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A MEAN ERGODIC THEOREM FOR NONLINEAR SEMIGROUPS ON THE HILBERT BALL

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Abstract. We establish a dual mean ergodic theorem for nonlinear continuous semigroups of ρ-nonexpansive self-mappings of the Hilbert ball $(\mathbb{B}, \rho)$.

1. Introduction

The main purpose of this note is to establish a dual mean ergodic theorem (Theorem 5.1 below) for nonlinear continuous semigroups of ρ-nonexpansive self-mappings of the Hilbert ball $(\mathbb{B}, \rho)$. This theorem may be considered a Hilbert ball analogue of a recently established dual mean ergodic theorem [32, Theorem 4.1] for nonlinear continuous semigroups of nonexpansive self-mappings of uniformly smooth Banach spaces. Dual mean ergodic theorems for a single nonexpansive mapping go back to [5] and [36]. Recent results regarding nonlinear continuous semigroups on the Hilbert ball and their asymptotic behavior can be found, for example, in [18], [7], [6] and [8].

In the next section we recall several pertinent properties of the hyperbolic metric $\rho : \mathbb{B} \times \mathbb{B} \mapsto \mathbb{R}^+$. Section 3 is devoted to nonlinear ρ-nonexpansive semigroups on $\mathbb{B}$ and to their generation. In Section 4 we discuss Banach limits on the Banach space of all real bounded functions defined on $\mathbb{R}^+$ and the concept of almost convergence. The fifth and last section contains our main result as well as two corollaries.

2. The hyperbolic metric

In this section we collect several relevant properties of the hyperbolic metric on the Hilbert ball. See [2, Section 9], [14, Theorem 2.10], [15] and [19] for more recent results concerning $(\mathbb{B}, \rho)$.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $| \cdot |$, and let $\mathbb{B} := \{x \in H : |x| < 1\}$ be its open unit ball. We denote the sets of natural numbers, the real line, the interval $[0, \infty)$ and the complex plane by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$, respectively. The *hyperbolic metric* $\rho : \mathbb{B} \times \mathbb{B} \mapsto \mathbb{R}^+$ [9, page 98] is

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defined by

$$\rho(x,y) := \operatorname{argtanh}\big(1-\sigma(x,y)\big)^{\frac{1}{2}}, \tag{2.1}$$

where

$$\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-\langle x,y\rangle|^2}, \quad x,y \in \mathbb{B}. \tag{2.2}$$

This metric is the infinite-dimensional analogue of the Poincaré metric on the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We let $B(a,r) := \{x \in \mathbb{B} : \rho(a,x) < r\}$ stand for the ρ-ball of center a and radius r. A subset of $\mathbb{B}$ is called *ρ-bounded* if it is contained in a ρ-ball. We say that a mapping $e : \mathbb{R} \mapsto \mathbb{B}$ is a *metric embedding* of the real line $\mathbb{R}$ into $\mathbb{B}$ if $\rho(e(s), e(t)) = |s-t|$ for all real s and t. The image of $\mathbb{R}$ under a metric embedding is called a *metric line*. The image of a real interval $[a,b] = \{t \in \mathbb{R} : a \le t \le b\}$ under such a mapping is called a *metric segment*. It is known [9, page 102] that for any two distinct points x and y in $\mathbb{B}$, there is a unique metric line (also called a *geodesic*) which passes through x and y. This metric line determines a unique metric segment joining x and y. For each $0 \le t \le 1$, there is a unique point z on this metric segment such that $\rho(x,z) = t\rho(x,y)$ and $\rho(z,y) = (1-t)\rho(x,y)$. This point will be denoted by $(1-t)x \oplus ty$. Similarly, for $r \ge 0$, we let $(1+r)x \ominus ry$ stand for the unique point $z \in \mathbb{B}$ that satisfies $\rho(z,x) = r\rho(x,y)$ and $\rho(z,y) = (1+r)\rho(x,y)$. This point lies on the unique geodesic determined by x and y.

The following inequality [9, page 104] shows that the metric space $(\mathbb{B}, \rho)$ is *hyperbolic* in the sense of [27].

Lemma 2.1. *For any four points a, b, x and y in $\mathbb{B}$, and any number $t \in [0,1]$,*

$$\rho((1-t)a \oplus tx, (1-t)b \oplus ty) \le (1-t)\rho(a,b) + t\rho(x,y). \tag{2.3}$$

Next, we mention another useful property of the hyperbolic metric.

Lemma 2.2. *For any three points $u, v, w \in \mathbb{B}$ and any number $0 \le t \le 1$,*

$$\rho(tv \oplus (1-t)w, u)^2 \le t\rho(v,u)^2 + (1-t)\rho(w,u)^2 - t(1-t)\rho(v,w)^2. \tag{2.4}$$

This is Lemma 2.3 on page 315 of [36]. It shows that the hyperbolic metric ρ is hyperbolically uniformly convex [27, page 541].

Recall that the Möbius transformations of $\mathbb{B}$ [9, page 98] are biholomorphic mappings $M_a : \mathbb{B} \mapsto \mathbb{B}$ of the form

$$M_a(z) = \big(\sqrt{(1-|a|^2)}Q_a + P_a\big)m_a(z), \quad z \in \mathbb{B}, \tag{2.5}$$

where $a \in \mathbb{B}$, P_a is the orthogonal projection of H onto the one-dimensional subspace spanned by a, $Q_a = I - P_a$, and $m_a(z) := (z+a)/(1+\langle z,a\rangle)$. Every Möbius transformation is an automorphism of $\mathbb{B}$ and hence a ρ-isometry. As a matter of fact, any automorphism of $\mathbb{B}$ is of the form $U \circ M_a$ for some unitary operator U on H and a point $a \in \mathbb{B}$ [9, Theorem 14.1].

To each $x \in \mathbb{B}$, we associate a Hilbert space H_x the elements of which are denoted by $\{[x,y] : y \in \mathbb{B}\}$ [34, page 638]. Both the vector space structure and the inner product of H_x are determined by the mapping $i : H_x \mapsto H$ defined by

$$i([x,y]) := \big(\rho(x,y)/|M_{-x}(y)|\big)M_{-x}(y) \tag{2.6}$$

when $y \neq x$ and by $i([x,y]) := 0$ when $y = x$. In particular, the inner product in H_x is given by

$$\langle [x,y],[x,z]\rangle = \frac{\rho(x,y)\cdot\rho(x,z)}{|M_{-x}(y)||M_{-x}(z)|}\langle M_{-x}(y), M_{-x}(z)\rangle, \tag{2.7}$$

where $y \neq x$ and $z \neq x$, and the norm of the element $[x,y] \in H_x$ is $\rho(x,y)$. The spaces H_x and H_y, where $x, y \in \mathbb{B}$, are isometric Hilbert spaces via, for example, the isometry $U_{x,y} : H_x \mapsto H_y$ defined by

$$U_{x,y}[x,z] := [y, M_y(M_{-x}(z))], \quad z \in \mathbb{B}. \tag{2.8}$$

The vector $[x,y] \in H_x$ may be identified with the vector v in the tangent space at x for which $\exp_x(v) = y$, where $\exp_x$ is the exponential map at x.

The following lemma is a special case of Corollary 2.6(a) on page 640 of [34].

Lemma 2.3. *For any $u, v, w \in \mathbb{B}$, define the function $\phi : [0,1] \mapsto \mathbb{R}$ by*

$$\phi(s) := \rho(sv \oplus (1-s)w, u)^2. \tag{2.9}$$

Then the derivative of ϕ at 1^-,

$$\phi'(1^-) := \lim_{s\to 1^-}[\phi(1) - \phi(s)]/(1-s), \tag{2.10}$$

exists and equals $2\,\mathrm{Re}\langle [v,u],[v,w]\rangle$. Moreover, the convergence in

$$\lim_{s\to 1^-}[\phi(1) - \phi(s)]/(1-s) \tag{2.11}$$

is uniform for $u \in D$, where D is any ρ-bounded subset of $\mathbb{B}$.

The following "law of cosines" is Lemma 2.2 on page 638 of [34].

Lemma 2.4. *For any three points $u, v, w \in \mathbb{B}$,*

$$\rho(v,w)^2 \geq \rho(u,v)^2 + \rho(u,w)^2 - 2\,\mathrm{Re}\langle [u,v],[u,w]\rangle. \tag{2.12}$$

The last preparatory lemma we include in this section, the "second law of cosines", is Lemma 2.3 on page 639 of [34].

Lemma 2.5. *For any three points $u, v, w \in \mathbb{B}$,*

$$\rho(u,v)^2 \leq \mathrm{Re}\langle [u,v],[u,w]\rangle + \mathrm{Re}\langle [v,u],[v,w]\rangle. \tag{2.13}$$

Proof. By Lemma 2.4, we have

$$\rho(v,w)^2 \geq \rho(u,v)^2 + \rho(u,w)^2 - 2\,\mathrm{Re}\langle [u,v],[u,w]\rangle \tag{2.14}$$

and

$$\rho(u,w)^2 \geq \rho(u,v)^2 + \rho(v,w)^2 - 2\,\mathrm{Re}\langle [v,u],[v,w]\rangle. \tag{2.15}$$

Adding these two inequalities, we obtain inequality (2.13). □

3. Nonlinear semigroups

This section is devoted to nonlinear ρ-nonexpansive semigroups on $\mathbb{B}$ and to their generation. We start with a discussion of coaccretive operators.

Recall (see [27] and [34]) that a possibly set-valued operator $T \subset \mathbb{B} \times \mathbb{B}$ with domain $D(T)$ and range $R(T)$ is said to be *coaccretive* if

$$\rho(x_1, x_2) \le \rho((1+r)x_1 \ominus ry_1, (1+r)x_2 \ominus ry_2) \tag{3.1}$$

for all $y_1 \in Tx_1,\ y_2 \in Tx_2$, and $r > 0$. Such operators are the Hilbert ball analogues of the operators of the form $T = I - A$, where I denotes the identity operator and A is an accretive operator on a Banach space. In this case, the operator T is also said to be pseudo-contractive [4, page 876]. Let D be a subset of $\mathbb{B}$. A mapping $T : D \mapsto \mathbb{B}$ is called *ρ-nonexpansive* if $\rho(Tx_1, Tx_2) \le \rho(x_1, x_2)$ whenever x_1 and x_2 belong to D. It is known (see, for example, [9, page 91]) that each holomorphic self-mapping of $\mathbb{B}$ is ρ-nonexpansive. Using Lemma 2.1, one can check that all ρ-nonexpansive mappings are coaccretive. An interesting family of (possibly set-valued) coaccretive operators is described on page 641 of [34]. These operators are analogues of subdifferentials of convex functions in Hilbert space. In particular, if $R_C : \mathbb{B} \mapsto C$ is the nearest point projection of $\mathbb{B}$ onto an arbitrary ρ-closed and ρ-convex subset C of $\mathbb{B}$, then the operator $\{(R_C z, 2R_C z \ominus z) : z \in \mathbb{B}\} \subset \mathbb{B} \times \mathbb{B}$ is coaccretive.

When the operator T is coaccretive, one can define, for each positive r, a single-valued ρ-nonexpansive mapping $J_r : R((1+r)I \ominus rT) \mapsto D(T)$, the *resolvent* of T, by

$$J_r((1+r)x \ominus ry) = x, \tag{3.2}$$

where $x \in D(T)$ and $y \in Tx$. These mappings (which in normed linear spaces are indeed the resolvents of the accretive operator $A = I - T$) satisfy the following resolvent identity for all $t \ge s > 0$ and $x \in D(J_t)$:

$$J_t x = J_s((s/t)x \oplus (1 - s/t)J_t x). \tag{3.3}$$

Recall that a mapping $T : D \mapsto \mathbb{B}$ is said to be *firmly nonexpansive of the first kind* [9, page 124] if for each x and y in D, the function $\phi : [0, 1] \mapsto [0, \infty)$, defined by

$$\phi(s) := \rho((1 - s)x \oplus sTx, (1 - s)y \oplus sTy), \quad 0 \le s \le 1, \tag{3.4}$$

is decreasing.

A proof of the following lemma (based on the resolvent identity (3.3)) can be found in [16, Section 2].

Lemma 3.1. *Any resolvent of a coaccretive operator is firmly nonexpansive of the first kind.*

We say that a coaccretive operator $T \subset \mathbb{B} \times \mathbb{B}$ is *m-coaccretive* if

$$R((1 + r)I \ominus rT) = \mathbb{B} \tag{3.5}$$

for all positive r.

Actually, given a coaccretive operator T, the assumption that (3.5) holds when $r = 1$ already implies that it holds for all $r > 0$. Any ρ-nonexpansive mapping $T : \mathbb{B} \mapsto \mathbb{B}$ is m-coaccretive.

A family $\mathcal{S} = \{S(t)\}_{t\geq 0}$ of ρ-nonexpansive self-mappings of the Hilbert ball $(\mathbb{B}, \rho)$ is called a *ρ-nonexpansive continuous (nonlinear) semigroup* on $\mathbb{B}$ if it satisfies the following four conditions:

$$S(r+t) = S(r)S(t) \text{ for all } r, t \geq 0; \tag{3.6}$$

$$S(0) = I; \tag{3.7}$$

$$S(t)x \text{ is continuous in } t \text{ for each } x \in \mathbb{B}; \tag{3.8}$$

$$\rho(S(t)x, S(t)y) \leq \rho(x, y) \text{ for each } t \in \mathbb{R}^+ \text{ and for all } x, y \in \mathbb{B}. \tag{3.9}$$

The following proposition, which is a special case of [27, Theorem 8.1], shows how an m-coaccretive operator generates a nonlinear semigroup on $\mathbb{B}$. We denote the ρ-closure of a subset D of $\mathbb{B}$ by ρ-cl(D).

Proposition 3.2. *Let $T \subset \mathbb{B} \times \mathbb{B}$ be an m-coaccretive operator with domain $D(T)$ and resolvent J_r. If ρ-cl$(D) = \mathbb{B}$, then T generates a ρ-nonexpansive continuous semigroup on $\mathbb{B}$ via the exponential formula*

$$S(t)x = \lim_{n\to\infty} J^n_{t/n} x, \tag{3.10}$$

where $t \geq 0$ and $x \in \mathbb{B}$.

There are other ways to generate ρ-nonexpansive (and, in particular, holomorphic) continuous semigroups on $(\mathbb{B}, \rho)$ (and on other domains in Banach spaces). See, for instance, the papers [26, 28, 29, 30, 18, 13] and the book [31]. We mention, in particular, those semigroups which are generated via Cauchy problems (see, for example, [28] and [18, Corollary 4.2]), and via exponential and other product formulae (see, for example, [26], [28], [29] and [30]). More precisely, if $f : \mathbb{B} \mapsto H$ is bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$, then f is the generator of a ρ-nonexpansive continuous semigroup $\{S(t)\}_{t\geq 0}$ on $\mathbb{B}$ if and only if f is *hyperbolically monotone* ([29] and [13]). In this case,

$$S(t)x = \lim_{n\to\infty} \big(I + (t/n)f\big)^{-n} x, \tag{3.11}$$

where $t \geq 0$ and $x \in \mathbb{B}$. Here the resolvent $T_r = (I+rf)^{-1}$ of f is *firmly nonexpansive of the second kind* [9, page 129]. That is, for each x and y in $\mathbb{B}$, the function $\psi : [0, 1] \mapsto [0, \infty)$, defined by

$$\psi(s) := \rho((1-s)x + sT_r x, (1-s)y + T_r y), \quad 0 \leq s \leq 1, \tag{3.12}$$

is decreasing [13, Lemma 4.2].

We denote by $\mathrm{Fix}(\mathcal{S})$ the set of all *fixed points* (equivalently, stationary points) of a semigroup $\mathcal{S}$, that is, the set of all points $x \in \mathbb{B}$ such that $S(t)x = x$ for all $t \in \mathbb{R}^+$. It is known (see, for example, [38, Theorem 2.9] and [35, Theorem 3.2] for even more general results) that if there is a ρ-bounded subset of $\mathbb{B}$ which is invariant under a ρ-nonexpansive semigroup $\mathcal{S}$, then $\mathcal{S}$ has a stationary point. This fact is a Hilbert ball analogue of [21, Theorem 2] which is concerned with nonexpansive semigroups in Banach spaces.

We end this section with another lemma which will be used in the proof of Theorem 5.1, our main result.

Lemma 3.3. *Let $\mathcal{S} = \{S(t)\}_{t\geq 0}$ be a ρ-nonexpansive continuous semigroup on $\mathbb{B}$ with a nonempty fixed point set $\mathrm{Fix}(\mathcal{S})$. Let $v \in \mathrm{Fix}(\mathcal{S})$ and let M_{-v} be the corresponding Möbius transformation. Then for each point $x \in \mathbb{B}$, both the limits $\lim_{t\to\infty} \rho(v, S(t)x)$ and $\lim_{t\to\infty} |M_{-v}(S(t)x)|$ exist.*

Proof. Let the function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be defined by $g(t) := \rho(v, S(t)x)$, $t \in \mathbb{R}^+$. Then g is decreasing because for each $r \geq 0$ and $t \geq 0$, we have $g(t + r) = \rho(v, S(t+r)x) = \rho(S(t+r)v, S(t+r)x) \leq \rho(S(t)v, S(t)x) = \rho(v, S(t)x) = g(t)$. Since $\rho(v, S(t)x) = \rho(M_{-v}(v), M_{-v}(S(t)x)) = \rho(0, M_{-v}(S(t)x)) = \mathrm{argtanh}|M_{-v}(S(t)x)|$, we also have $|M_{-v}(S(t)x)| = \tanh \rho(v, S(t)x) = \tanh g(t)$ and hence $|M_{-v}(S(t)x)|$ decreases too. $\square$

4. Almost convergence

In this section we briefly discuss Banach limits and the concept of almost convergence.

Let $(B, |\cdot|)$ be the Banach space of all real bounded functions defined on the interval $\mathbb{R}^+ = [0, \infty)$, equipped with the supremum norm. That is, for $x \in B$,

$$|x| := \sup\{|x(s)| : s \in \mathbb{R}^+\}. \tag{4.1}$$

It follows from the Hahn-Banach theorem that there are (linear) functionals Lim, called *Banach limits*, which belong to the unit sphere of the dual space B^*, and have the following four properties for every $x, y \in B$, all $a, b \in \mathbb{R}$, and for each $s_0 \in \mathbb{R}^+$ [1, page 33]:

$$\mathrm{Lim}\,(ax + by) = a\,\mathrm{Lim}\,(x) + b\,\mathrm{Lim}\,(y); \tag{4.2}$$

$$\mathrm{Lim}\,(x) \geq 0 \ \text{ when } x(s) \geq 0 \ \text{ for all } s \in \mathbb{R}^+; \tag{4.3}$$

$$\mathrm{Lim}\,(z) = \mathrm{Lim}\,(x), \text{ where } z(s) = x(s + s_0) \ \text{ for all } s \in \mathbb{R}^+; \tag{4.4}$$

$$\mathrm{Lim}\,(1) = 1. \tag{4.5}$$

It follows that if $x \in B$ and $\lambda = \lim_{s\to\infty} x(s)$ exists, then Lim $(x) = \lambda$. Therefore we will also sometimes write Lim $(x) = \operatorname*{Lim}_{s\to\infty} x(s)$.

Following in the footsteps of Lorentz [20], who introduced this notion for elements of ℓ^∞, we say that an element $x \in B$ is *almost convergent* and the number λ is called its *B-limit* if $\lambda =$ Lim (x) for every Banach limit Lim.

We now quote Theorem 3.2 in [32].

Proposition 4.1. *Let $g \in B$ be a measurable function. If g is almost convergent, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_c^{c+T} g(s)ds \tag{4.6}$$

exists uniformly in $c \geq 0$. Moreover, this limit coincides with the B-limit of g.

We also quote a result in the other direction [32, Theorem 3.3].

Proposition 4.2. *Let $g \in B$ be uniformly continuous. If the limit (4.6) exists uniformly in $c \geq 0$, then g is almost convergent.*

Remark. An example due to I. Shafrir [37] shows that Proposition 4.2 does not hold for all measurable $g \in B$.

We proceed with a Tauberian result.

Proposition 4.3. *Let $g \in B$ be uniformly continuous and almost convergent. If*

$$\lim_{t\to\infty} [g(t+r) - g(t)] = 0 \tag{4.7}$$

for each $r > 0$, then $\lim_{t\to\infty} g(t)$ exists.

Proof. We may and shall assume that the B-limit of g is zero. Given $\epsilon > 0$, there is a number $T_1 > 0$ such that

$$\left| \frac{1}{T_1} \int_c^{c+T_1} g(t)dt \right| < \epsilon/2 \tag{4.8}$$

for all $c \geq 0$. Since g is uniformly continuous, we can also find $T_2 > 0$ so that $|g(t+r) - g(t)| < \epsilon/2$ for all $t \geq T_2$ and $0 \leq r \leq T_1$. Since obviously,

$$g(t) = \frac{1}{T_1} \int_0^{T_1} g(t+r)dr + \frac{1}{T_1} \int_0^{T_1} [g(t) - g(t+r)]dr, \tag{4.9}$$

it follows that

$$|g(t)| \leq \left| \frac{1}{T_1} \int_t^{t+T_1} g(r)dr \right| + \epsilon/2 < \epsilon \tag{4.10}$$

for all $t \geq T_2$. $\square$

Recall that a function $K : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}$ is said to be a *strongly regular kernel* (*cf.*, for example, [10, page 50], [22, page 326], [23, page 270], [24, page 550] and [33, page 58]) if it has the following three properties:

$$\sup \{\int_0^\infty |K(s,t)|dt : s \geq 0\} < \infty; \tag{4.11}$$

$$\lim_{s\to\infty} \int_0^\infty K(s,t)dt = 1; \tag{4.12}$$

$$\lim_{s\to\infty} \int_0^\infty |K(s,t+r) - K(s,t)|dt = 0 \tag{4.13}$$

for each $r \geq 0$.

The following proposition follows, for example, from an argument on page 550 of [24].

Proposition 4.4. *If $g \in B$ is almost convergent to $\lambda \in \mathbb{R}$ and K is a strongly regular kernel, then*

$$\lim_{s\to\infty} \int_0^\infty K(s,r)g(r)dr = \lambda. \tag{4.14}$$

5. A mean ergodic theorem

In this section we state and prove our main result (Theorem 5.1 below). It is a dual mean ergodic theorem for nonlinear continuous semigroups of ρ-nonexpansive self-mappings of the Hilbert ball $(\mathbb{B}, \rho)$. This is indeed a result of a dual nature because, as can be seen from its proof, it deals, in fact, with convergence in the tangent bundle of $(\mathbb{B}, \rho)$.

Theorem 5.1. *Let $(\mathbb{B}, \rho)$ be the Hilbert ball equipped with the hyperbolic metric and let $\mathcal{S} = \{S(t)\}_{t\geq 0}$ be a ρ-nonexpansive continuous semigroup on $\mathbb{B}$. Assume that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Then for each $x \in \mathbb{B}$, there exists a unique point $v \in \mathrm{Fix}(\mathcal{S})$ such that, for each $y \in H$,*

$$\lim_{T\to\infty} \left\langle \frac{1}{T} \int_c^{c+T} M_{-v}(S(t)x)dt,\ y \right\rangle = 0, \tag{5.1}$$

uniformly in $c \geq 0$.

Proof. Fix a point $x \in \mathbb{B}$ and consider the functional $g : \mathbb{B} \times [0, \infty) \mapsto \mathbb{R}$ defined by

$$g(z,t) := \rho(z, S(t)x)^2, \quad z \in \mathbb{B},\ t \geq 0. \tag{5.2}$$

Since the ρ-nonexpansive semigroup $\mathcal{S}$ has a fixed point by assumption, the set $\{S(t)x : t \geq 0\}$ is ρ-bounded and hence, for each fixed $z \in \mathbb{B}$, the function $g(z,t)$ is bounded in the variable $t \in [0, \infty)$. It is also hyperbolically uniformly convex, for each fixed $t \geq 0$, by Lemma 2.2.

Now let Lim be an arbitrary Banach limit on B and define a functional $f : \mathbb{B} \mapsto \mathbb{R}$ by

$$f(z) := \operatorname*{Lim}_{t\to\infty} \rho(z, S(t)x)^2, \quad z \in \mathbb{B}. \tag{5.3}$$

This continuous functional is also hyperbolically uniformly convex and $f(z) \to \infty$ as $|z| \to 1$. Therefore it attains its infimum over $\mathbb{B}$ at a unique point $v \in \mathbb{B}$ [9, Proposition 18.2 on page 108]. We now observe that f is a Lyapunov function for the semigroup $\mathcal{S}$. Indeed, for any $r \geq 0$ and $z \in \mathbb{B}$, we have $f(S(r)z) = \operatorname*{Lim}_{t\to\infty} \rho(S(r)z, S(t)x)^2 = \operatorname*{Lim}_{t\to\infty} \rho(S(r)z, S(t+r)x)^2 \leq \operatorname*{Lim}_{t\to\infty} \rho(z, S(t)x)^2 = f(z)$. Hence v is a fixed point of $\mathcal{S}$.

Next, consider another Banach limit $\widetilde{\mathrm{Lim}}$ on B and the corresponding functional $\tilde{f} : \mathbb{B} \mapsto \mathbb{R}$ defined by

$$\tilde{f}(z) := \widetilde{\operatorname*{Lim}}_{t\to\infty} \rho(z, S(t)x)^2, \quad z \in \mathbb{B}. \tag{5.4}$$

This functional attains its infimum over $\mathbb{B}$ at $\tilde{v} \in \mathrm{Fix}(\mathcal{S})$. Since $\lim\limits_{t\to\infty} \rho(w, S(t)x)^2$ exists for each $w \in \mathrm{Fix}(\mathcal{S})$, we have $f(v) \leq f(\tilde{v}) = \tilde{f}(\tilde{v}) \leq \tilde{f}(v) = f(v)$. Hence $f(v) = \tilde{f}(\tilde{v})$ and so f attains its infimum over $\mathbb{B}$ at the same point $v \in \mathrm{Fix}(\mathcal{S})$ for any Banach limit Lim.

Consider now the function $\phi : [0,1] \mapsto \mathbb{R}$ defined by

$$\phi(s) := \rho(sv \oplus (1-s)w, S(t)x)^2, \quad s \in [0,1], \tag{5.5}$$

where $w \in \mathbb{B}$. We know, by Lemma 2.3, that

$$\lim_{s\to 1^-} [\phi(1) - \phi(s)]/(1-s) = 2\,\mathrm{Re}\langle [v, S(t)x], [v, w]\rangle, \tag{5.6}$$

uniformly in $t \geq 0$. Therefore

$$\lim_{s\to 1^-} [f(v) - f(sv \oplus (1-s)w)]/(1-s) = 2 \operatorname*{Lim}_{t\to\infty} \mathrm{Re}\langle [v, S(t)x], [v, w]\rangle. \tag{5.7}$$

Since the point v minimizes f and since this last equality is true for all $w \in \mathbb{B}$, it follows that

$$\operatorname*{Lim}_{t\to\infty} \mathrm{Re}\langle [v, S(t)x], [v, w]\rangle = 0 \tag{5.8}$$

for all $w \in \mathbb{B}$ and for all Banach limits Lim.

Fix a point $w \in \mathbb{B}$, and let c and T be two real numbers in the interval $[0, \infty)$. Consider the function $h : [0, \infty) \mapsto \mathbb{R}$, defined by

$$h(s) := \mathrm{Re}\langle [v, w], [v, S(s)x]\rangle, \quad s \in [0, \infty). \tag{5.9}$$

This continuous function belongs to B and $\operatorname*{Lim}_{s\to\infty} h(s) = 0$ for any Banach limit Lim. Therefore $h \in B$ is almost convergent to zero by definition and hence

$$\lim_{T\to\infty} \frac{1}{T} \int_c^{c+T} h(t)\,dt = 0, \tag{5.10}$$

uniformly in $c \geq 0$, by Proposition 4.1. Now recall that (see (2.7))

(5.11)
$$h(t) = \mathrm{Re}\langle [v,w],[v,S(t)x]\rangle = \frac{\rho(v,w)\rho(v,S(t)x)}{|M_{-v}(S(t)x)|\cdot|M_{-v}(w)|}\mathrm{Re}\langle M_{-v}(S(t)x), M_{-v}(w)\rangle$$

for all $t \geq 0$. Since both $\lim_{t\to\infty} \rho(v,S(t)x)$ and $\lim_{t\to\infty} |M_{-v}(S(t)x)|$ exist by Lemma 3.3, it follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_c^{c+T} \mathrm{Re}\langle M_{-v}(S(t)x), M_{-v}(w)\rangle dt = 0, \tag{5.12}$$

uniformly in $c \geq 0$. In other words, for each $y \in H$,

$$\lim_{T\to\infty} \frac{1}{T}\int_c^{c+T} \mathrm{Re}\langle M_{-v}(S(t)x), y\rangle dt = 0, \tag{5.13}$$

uniformly in $c \geq 0$. The function $p : [0,\infty) \mapsto H$, defined by

$$p(t) := M_{-v}(S(t)x),\ 0 \leq t < \infty, \tag{5.14}$$

is continuous and Bochner integrable [3] on each interval $[c, c+T]$. Hence

$$\lim_{T\to\infty} \left\langle \frac{1}{T}\int_c^{c+T} M_{-v}(S(t)x)dt,\ y\right\rangle = 0, \tag{5.15}$$

uniformly in $c \geq 0$, as asserted.

To prove the uniqueness of $v \in \mathrm{Fix}(\mathcal{S})$, suppose that there is another point $\tilde{v} \in \mathrm{Fix}(\mathcal{S})$ such that

$$\lim_{T\to\infty} \left\langle \frac{1}{T}\int_c^{c+T} M_{-\tilde{v}}(S(t)x)dt,\ y\right\rangle = 0, \tag{5.16}$$

uniformly in $c \geq 0$. Let

$$h(t) := \mathrm{Re}\langle [v,\tilde{v}],[v,S(t)x]\rangle,\quad 0 \leq t < \infty, \tag{5.17}$$

and

$$\tilde{h}(t) := \mathrm{Re}\langle [\tilde{v},v],[\tilde{v},S(t)x]\rangle,\quad 0 \leq t < \infty. \tag{5.18}$$

Then

$$\lim_{T\to\infty} \frac{1}{T}\int_c^{c+T} h(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_c^{c+T} \tilde{h}(t)dt = 0. \tag{5.19}$$

Adding and using Lemma 2.5, we obtain

$$\rho(v,\tilde{v})^2 \leq \lim_{T\to\infty} \frac{1}{T}\int_c^{c+T} (h(t)+\tilde{h}(t))dt = 0 \tag{5.20}$$

and so $v = \tilde{v}$, as claimed. This completes the proof of Theorem 5.1. $\square$

As far as we know, the first dual mean ergodic theorem for a single nonexpansive self-mapping of a Banach space appeared in [5] and its Hilbert ball analogue was established on pages 316–317 of [36]. Theorem 5.1 confirms an indication, given on page 327 of [36], that a result in its spirit is possible for those ρ-nonexpansive semigroups generated by the exponential formula (3.10). We emphasize, however, that our Theorem 5.1 holds for all those ρ-nonexpansive semigroups on $\mathbb{B}$ which have a stationary point, irrespective of the way they are generated. It seems to be new even in the finite-dimensional case. The proof of Theorem 5.1 may be considered an application of the "optimization method" [25]. It would be of interest to determine if (dual) mean ergodic theorems can be established for semigroups of ρ-nonexpansive and holomorphic self-mappings of other domains in Banach spaces, and, in particular, of (finite) powers $\mathbb{B}^n$ of the Hilbert ball. The papers [11] and [12] contain certain results in this direction for nonlinear semigroups with holomorphic generators. Information on the asymptotic behavior of fixed point free ρ-nonexpansive semigroups on $\mathbb{B}$ can be found, for instance, in [27], [26], [7] and [8].

We conclude this note with two corollaries. The first one provides a Tauberian condition for weak convergence of semigroup trajectories.

Corollary 5.2. *Let $\mathcal{S} = \{S(t)\}_{t\geq 0}$ be a ρ-nonexpansive continuous semigroup on the Hilbert ball $(\mathbb{B}, \rho)$ with a nonempty fixed point set* $\mathrm{Fix}(\mathcal{S})$*, and let $x \in \mathbb{B}$. If*

$$\lim_{t\to\infty} [S(t+r)x - S(t)x] = 0 \tag{5.21}$$

for each positive r, then $S(t)x$ converges weakly as $t \to \infty$ to a fixed point of $\mathcal{S}$.

Proof. Let $v \in \mathrm{Fix}(\mathcal{S})$ be the fixed point the existence of which is guaranteed by Theorem 5.1. It follows from our assumptions that, for each $r > 0$,

$$\lim_{t\to\infty} \rho(S(t+r)x, S(t)x) = 0, \tag{5.22}$$

and since the Möbius transformation M_{-v} is a ρ-isometry, that

$$\lim_{t\to\infty} \rho\big(M_{-v}(S(t+r)x), M_{-v}(S(t)x)\big) = 0 \tag{5.23}$$

and

$$\lim_{t\to\infty} [M_{-v}(S(t+r)x) - M_{-v}(S(t)x)] = 0. \tag{5.24}$$

Theorem 5.1 and Proposition 4.3 now imply that $M_{-v}(S(t)x)$ converges weakly as $t \to \infty$ to the origin of H. Since M_v is weakly continuous [9, Lemma 21.3 on page 116], $M_v \circ M_{-v} = I$, and $M_v(0) = v$, we conclude that the weak $\lim\limits_{t\to\infty} S(t)x = v$, as asserted. □

Since the hyperbolic metric ρ is lower semicontinuous with respect to the weak topology [17], this result provides a ρ-nonexpansive retraction of $\mathbb{B}$ onto the fixed point set of $\mathcal{S}$. It also yields an alternative proof of [26, Corollary 1].

Our second corollary follows from Proposition 4.4.

Corollary 5.3. *Let $\mathcal{S} = \{S(t)\}_{t\geq 0}$ be a ρ-nonexpansive continuous semigroup on the Hilbert ball $(\mathbb{B}, \rho)$ with a nonempty fixed point set* $\mathrm{Fix}(\mathcal{S})$*, and let $x \in \mathbb{B}$. Let*

$v \in \mathrm{Fix}(\mathcal{S})$ be the fixed point the existence of which is guaranteed by Theorem 5.1. If the kernel K is strongly regular and

$$R(s)x := \int_0^\infty K(s,r)M_{-v}(S(r)x)dr, \tag{5.25}$$

then $R(s)x$ converges weakly as $s \to \infty$ to the origin of H.

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