psi}(0)$ and c with $|c|=1$ satisfying

$$f=\begin{pmatrix} 1 \\ c(1+a) \end{pmatrix}.$$

By $a\in\partial\widetilde{\psi}(0)$ and $a\in\partial\widetilde{\psi}(t)$, we obtain $\psi(0)-\psi(t)=a(0-t)$, which implies $1=\psi(t)-at$ and $1+a=\psi(t)+a(1-t)$. Similarly, by $-1\in\partial\widetilde{\psi}(0)=\partial\widetilde{\psi}(t)$, we obtain $1 = \psi(t) + t$ and $0 = \psi(t) - (1-t)$. Hence $a = -1$, that is, $c(1+a) = 0 = \psi(t) + a(1-t)$. Hence we have from Lemma 1.4,

$$f = \begin{pmatrix} \psi(t) - at \\ \psi(t) + a(1-t) \end{pmatrix} \in D\big(\mathbb{C}^2, x(t)\big).$$

In other cases, we similarly have $f \in D\big(\mathbb{C}^2, x(t)\big)$. Hence we have $D\big(\mathbb{C}^2, x(s)\big) \subset D\big(\mathbb{C}^2, x(t)\big)$. Similarly, we have $D\big(\mathbb{C}^2, x(s)\big) \supset D\big(\mathbb{C}^2, x(t)\big)$. Hence $D\big(\mathbb{C}^2, x(s)\big) = D\big(\mathbb{C}^2, x(t)\big)$. Namely,

$$D\big(\mathbb{C}^2, (|x_0|, |x_1|)\big) = D\big(\mathbb{C}^2, (|y_0|, |y_1|)\big) \tag{2.1}$$

Similarly, we consider the points x and $\frac{x+y}{2}$, and we can obtain

$$D\big(\mathbb{C}^2, (|x_0|, |x_1|)\big) = D\big(\mathbb{C}^2, (|(x_0+y_0)/2|, |(x_1+y_1)/2|)\big). \tag{2.2}$$

We next show $D\big(\mathbb{C}^2, x\big) = D\big(\mathbb{C}^2, y\big)$. Fix $f = (\alpha_0, \alpha_1) \in D\big(\mathbb{C}^2, x\big)$. Put ρ_i, η_i and ξ_i as $\rho_i = \arg x_i \in [0, 2\pi), \eta_i = \arg y_i \in [0, 2\pi)$ and $\xi_i = \arg\,(x_i + y_i)/2 \in [0, 2\pi)$ for all $i = 0, 1$, where $\arg 0 = 0$. Then we have from (2.1) and (2.2),

$$g := (e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1) \in D\big(\mathbb{C}^2, (|y_0|, |y_1|)\big)$$

and

$$h := (e^{i(\rho_0-\xi_0)}\alpha_0, e^{i(\rho_1-\xi_1)}\alpha_1) \in D\big(\mathbb{C}^2, (x+y)/2\big).$$

Note that $D\big(\mathbb{C}^2, (x+y)/2\big) \subset D\big(\mathbb{C}^2, x\big) \cap D\big(\mathbb{C}^2, y\big)$. Then by $h \in D\big(\mathbb{C}^2, x\big)$, we have

$$\begin{aligned} 1 = \mathrm{Re}\langle h, x\rangle &= \mathrm{Re}(e^{i(\rho_0-\xi_0)}\alpha_0 x_0) + \mathrm{Re}(e^{i(\rho_1-\xi_1)}\alpha_1 x_1) \\ &\le |e^{i(\rho_0-\xi_0)}\alpha_0 x_0| + |e^{i(\rho_1-\xi_1)}\alpha_1 x_1| \le \|f\|\|x\|_\psi = 1, \end{aligned}$$

which implies $e^{i(\rho_i-\xi_i)}\alpha_i x_i \ge 0$ for all $i = 0, 1$. We similarly have from $h \in D\big(\mathbb{C}^2, y\big)$, $e^{i(\rho_i-\xi_i)}\alpha_i y_i \ge 0$ for all $i = 0, 1$.

From these results we show $e^{i\eta_i}\alpha_i = e^{i\rho_i}\alpha_i$ for all i with $x_i \ne 0$ and $y_i \ne 0$. Let $x_i \ne 0$ and $y_i \ne 0$. We may assume $\alpha_i \ne 0$. Then we have $e^{i(\rho_i-\xi_i)}\alpha_i x_i = e^{i(2\rho_i-\xi_i)}\alpha_i|x_i| \ge 0$. Hence we obtain

$$e^{i(2\rho_i-\xi_i)}\alpha_i \ge 0. \tag{2.3}$$

Also, we have $e^{i(\rho_i-\xi_i)}\alpha_i y_i = e^{i(\rho_i-\xi_i+\eta_i)}\alpha_i|y_i| \ge 0$. Hence we obtain

$$e^{i(\rho_i-\xi_i+\eta_i)}\alpha_i \ge 0. \tag{2.4}$$

From (2.3) and (2.4) we have $e^{i(\rho_i-\eta_i)} \ge 0$ and so $\rho_i = \eta_i$. Thus $e^{i\eta_i}\alpha_i = e^{i\rho_i}\alpha_i$.

Moreover we show $(e^{i\eta_0}\alpha_0, e^{i\eta_1}\alpha_1) \in D(\mathbb{C}^2, (|y_0|, |y_1|))$. To do it, we show

$$(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1) \in D(\mathbb{C}^2, (|y_0|, |y_1|)). \tag{2.5}$$

We consider the case where $x_0 \ne 0$ and $y_0 \ne 0$. By $e^{i\eta_0}\alpha_0 = e^{i\rho_0}\alpha_0$, we have $g = (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1)$. Hence we get (2.5). We consider the case where $x_0 = 0$. Then by $(e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1) \in D(\mathbb{C}^2, (|x_0|, |x_1|))$, we have

$$\langle (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1), (0, |x_1|)\rangle = \langle (e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1), (0, |x_1|)\rangle = 1.$$

Also, we have

$$\|(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1)\| = \|(e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1)\| = 1.$$

Hence $(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1) \in D(\mathbb{C}^2, (|x_0|, |x_1|))$. Thus by (2.1), we have (2.5). We consider the case where $y_0 = 0$. Then by $(e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1) \in D(\mathbb{C}^2, (|y_0|, |y_1|))$, we have

$$\langle (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1), (0, |y_1|)\rangle = \langle (e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1), (0, |y_1|)\rangle = 1.$$

Hence we obtain (2.5). Thus (2.5) holds for all cases. Similarly, we can obtain $(e^{i\eta_0}\alpha_0, e^{i\eta_1}\alpha_1) \in D(\mathbb{C}^2, (|y_0|, |y_1|))$ by using (2.5). Hence $f = (\alpha_0, \alpha_1) \in D(\mathbb{C}^2, y)$. Therefore we have $D(\mathbb{C}^2, x) \subset D(\mathbb{C}^2, y)$. Similarly, $D(\mathbb{C}^2, x) \supset D(\mathbb{C}^2, y)$. Thus we have (ii)$\Rightarrow$(i).

(ii)$\Rightarrow$(iii): Assume that assertion (ii) holds. Put s and t with $0 \leq s < t \leq 1$. Let ψ be affine on $[s, t]$. Then it follows from the assumption that $\partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$. By $\widetilde{\psi}'_L(s) \in \partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$, we have $\widetilde{\psi}'_L(t) \leq \widetilde{\psi}'_L(s) \leq \widetilde{\psi}'_R(t)$, where $\widetilde{\psi}'_L(t)$ (resp. $\widetilde{\psi}'_R(t)$) is the left (resp. right) derivative of $\widetilde{\psi}$ at t. We also have by the convexity of $\widetilde{\psi}$, $\widetilde{\psi}'_L(s) \leq \widetilde{\psi}'_R(s) \leq \widetilde{\psi}'_L(t) \leq \widetilde{\psi}'_R(t)$. These imply that $\widetilde{\psi}$ is differentiable at s. Similarly, $\widetilde{\psi}$ is differentiable at t. Thus we have (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii): Assume that ψ is a u-function. Take s and t with $0 \leq s < t \leq 1$. Let ψ be affine on $[s, t]$. From $\widetilde{\psi}'_R(s) = \widetilde{\psi}'_L(t)$ and the differentiability, we get $\partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$. Thus we have (iii)$\Rightarrow$(ii). □

3. U-convexity of absolute norms on $\mathbb{C}^n$

In this section, we characterize the U-convexity of $(\mathbb{C}^n, \|\cdot\|_\psi)$. To do it, we shall introduce the following.

Definition 3.1. $\psi \in \Psi_n$ is said to be a *u-function* if, for any $s, t \in \Delta_n$ with $s \neq t$, then $\partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$, whenever ψ is affine on $[s, t]$.

When $n = 2$, this coincides with the notion of u-function in Definition 2.1, by Theorem 2.3.

Theorem 3.2. *Let $\psi \in \Psi_n$. Then the following are equivalent:*
(i) *$(\mathbb{C}^n, \|\cdot\|_\psi)$ is a u-space.*
(ii) *ψ is a u-function.*

Proof. (i)$\Rightarrow$(ii): For each $t = (t_1, t_2, \ldots, t_{n-1}) \in \Delta_n$, we put

$$t_0 = 1 - \sum_{i=1}^{n-1} t_i \quad \text{and} \quad x(t) = \frac{1}{\psi(t)}\,(t_0, t_1, \ldots, t_{n-1})\,.$$

Assume that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is a u-space. Fix $s = (s_1, \ldots, s_{n-1}) \in \Delta_n$ and $t = (t_1, \ldots, t_{n-1}) \in \Delta_n$ with $s \neq t$. Let ψ be affine on $[s, t]$. As the proof of Theorem 2.3, we obtain $\|x(s) + x(t)\|_\psi = 2$. Hence we have from the assumption, $D\big(\mathbb{C}^n, x(s)\big) = D\big(\mathbb{C}^n, x(t)\big)$. We show $\partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$. Take any $a \in \partial\widetilde{\psi}(s)$. By

Lemma 1.5,

$$f := \begin{pmatrix} \psi(s) + \langle a, p_0 - s\rangle \\ \psi(s) + \langle a, p_1 - s\rangle \\ \vdots \\ \psi(s) + \langle a, p_{n-1} - s\rangle \end{pmatrix} \in D\big(\mathbb{C}^n, x(s)\big).$$

Hence it follows from $f \in D\big(\mathbb{C}^n, x(t)\big)$ and Lemma 1.5 that there exist $c_0, c_1, \ldots, c_{n-1}$ with $|c_j| = 1 (\forall j)$ and $b \in \partial\widetilde{\psi}(t)$ satisfying

$$f = \begin{pmatrix} c_0(\psi(t) + \langle b, p_0 - t\rangle) \\ c_1(\psi(t) + \langle b, p_1 - t\rangle) \\ \vdots \\ c_{n-1}(\psi(t) + \langle b, p_{n-1} - t\rangle) \end{pmatrix}.$$

So we have $\psi(s) + \langle a, p_j - s\rangle = c_j(\psi(t) + \langle b, p_j - t\rangle)$ for all $j \in I_n$. Note that since $\psi(t) + \langle b, p_j - t\rangle \geq 0$ for all $j \in I_n$, we have for all $j \in I_n$, $\psi(s) + \langle a, p_j - s\rangle = \psi(t) + \langle b, p_j - t\rangle$. Hence we have for all $u \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
&\widetilde{\psi}(u) - \widetilde{\psi}(t) - \langle a, u - t\rangle \\
&= \widetilde{\psi}(u) - \widetilde{\psi}(s) - \langle a, u - s\rangle + \widetilde{\psi}(s) - \widetilde{\psi}(t) + \langle a, t - s\rangle \\
&\geq \psi(s) - \psi(t) + \langle a, t - s\rangle && (\text{ by } a \in \partial\widetilde{\psi}(s)) \\
&= \psi(s) - \psi(t) + \langle a, \sum_{t=0}^{n-1} t_j p_j - s\rangle && (\text{ by } t = \sum_{t=0}^{n-1} t_j p_j) \\
&= \sum_{t=0}^{n-1} t_j(\psi(s) + \langle a, p_j - s\rangle) - \psi(t) \\
&= \sum_{t=0}^{n-1} t_j(\psi(t) + \langle b, p_j - t\rangle) - \psi(t) \\
&= \psi(t) + \langle b, \sum_{t=0}^{n-1} t_j p_j - t\rangle - \psi(t) = 0,
\end{aligned}$$

which implies $a \in \partial\widetilde{\psi}(t)$. Hence $\partial\widetilde{\psi}(s) \subset \partial\widetilde{\psi}(t)$. Similarly, $\partial\widetilde{\psi}(s) \supset \partial\widetilde{\psi}(t)$. Thus ψ is a u-function.

(ii)⇒(i): For each $x = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{C}^n$, we put $|x|$ as $|x| = (|x_0|, |x_1|, \ldots, |x_{n-1}|)$. Assume that ψ is a u-function. We show that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is a u-space. Take any $x = (x_0, x_1, \ldots, x_{n-1}) \in S_{(\mathbb{C}^n, \|\cdot\|_\psi)}$ and $y = (y_0, y_1, \ldots, y_{n-1}) \in S_{(\mathbb{C}^n, \|\cdot\|_\psi)}$ with $\|x + y\|_\psi = 2$. For each i, put s_i and t_i as

$$s_i = \frac{|x_i|}{\sum_{j=0}^{n-1} |x_j|} \quad \text{and} \quad t_i = \frac{|y_i|}{\sum_{j=0}^{n-1} |x_j|}.$$

We also put $s = (s_1, \ldots, s_{n-1}) \in \Delta_n$, $t = (t_1, \ldots, t_{n-1}) \in \Delta_n$ and

$$\lambda = \frac{\sum_{j=0}^{n-1} |y_j|}{\sum_{j=0}^{n-1}(|x_j| + |y_j|)}.$$

Then since

$$\begin{aligned} 2 &\le \|(|x_0| + |y_0|, |x_1| + |y_1|, \ldots, |x_{n-1}| + |y_{n-1}|)\|_\psi \\ &= \sum_{j=0}^{n-1}(|x_j| + |y_j|)\psi\big((1-\lambda)s + \lambda t\big) \\ &\le \sum_{j=0}^{n-1}(|x_j| + |y_j|)\{(1-\lambda)\psi(s) + \lambda\psi(t)\} \\ &= \|x\|_\psi + \|y\|_\psi = 2, \end{aligned}$$

it follows that ψ is affine on $[s, t]$. So $\partial\widetilde{\psi}(s) = \partial\widetilde{\psi}(t)$. We first show $D\big(\mathbb{C}^n, x(s)\big) = D\big(\mathbb{C}^n, x(t)\big)$. Fix $f \in D\big(\mathbb{C}^n, x(s)\big)$. Then there exist $a \in \partial\widetilde{\psi}(s)$ and $\{c_j\}_{j=0}^{n-1}$ such that

$$f = \begin{pmatrix} c_0\big(\psi(s) + \langle a, p_0 - s\rangle\big) \\ c_1\big(\psi(s) + \langle a, p_1 - s\rangle\big) \\ \vdots \\ c_{n-1}\big(\psi(s) + \langle a, p_{n-1} - s\rangle\big) \end{pmatrix},$$

where $c_j = 1$ for $j \in I_n$ with $s_j > 0$, and $|c_j| = 1$ for $j \in I_n$ with $s_j = 0$. In order to show $f \in D\big(\mathbb{C}^n, x(t)\big)$, from Lemma 1.5, it is enough to show the following:
(a): $a \in \partial\widetilde{\psi}(t)$,
(b): For all $i \in I_n$, we have $\psi(s) + \langle a, p_i - s\rangle = \psi(t) + \langle a, p_i - t\rangle$,
(c): For all $i \in I_n$ with $t_i > 0$, we have $c_i = 1$ or $\psi(s) + \langle a, p_i - s\rangle = 0$.
The assertion (a) is clear. We also have from $a \in \partial\widetilde{\psi}(s)$ and $a \in \partial\widetilde{\psi}(t)$, $\psi(t) - \psi(s) = \langle a, t - s\rangle$, which implies (b). We show the assertion (c). Assume that $t_0 > 0$. Then we show that $c_0 = 1$ or $\psi(s) + \langle a, p_0 - s\rangle = 0$. If $s_0 > 0$, then $c_0 = 1$. Let $s_0 = 0$. Put g and h as

$$g = \begin{pmatrix} \psi(s) + \langle a, p_0 - s\rangle \\ \psi(s) + \langle a, p_1 - s\rangle \\ \vdots \\ \psi(s) + \langle a, p_{n-1} - s\rangle \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 0 \\ \psi(s) + \langle a, p_1 - s\rangle \\ \vdots \\ \psi(s) + \langle a, p_{n-1} - s\rangle \end{pmatrix}.$$

From Lemma 1.5, (a) and (b), we have $g \in D\big(\mathbb{C}^n, x(s)\big)$ and $g \in D\big(\mathbb{C}^n, x(t)\big)$. By $s_0 = 0$, we have $\langle h, x(s)\rangle = \langle g, x(s)\rangle = 1$, which implies $h \in D\big(\mathbb{C}^n, x(s)\big)$. Hence, by Lemma 1.5, there exists $c \in \partial\widetilde{\psi}(s)$ such that

$$h = \begin{pmatrix} \psi(s) + \langle c, p_0 - s\rangle \\ \vdots \\ \psi(s) + \langle c, p_{n-1} - s\rangle \end{pmatrix}.$$

Since $c \in \partial\widetilde{\psi}(t)$ and $\psi(t) + \langle c, p_i - t\rangle = \psi(s) + \langle c, p_i - s\rangle$ for every $i \in I_n$, we have $h \in D\big(\mathbb{C}^n, x(t)\big)$. Hence $0 = \langle g, x(t)\rangle - \langle h, x(t)\rangle = t_0(\psi(t) + \langle a, p_0 - t\rangle)$. Therefore we have by $t_0 > 0$, $\psi(s) + \langle a, p_0 - s\rangle = \psi(t) + \langle a, p_0 - t\rangle = 0$. Similarly, for all $i = 1, 2, \ldots, n-1$ with $t_i > 0$, we have $c_i = 1$ or $\psi(s) + \langle a, p_i - s\rangle = 0$. Thus $D\big(\mathbb{C}^n, x(s)\big) = D\big(\mathbb{C}^n, x(t)\big)$, that is,

$$D\big(\mathbb{C}^n, |x|\big) = D\big(\mathbb{C}^n, |y|\big). \tag{3.1}$$

Similarly,

$$D\big(\mathbb{C}^n, |x|\big) = D\big(\mathbb{C}^n, |(x+y)/2|\big). \tag{3.2}$$

We next show $D\big(\mathbb{C}^n, x\big) = D\big(\mathbb{C}^n, y\big)$. Fix $f = (\alpha_1, \ldots, \alpha_{n-1}) \in D\big(\mathbb{C}^n, x\big)$. Put ρ_i, η_i and ξ_i as $\rho_i = \arg x_i \in [0, 2\pi), \eta_i = \arg y_i \in [0, 2\pi)$ and $\xi_i = \arg\,(x_i + y_i)/2 \in [0, 2\pi)$ for all i, where $\arg 0 = 0$. Then by (3.1) and (3.2), we have

$$g := (e^{i\rho_0}\alpha_0, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |x|\big) = D\big(\mathbb{C}^n, |y|\big),$$

and

$$h := (e^{i(\rho_0 - \xi_0)}\alpha_0, \ldots, e^{i(\rho_{n-1} - \xi_{n-1})}\alpha_{n-1}) \in D\big(\mathbb{C}^n, (x+y)/2\big).$$

Note that $D\big(\mathbb{C}^n, (x+y)/2\big) \subset D\big(\mathbb{C}^n, x\big) \cap D\big(\mathbb{C}^n, y\big)$. Then by $h \in D\big(\mathbb{C}^n, x\big)$, we have

$$\begin{aligned} 1 = \mathrm{Re}\langle h, x\rangle &= \sum_{i=0}^{n-1} \mathrm{Re}(e^{i(\rho_i - \xi_i)}\alpha_i x_i) \\ &\le \sum_{i=0}^{n-1} |e^{i(\rho_i - \xi_i)}\alpha_i x_i| \le \|f\|\|x\|_\psi = 1, \end{aligned}$$

which implies $e^{i(\rho_i - \xi_i)}\alpha_i x_i \ge 0$ for all i. Similarly, we have by $h \in D\big(\mathbb{C}^n, y\big)$, $e^{i(\rho_i - \xi_i)}\alpha_i y_i \ge 0$ for all i.

Moreover we show that $e^{i\rho_i}\alpha_i = e^{i\eta_i}\alpha_i$ for all i with $x_i \ne 0$ and $y_i \ne 0$. Let $x_i \ne 0$ and $y_i \ne 0$. We may assume $\alpha_i \ne 0$. Then $e^{i(\rho_i - \xi_i)}\alpha_i x_i = e^{i(2\rho_i - \xi_i)}\alpha_i|x_i| \ge 0$ and hence $e^{i(2\rho_i - \xi_i)}\alpha_i \ge 0$. We also have $e^{i(\rho_i - \xi_i)}\alpha_i y_i = e^{i(\rho_i - \xi_i + \eta_i)}\alpha_i|y_i| \ge 0$ and hence $e^{i(\rho_i - \xi_i + \eta_i)}\alpha_i \ge 0$. These imply $e^{i(\rho_i - \eta_i)} \ge 0$. Hence $\rho_i = \eta_i$ and so $e^{i\rho_i}\alpha_i = e^{i\eta_i}\alpha_i$.

From this result, we show $(e^{i\eta_0}\alpha_0, \ldots, e^{i\eta_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big)$. To do it, we show

$$(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big). \tag{3.3}$$

Let $x_0 \ne 0$ and $y_0 \ne 0$. Then by $e^{i\rho_0}\alpha_0 = e^{i\eta_0}\alpha_0$, we have

$$g = (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}).$$

By $g \in D\big(\mathbb{C}^n, |y|\big)$, we have (3.3). Let $x_0 = 0$. By $(e^{i\rho_0}\alpha_0, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |x|\big)$ we have

$$\begin{aligned} &\langle (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}), (0, |x_1|, \ldots, |x_{n-1}|)\rangle \\ &\quad = \langle (e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1}), (0, |x_1|, \ldots, |x_{n-1}|)\rangle = 1 \end{aligned}$$

and

$$\|(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1})\| = \|(e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1, \ldots, e^{i\rho_{n-1}}\alpha_{n-1})\| = 1.$$

and hence

$$(e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \dots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |x|\big).$$

Thus by (3.1) we obtain (3.3). If $y_0 = 0$, then by $(e^{i\rho_0}\alpha_0, \dots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big)$, we have

$$\begin{aligned}&\langle (e^{i\eta_0}\alpha_0, e^{i\rho_1}\alpha_1, \dots, e^{i\rho_{n-1}}\alpha_{n-1}), (0, |y_1|, \dots, |y_{n-1}|)\rangle \\ &= \langle (e^{i\rho_0}\alpha_0, e^{i\rho_1}\alpha_1, \dots, e^{i\rho_{n-1}}\alpha_{n-1}), (0, |y_1|, \dots, |y_{n-1}|)\rangle = 1.\end{aligned}$$

Hence we have (3.3). Thus (3.3) holds for all cases. Similarly, we obtain

$$(e^{i\eta_0}\alpha_0, e^{i\eta_1}\alpha_1, e^{i\rho_2}\alpha_2, \dots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big)$$

by using (3.3). In the same way, we can show that for each i,

$$(e^{i\eta_0}\alpha_0, \dots, e^{i\eta_{i-1}}\alpha_{i-1}, e^{i\eta_i}\alpha_i, e^{i\rho_{i+1}}\alpha_{i+1}, \dots, e^{i\rho_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big).$$

Hence we have

$$(e^{i\eta_0}\alpha_0, e^{i\eta_1}\alpha_1, \dots, e^{i\eta_{n-1}}\alpha_{n-1}) \in D\big(\mathbb{C}^n, |y|\big)$$

and so $f = (\alpha_0, \dots, \alpha_{n-1}) \in D\big(\mathbb{C}^n, y\big)$. Therefore we have $D(\mathbb{C}^n, x) \subset D(\mathbb{C}^n, y)$. Similarly, $D\big(\mathbb{C}^n, x\big) \supset D\big(\mathbb{C}^n, y\big)$. Thus $(\mathbb{C}^n, \|\cdot\|_\psi)$ is a u-space and this completes the proof.

□

As a direct consequence of Proposition 1.1 and Theorem 3.2, we obtain the following.

Theorem 3.3. *Let $\psi \in \Psi_n$. Then the following are equivalent:*
(i) $(\mathbb{C}^n, \|\cdot\|_\psi)$ *is a U-space.*
(ii) ψ *is a u-function.*

4. U-convexity of $(X_1 \oplus \cdots \oplus X_n)_\psi$

In this section, we characterize the U-convexity of $(X_1 \oplus \cdots \oplus X_n)_\psi$.

Theorem 4.1. *Let $X_1, X_2, \dots, X_n$ be Banach spaces and $\psi \in \Psi_n$. Then the following are equivalent:*
(i) $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ *is a u-space.*
(ii) $X_1, X_2, \dots, X_n$ *and* $(\mathbb{C}^n, \|\cdot\|_\psi)$ *are u-spaces.*
(iii) $X_1, X_2, \dots, X_n$ *are u-spaces and* ψ *is a u-function.*

Proof. From Theorem 3.2, it is enough to show (ii) ⇒ (i). Assume that $X_1, X_2, \dots, X_n$ and $(\mathbb{C}^n, \|\cdot\|_\psi)$ are u-spaces. We put $X = (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$. Fix $x = (x_1, x_2, \dots, x_n) \in S_X$ and $y = (y_1, y_2, \dots, y_n) \in S_X$ with $\|x + y\|_\psi = 2$. We also put $z = (\|x_1\|, \|x_2\|, \dots, \|x_n\|)$ and $w = (\|y_1\|, \|y_2\|, \dots, \|y_n\|)$. We shall show $D\big(X, x\big) \subset D\big(X, y\big)$. Fix

$$f = \begin{pmatrix} a_1 f_1 \\ a_2 f_2 \\ \vdots \\ a_n f_n \end{pmatrix} \in D(X, x),$$

where $(a_1, \ldots, a_n) \in D(\mathbb{C}^n, z)$, $f_i \in D(X_i, x_i)$ for $i \in I_n$ with $x_i \neq 0$, and $f_i \in S_{X_i^*}$ for $i \in I_n$ with $x_i = 0$. For each $i \in I_n$ with $y_i \neq 0$, take a $g_i \in D(X_i, y_i)$. For each $i \in I_n$ with $y_i = 0$, take a $g_i' \in S_{X_i}$. Moreover, we put

$$h_i = \begin{cases} f_i, \text{ if } & a_i \neq 0, \\ g_i, \text{ if } & a_i = 0 \text{ and } y_i \neq 0, \\ g_i', \text{ if } & a_i = 0 \text{ and } y_i = 0, \end{cases}$$

for each $i \in I_n$. Note that

$$f = \begin{pmatrix} a_1 h_1 \\ \vdots \\ a_n h_n \end{pmatrix}.$$

In order to show $f \in D(X, y)$, from Lemma 1.6, it is enough to show the following:
(a): $(a_1, a_2, \ldots, a_n) \in D(\mathbb{C}^n, w)$,
(b): For $i \in I_n$ with $y_i \neq 0$, $h_i \in D(X_i, y_i)$,
(c): For $i \in I_n$ with $y_i = 0$, $h_i \in S_{X_i}$.
Obviously, the assertion (c) holds. We show (a). From $2 = \|x+y\|_\psi \leq \|z+w\|_\psi \leq 2$, we obtain $\|z+w\|_\psi = 2$ and $\|z\|_\psi = \|w\|_\psi = 1$. Hence, since $(\mathbb{C}^n, \|\cdot\|_\psi)$ is a u-space, we have

$$D(\mathbb{C}^n, z) = D(\mathbb{C}^n, w), \tag{4.1}$$

which implies (a). Similarly, we have

$$D\big(\mathbb{C}^n, (\|(x_1 + y_1)/2\|, \ldots, \|(x_n + y_n)/2\|)\big) = D(\mathbb{C}^n, z). \tag{4.2}$$

We next show (b). Fix $i \in I_n$ with $y_i \neq 0$. If $a_i = 0$, then $h_i = g_i \in D(X_i, y_i)$. Let $a_i \neq 0$. Assume that $x_i = 0$. Then both $(a_1, \ldots, a_n)$ and $(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$ belong to $D(\mathbb{C}^n, z)$. From (4.1), they also belong to $D(\mathbb{C}^n, w)$. Hence

$$a_i\|y_i\| = \langle (a_1, \ldots, a_n), w\rangle - \langle (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n), w\rangle = 0,$$

which is a contradiction. So $x_i \neq 0$. Note that $a_i \neq 0$ and

$$(a_1, \ldots, a_n) \in D\big(\mathbb{C}^n, (\|(x_1 + y_1)/2\|, \ldots, \|(x_n + y_n)/2\|)\big).$$

Hence, by Lemma 1.6, we can take an element

$$k = (k_1, \ldots, k_n) \in D\Big(X, \frac{x+y}{2}\Big)$$

with $k_i \neq 0$. Note that $D\Big(X, \frac{x+y}{2}\Big) \subset D(X, x) \cap D(X, y)$. Hence by $k \in D(X, x)$, we have

$$1 = \sum_{i=1}^n \mathrm{Re}(k_i(x_i)) \leq \sum_{i=1}^n |k_i(x_i)| \leq \sum_{i=1}^n \|k_i\|\|x_i\| \leq 1,$$

which implies $k_i(x_i) = \|k_i\|\|x_i\|$. Similarly $k_i(y_i) = \|k_i\|\|y_i\|$. These imply

$$2 = \frac{k_i}{\|k_i\|}\left(\frac{x_i}{\|x_i\|} + \frac{y_i}{\|y_i\|}\right) \leq \left\|\frac{x_i}{\|x_i\|} + \frac{y_i}{\|y_i\|}\right\| \leq 2,$$

and so

$$\left\|\frac{x_i}{\|x_i\|} + \frac{y_i}{\|y_i\|}\right\| = 2.$$

Since X_i is a u-space, we have $D\left(X_i, \frac{x_i}{\|x_i\|}\right) = D\left(X_i, \frac{y_i}{\|y_i\|}\right)$, which implies $D(X_i, x_i) = D(X_i, y_i)$. Hence $h_i = f_i \in D(X_i, y_i)$. Therefore we have (b). Thus $f \in D(X, y)$, and so $D(X, x) \subset D(X, y)$. We similarly have $D(X, x) \supset D(X, y)$. Thus X is a u-space. This completes the proof. □

Corollary 4.2 (cf. [3]). *Let X and Y be Banach spaces and $\psi \in \Psi_2$. Then the following are equivalent:*
(i) *$(X \oplus Y)_\psi$ is a u-space.*
(ii) *X, Y and $(\mathbb{C}^2, \|\cdot\|_\psi)$ are u-spaces.*
(iii) *X and Y are u-spaces and ψ is a u-function.*

As a direct consequence of Proposition 1.1, Proposition 1.2 and Theorem 4.1, we obtain the following.

Theorem 4.3. *Let $X_1, X_2, \ldots, X_n$ be Banach spaces and $\psi \in \Psi_n$. Then the following are equivalent:*
(i) *$(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is a U-space.*
(ii) *$X_1, X_2, \ldots, X_n$ and $(\mathbb{C}^n, \|\cdot\|_\psi)$ are U-spaces.*
(iii) *$X_1, X_2, \ldots, X_n$ are U-spaces and ψ is a u-function.*

Corollary 4.4 (cf. [3]). *Let X and Y be Banach spaces and $\psi \in \Psi_2$. Then the following are equivalent:*
(i) *$(X \oplus Y)_\psi$ is a U-space.*
(ii) *X, Y and $(\mathbb{C}^2, \|\cdot\|_\psi)$ are U-spaces.*
(iii) *X and Y are U-spaces, and ψ is a u-function.*

References

[1] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, 2nd ed., North-Holland, Amsterdam-New York-Oxford, 1985.

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Series, Vol.10, 1973.

[3] S. Dhompongsa, A. Kaewkhao and S. Saejung, *Uniform smoothness and U-convexity of ψ-direct sums*, J. Nonlinear Convex Anal. **6** (2005), 327–338.

[4] J. Gao and K. S. Lau, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99** (1991), 41–56.

[5] M. Kato, K. -S. Saito and T. Tamura, *On ψ-direct sums of Banach spaces and convexity*, J. Austral. Math. Soc. **75** (2003), 413–422.

[6] M. Kato, K. -S. Saito and T. Tamura, *Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, Math. Inequal. Appl. **7** (2004), 429–437.

[7] K. S. Lau, *Best approximation by closed sets in Banach spaces*, J. Approx. Theory. **23** (1978), 29–36.

[8] K. Mitani, K. -S. Saito and T. Suzuki, *Smoothness of absolute norms on $\mathbb{C}^n$*, J. Convex. Anal. **10** (2003), 89–107.

[9] K. Mitani, S. Oshiro and K. -S. Saito, *Smoothness of ψ-direct sums of Banach spaces*, Math. Inequal. Appl. **8** (2005), 147–157.

[10] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[11] K. -S. Saito and M. Kato, *Uniform convexity of ψ-direct sums of Banach spaces*, J. Math. Anal. Appl. **277** (2003), 1–11.

[12] K. -S. Saito, M. Kato and Y. Takahashi, *Absolute norms on $\mathbb{C}^n$*, J. Math. Anal. Appl. **252** (2000), 879–905.

[13] Y. Takahashi, M. Kato and K. -S. Saito, *Strict convexity of absolute normes on $\mathbb{C}^2$ and direct sums of Banach spaces*, J. Inequal. Appl. **7** (2002), 179–186.

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