Journal of Nonlinear and Convex Analysis
Volume 11, Number 1, 2010, 89–99

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ALMOST CONVERGENCE AND A DUAL ERGODIC THEOREM FOR NONLINEAR SEMIGROUPS

SIMEON REICH AND AYA WALLWATER

Abstract. We use Banach limits and the concept of almost convergence to establish a dual mean ergodic theorem for nonexpansive nonlinear semigroups in uniformly smooth Banach spaces.

1. Introduction

The main purpose of this note is to establish a dual mean ergodic theorem for continuous nonlinear semigroups of nonexpansive mappings in uniformly smooth Banach spaces (Theorem 4.1 below). This is a continuous analog of the discrete dual ergodic theorem presented in [4]. Since the proof uses Banach limits, we devote the next section of our note to them. Section 3 is devoted to the concept of almost convergence introduced by Lorentz in [5]. After proving our main result in Section 4, we deduce from it a version of the mean ergodic theorem for nonlinear semigroups in Hilbert space (Corollary 4.2).

2. Banach limits

Consider the space $(B, \|\cdot\|)$ of all real bounded functions defined on the interval $[0, \infty)$, equipped with the supremum norm. That is, for each $x \in B$,

$$\|x\| := \sup\{|x(s)| : s \geq 0\}. \tag{2.1}$$

It is well known that B is a Banach space.

Let $p : B \to \mathbb{R}$ be the functional defined by

$$p(x) := \inf_{n;\, \alpha_1,\ldots,\alpha_n} \left[\limsup_{s\to\infty} \frac{1}{n} \sum_{k=1}^{n} x(s+\alpha_k) \right], \tag{2.2}$$

where n is a natural number and $\{\alpha_1, ..., \alpha_n\}$ is an arbitrary set of n positive numbers.

It is not difficult to check that p is a sublinear and positively homogeneous functional over B.

It now follows from the Hahn-Banach theorem [1, p. 33] that to each function $x \in B$, one can assign a real number $\operatorname*{Lim}_{s\to\infty} x(s)$ so that for every $x, y \in B$ and for

2000 *Mathematics Subject Classification.* 46T25, 47H09, 47H10, 47H20, 47H25.

Key words and phrases. Banach limit, Banach space, duality mapping, fixed point, mean ergodic theorem, nonexpansive mapping, nonlinear semigroup.

The first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion President's Research Fund.

every $a, b \in \mathbb{R}$ and $s_0 \geq 0$,

$$-p\,(-x) \leq \operatorname*{Lim}_{s\to\infty} x\,(s) \leq p\,(x)\,; \tag{2.3}$$

$$\operatorname*{Lim}_{s\to\infty} [ax\,(s) + by\,(s)] = a\operatorname*{Lim}_{s\to\infty} x\,(s) + b\operatorname*{Lim}_{s\to\infty} y\,(s)\,; \tag{2.4}$$

$$\operatorname*{Lim}_{s\to\infty} x\,(s) \geq 0 \text{ when } x\,(s) \geq 0 \text{ for all } s \geq 0; \tag{2.5}$$

$$\operatorname*{Lim}_{s\to\infty} x\,(s+s_0) = \operatorname*{Lim}_{s\to\infty} x\,(s)\,; \tag{2.6}$$

$$\operatorname*{Lim}_{s\to\infty} 1 = 1. \tag{2.7}$$

Defining the linear functional Lim on B by

$$\operatorname{Lim}(x) := \operatorname*{Lim}_{s\to\infty} x\,(s)\,, \quad x \in B,$$

we see that Lim belongs to the dual space B^* of B and that $\|\operatorname{Lim}\| = 1$. We call any such functional a *Banach limit* on B. If $x \in B$ and $L = \lim_{s\to\infty} x(s)$ exists, then $\operatorname{Lim}(x) = L$.

Next, we relate Banach limits to Gâteaux differentiability. Let E be an arbitrary real Banach space.

Definition 2.1. We say that $g : E \times [0, \infty) \to \mathbb{R}$ is *Gâteaux differentiable* in the variable x in E, uniformly in $t \geq 0$, if for each x in E and y in the unit sphere of E, the limit

$$g'_x\,(x,t)\,(y) = \lim_{r\to 0} \frac{g\,(x+ry,t) - g\,(x,t)}{r}$$

exists uniformly in $t \geq 0$.

Lemma 2.2. *Let $g : E \times [0, \infty) \to \mathbb{R}$ be Gâteaux differentiable in the variable x in E, uniformly in $t \geq 0$, and bounded in the variable $t \in [0, \infty)$. Let* Lim *be a Banach limit and define $f : E \to \mathbb{R}$ by $f\,(x) = \operatorname*{Lim}_{t\to\infty} g\,(x,t)$. Then f is Gâteaux differentiable at every point x in E and*

$$f'\,(x)\,(y) = \operatorname*{Lim}_{t\to\infty} g'_x\,(x,t)\,(y)$$

for each x in E and y in the unit sphere of E.

Proof. Fix $x \in E$, y in the unit sphere of E, and $\epsilon > 0$. There is a positive number $\delta = \delta\,(\epsilon)$ such that, if $|r| < \delta$, then

$$\left| \frac{g\,(x+ry,t) - g\,(x,t)}{r} - g'_x\,(x,t)\,(y) \right| < \frac{\epsilon}{2}.$$

for all $t \geq 0$. Hence

$$\left| \operatorname*{Lim}_{t\to\infty} \left[\frac{g\,(x+ry,t) - g\,(x,t)}{r} - g'_x\,(x,t)\,(y) \right] \right| \leq \frac{\epsilon}{2}$$

and

$$\left|\frac{f(x+ry)-f(x)}{r}-\operatorname*{Lim}_{t\to\infty} g_x'(x,t)(y)\right|<\epsilon$$

for all $|r|<\delta$. Thus f is indeed Gâteaux differentiable at x and

$$f'(x)(y)=\operatorname*{Lim}_{t\to\infty} g_x'(x,t)(y)$$

as claimed. □

3. Almost convergence

Following G. G. Lorentz [5], who studied Banach limits on l^∞, we say that $x\in B$ is *almost convergent* if $\operatorname*{Lim}_{s\to\infty} x(s)$ is the same for every Banach limit Lim on B. In this case we call $\operatorname*{Lim}_{s\to\infty} x(s)$ the B-limit of x. If $L=\lim_{s\to\infty} x(s)$ exists, then x is clearly almost convergent and its B-limit coincides with L.

The following proposition provides us with a characterization of almost convergence in terms of the functional $p:B\to\mathbb{R}$.

Proposition 3.1. *The function $x\in B$ is almost convergent if and only if*

$$p(x)=-p(-x).$$

The discrete analog of this proposition is presented by Lorentz in [5, p. 169]. Proposition 3.1 can be proved using arguments analogous to his [12, p. 11].

Theorem 3.2. *Let $x\in B$ be a measurable function. If x is almost convergent, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_c^{c+T} x(s)\,ds$$

exists uniformly in $c\geq 0$. Moreover, this limit coincides with the B-limit of x.

Proof. Assume that $x\in B$ is almost convergent. Then there exists $a\in\mathbb{R}$ such that $p(x)=-p(-x)=a$. Fix $\epsilon>0$. The equality

$$p(x)=\inf_{n;\,\alpha_1,\ldots,\alpha_n}\left[\limsup_{s\to\infty}\frac{1}{n}\sum_{k=1}^{n} x(s+\alpha_k)\right]=a$$

implies that there exist $n;\alpha_1,\ldots,\alpha_n$ such that

$$\lim_{s\to\infty}\left(\sup_{t\geq s}\tfrac{1}{n}\sum_{k=1}^{n} x(t+\alpha_k)\right)=\limsup_{s\to\infty}\tfrac{1}{n}\sum_{k=1}^{n} x(s+\alpha_k)<a+\epsilon.$$

Therefore, for sufficiently large s, say $s\geq s_0$, we have

$$\frac{1}{n}\sum_{k=1}^{n} x(s+\alpha_k)<a+\epsilon.$$

Substituting $\beta_k=\alpha_k+s_0$ and $t=s-s_0$, we get

$$\frac{1}{n}\sum_{k=1}^{n} x(t+\beta_k)<a+\epsilon,\qquad \forall t\geq 0.$$

We now obtain, for every $c\geq 0$,

$$T\,(a+\epsilon) \geq \int\limits_{c}^{c+T} \frac{1}{n}\sum_{k=1}^{n} x\,(t+\beta_k)\,dt = \frac{1}{n}\sum_{k=1}^{n}\int\limits_{c}^{c+T} x\,(t+\beta_k)\,dt$$

$$\underset{t_k=t+\beta_k}{=} \frac{1}{n}\sum_{k=1}^{n}\int\limits_{c+\beta_k}^{c+T+\beta_k} x\,(t_k)\,dt_k$$

$$= \int\limits_{c}^{c+T} x\,(t)\,dt + \frac{1}{n}\sum_{k=1}^{n}\left[\int\limits_{c+T}^{c+T+\beta_k} x\,(t)\,dt - \int\limits_{c}^{c+\beta_k} x\,(t)\,dt\right]$$

$$\underset{M=\sup\limits_{s\geq 0}|x(s)|}{\geq} \int\limits_{c}^{c+T} x\,(t)\,dt - \frac{2M}{n}\sum_{k=1}^{n}\beta_k.$$

Hence, for every $c \geq 0$, we have

$$\frac{1}{T}\int\limits_{c}^{c+T} x\,(t)\,dt \leq a+\epsilon+\frac{1}{T}\left(\frac{2M}{n}\sum_{k=1}^{n}\beta_k\right).$$

Therefore, for any large enough T, we get

$$\frac{1}{T}\int\limits_{c}^{c+T} x\,(t)\,dt < a+2\epsilon, \qquad \forall c \geq 0 \tag{3.1}$$

On the other hand, the equality

$$-p\,(-x) = \sup_{n;\,\alpha_1,\ldots,\alpha_n}\left[\liminf_{s\to\infty}\frac{1}{n}\sum_{k=1}^{n} x\,(s+\alpha_k)\right] = a$$

implies that there exist $n; \alpha_1, ..., \alpha_n$ such that

$$\lim_{s\to\infty}\left(\inf_{t\geq s}\frac{1}{n}\sum_{k=1}^{n} x\,(t+\alpha_k)\right) = \liminf_{s\to\infty}\frac{1}{n}\sum_{k=1}^{n} x\,(s+\alpha_k) > a-\epsilon.$$

Therefore, for sufficiently large s, say $s \geq \tilde{s}_0$, we have

$$\frac{1}{n}\sum_{k=1}^{n} x\,(s+\alpha_k) > a-\epsilon.$$

Substituting $\beta_k = \alpha_k + \tilde{s}_0$ and $t = s - \tilde{s}_0$, we get

$$\frac{1}{n}\sum_{k=1}^{n} x\,(t+\beta_k) > a-\epsilon, \qquad \forall t \geq 0.$$

For every $c \geq 0$, we now have

$$T(a-\epsilon) \le \int_c^{c+T} \frac{1}{n}\sum_{k=1}^{n} x(t+\beta_k)\,dt = \frac{1}{n}\sum_{k=1}^{n}\int_c^{c+T} x(t+\beta_k)\,dt$$

$$\underset{t_k=t+\beta_k}{=} \frac{1}{n}\sum_{k=1}^{n}\int_{c+\beta_k}^{c+T+\beta_k} x(t_k)\,dt_k$$

$$= \int_c^{c+T} x(t)\,dt + \frac{1}{n}\sum_{k=1}^{n}\left[\int_{c+T}^{c+T+\beta_k} x(t)\,dt - \int_c^{c+\beta_k} x(t)\,dt\right]$$

$$\underset{M=\sup_{s\ge 0}|x(s)|}{\le} \int_c^{c+T} x(t)\,dt + \frac{2M}{n}\sum_{k=1}^{n}\beta_k.$$

Hence, for every $c \ge 0$,

$$\frac{1}{T}\int_c^{c+T} x(t)\,dt \ge a - \epsilon - \frac{1}{T}\left(\frac{2M}{n}\sum_{k=1}^{n}\beta_k\right).$$

Therefore, for any large enough T, we obtain

$$\frac{1}{T}\int_c^{c+T} x(t)\,dt > a - 2\epsilon \tag{3.2}$$

for all $c \ge 0$.

Combining (3.1) and (3.2), we see that if x is almost convergent to $a \in \mathbb{R}$, then

$$\lim_{T\to\infty}\frac{1}{T}\int_c^{c+T} x(t)\,dt = a,$$

uniformly in $c \ge 0$, as claimed. □

In the other direction we have the following result.

Theorem 3.3. *Let $x \in B$ be a measurable function. If x is uniformly continuous and the limit*

$$\lim_{T\to\infty}\frac{1}{T}\int_c^{c+T} x(s)\,ds$$

exists uniformly in $c \ge 0$, then x is almost convergent.

Proof. Let x be a uniformly continuous function on $[0,\infty)$. Fix $\epsilon > 0$ and assume that

$$\lim_{T\to\infty}\frac{1}{T}\int_c^{c+T} x(t)\,dt = a,$$

uniformly in $c \geq 0$. Then there exists T_0 such that for all $T \geq T_0$ and $c \geq 0$,

$$a - \epsilon < \frac{1}{T} \int\limits_{c}^{c+T} x(t)\, dt < a + \epsilon.$$

Without loss of generality we may assume that $T_0 \geq 1$. Since $x(s)$ is uniformly continuous, there exists $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow |x(s) - x(t)| < \frac{\epsilon}{T_0}$$

for all $s, t \geq 0$. Fix n_0 such that $\frac{T_0}{n_0} < \delta$. Let $c \geq 0$ and consider the interval $[c, c + T_0]$. Define a partition of $[c, c + T_0]$ by

$$c = t_0 < t_1 < \ldots < t_{n_0} = c + T_0,$$

where $t_k = c + \frac{T_0}{n_0} k = c + \alpha_k$ and $\alpha_k = \frac{T_0}{n_0} k$. Denote by $U(x)$ and $L(x)$ the upper and lower Darboux sums, respectively, that is,

$$U(x) = \sum_{i=1}^{n_0} x\left(t_i^U\right)(t_i - t_{i-1})$$

and

$$L(x) = \sum_{i=1}^{n_0} x\left(t_i^L\right)(t_i - t_{i-1}),$$

where

$$t_i^U = \text{argmax}\{x(t) : \ t \in [t_{i-1}, t_i]\}$$

and

$$t_i^L = \text{argmin}\{x(t) : \ t \in [t_{i-1}, t_i]\}.$$

Then

$$U(x) - L(x) \leq \sum_{i=1}^{n_0} \left| x\left(t_i^U\right) - x\left(t_i^L\right) \right| |t_i - t_{i-1}| < \sum_{i=1}^{n_0} \frac{\epsilon}{T_0} |t_i - t_{i-1}|$$

$$= \frac{\epsilon}{T_0} T_0 = \epsilon$$

and

$$L(x) \leq \int\limits_{c}^{c+T_0} x(t)\, dt \leq U(x).$$

On the one hand, we have

$$T_0(a + \epsilon) > \int\limits_{c}^{c+T_0} x(t)\, dt \geq L(x) > U(x) - \epsilon = \sum_{i=1}^{n_0} x\left(t_i^U\right) \underbrace{|t_i - t_{i-1}|}_{\frac{T_0}{n_0}} - \epsilon$$

$$\geq \frac{T_0}{n_0} \sum_{i=1}^{n_0} x(c + \alpha_i) - \epsilon$$

$\Rightarrow \frac{1}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) < a + \epsilon \left(1 + \frac{1}{T_0}\right).$

On the other hand,

$$T_0 (a - \epsilon) < \int_c^{c+T_0} x(t)\, dt \le U(x) < L(x) + \epsilon = \sum_{i=1}^{n_0} x\left(t_i^L\right) \underbrace{|t_i - t_{i-1}|}_{\frac{T_0}{n_0}} + \epsilon$$

$$\le \frac{T_0}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) + \epsilon$$

$\Rightarrow \frac{1}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) > a - \epsilon \left(1 + \frac{1}{T_0}\right).$

Therefore $\forall c \ge 0$,

$$a - 2\epsilon \le a - \epsilon \left(1 + \frac{1}{T_0}\right) < \frac{1}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) < a + \epsilon \left(1 + \frac{1}{T_0}\right) \le a + 2\epsilon.$$

It follows that

$$a - 2\epsilon \le \liminf_{c\to\infty} \frac{1}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) \le \limsup_{c\to\infty} \frac{1}{n_0} \sum_{i=1}^{n_0} x(c+\alpha_i) \le a + 2\epsilon.$$

Since

$$p(x) = \inf_{n;\, \gamma_1,\ldots,\gamma_n} \left[\limsup_{c\to\infty} \frac{1}{n} \sum_{k=1}^{n} x(c+\gamma_k)\right]$$

and

$$-p(-x) = \sup_{m;\, \beta_1,\ldots,\beta_m} \left[\liminf_{c\to\infty} \frac{1}{m} \sum_{k=1}^{m} x(c+\beta_k)\right]$$

we obtain

$$a - 2\epsilon \le -p(-x) \le p(x) \le a + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$p(x) = -p(-x)$$

and hence $x(s)$ is almost convergent by Proposition 3.1, as asserted. □

Remark. An example due to I. Shafrir [10] shows that Theorem 3.3 is not true for all measurable $x \in B$.

4. Ergodic theory

In this section we formulate and prove our main result. Recall that a *nonexpansive continuous (nonlinear) semigroup* on a Banach space E is a family $\mathcal{S} = \{S(t)\}_{t\ge 0}$ of (nonexpansive) self-mappings of E satisfying the following conditions:

(4.1) $$S(t+r) = S(t)\, S(r) \text{ for all } t, r \ge 0;$$

$$S(0) = I; \tag{4.2}$$

$$S(t)\,x \text{ is continuous in } t \text{ for each } x \in E; \tag{4.3}$$

$$\|S(t)\,x - S(t)\,y\| \le \|x - y\| \text{ for all } t \ge 0 \text{ and } x, y \in E. \tag{4.4}$$

We denote by $Fix\mathcal{S}$ the set of all common fixed points of $\mathcal{S}$. That is,

$$Fix\mathcal{S} := \{x \in E : \ S(t)\,x = x \text{ for all } t \ge 0\}.$$

Let E^* denote the dual of an arbitrary real Banach space $(E, \|\cdot\|)$. We denote the value of $x^* \in E^*$ at $x \in E$ either by $x^*(x)$ or by $\langle x, x^*\rangle$. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}. \tag{4.5}$$

We are now ready to formulate and prove our main result, where we use the integral introduced by Bochner in [2].

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a real uniformly smooth Banach space and let $\mathcal{S} = \{S(t)\}_{t\ge 0}$ be a nonexpansive continuous semigroup on E. Assume that $Fix\mathcal{S} \ne \emptyset$. Then for each $x \in E$, there exists a point $v \in Fix\mathcal{S}$ such that, for each $y \in E$,*

$$\lim_{T\to\infty} \left\langle y, \frac{1}{T} \int_c^{c+T} J(S(t)\,x - v)\,dt \right\rangle = 0,$$

uniformly in $c \ge 0$.

Proof. Fix $x \in E$ and consider the function $g : E \times [0, \infty) \to \mathbb{R}$ defined by

$$g(z, t) := \|S(t)\,x - z\|^2, \ z \in E, \ t \ge 0. \tag{4.6}$$

Since the nonexpansive semigroup $\mathcal{S}$ has a fixed point by assumption, this function is clearly bounded, for each fixed $z \in E$, in the variable $t \in [0, \infty)$.

Now let Lim be an arbitrary Banach limit on B and define the function $f : E \to \mathbb{R}$ by

$$f(z) := \operatorname*{Lim}_{t\to\infty} \|S(t)\,x - z\|^2, \ z \in E. \tag{4.7}$$

It is clear that f is a well-defined convex function and that

$$\lim_{\|z\|\to\infty} f(z) = \infty. \tag{4.8}$$

We denote the set of minimizers of f over E by A.

Using the uniform smoothness of $(E, \|\cdot\|)$ (which is equivalent to the uniform Fréchet differentiability of $\|\cdot\|^2$) and applying Lemma 2.2, we see that the duality mapping J of E is single-valued and continuous, the function f is Gâteaux differentiable at each point $z \in E$, and that

$$f'(z)(y) = 2\operatorname*{Lim}_{t\to\infty} \langle -y, J(S(t)\,x - z)\rangle \tag{4.9}$$

for all $y \in E$.

Since the space E is reflexive, and the continuous and convex function f satisfies (4.8), the set of minimizers A is non-empty, bounded, closed and convex.

We also observe that f is a Liapunov function for the semigroup $\mathcal{S}$. Indeed, for any $r \geq 0$ and $z \in E$, we have

$$f(S(r)z) = \operatorname*{Lim}_{t\to\infty} \|S(t)x - S(r)z\|^2 = \operatorname*{Lim}_{t\to\infty} \|S(t+r)x - S(r)z\|^2$$
$$= \operatorname*{Lim}_{t\to\infty} \|S(r)S(t)x - S(r)z\|^2 \leq \operatorname*{Lim}_{t\to\infty} \|S(t)x - z\|^2 = f(z).$$

Hence A is invariant under $\mathcal{S}$. Recalling that uniformly smooth Banach spaces have the fixed point property for nonexpansive mappings and invoking either a common fixed point theorem of Bruck [3, p. 59] or a more recent one by Suzuki [11, p. 1016], we conclude that A must contain a fixed point v of $\mathcal{S}$.

Consider now another Banach limit $\widetilde{\operatorname{Lim}}$ on B and the corresponding function $\widetilde{f} : E \to \mathbb{R}$ defined by

$$\widetilde{f}(z) := \widetilde{\operatorname*{Lim}}_{t\to\infty} \|S(t)x - z\|^2, \ z \in E. \tag{4.10}$$

This function attains its minimum over E at a fixed point $\widetilde{v}$ of $\mathcal{S}$. Since $\lim_{t\to\infty} \|S(t)x - w\|^2$ exists for all $w \in Fix\mathcal{S}$, we have

$$f(v) \leq f(\widetilde{v}) = \widetilde{f}(\widetilde{v}) \leq \widetilde{f}(v) = f(v).$$

Hence $f(v) = \widetilde{f}(\widetilde{v})$ and so f attains its minimum over E at the same point $v \in Fix\mathcal{S}$ for any Banach limit Lim. Therefore, by (4.9),

$$f'(v)(y) = 2\operatorname*{Lim}_{t\to\infty} \langle -y, J(S(t)x - v)\rangle = 0 \tag{4.11}$$

for any Banach limit Lim and each $y \in E$.

Now fix a point $y \in E$, and let c and T be two numbers in $[0,\infty)$. Consider the function $h : [0,\infty) \to E^*$ defined by

$$h(t) := J(S(t)x - v), \ 0 \leq t < \infty, \tag{4.12}$$

and the function $u : [0,\infty) \to \mathbb{R}$ defined by

$$u(s) := \langle y, h(s)\rangle, \ s \in [0,\infty). \tag{4.13}$$

The function h is Bochner integrable and continuous, and the function u belongs to B. Hence

$$\left\langle y, \int_c^{c+T} h(t)\,dt \right\rangle = \int_c^{c+T} \langle y, h(t)\rangle\,dt = \int_c^{c+T} u(t)\,dt. \tag{4.14}$$

Since $\operatorname*{Lim}_{s\to\infty} u(s) = 0$ for any Banach limit Lim, the function $u \in B$ is almost convergent to 0 by definition and

$$\lim_{T\to\infty} \frac{1}{T} \int_c^{c+T} u(t)\,dt = 0, \tag{4.15}$$

uniformly in $c \geq 0$, by Theorem 3.2.

It now follows from (4.14) that

$$\lim_{T\to\infty}\left\langle y, \frac{1}{T}\int_{c}^{c+T} J\left(S\left(t\right)x - v\right)dt\right\rangle = 0, \tag{4.16}$$

uniformly in $c \geq 0$, as asserted. □

The proof of Theorem 4.1 may be considered an application of the so-called "optimization method" [9].

In the special case where E is a Hilbert space, we see that the semigroup $\mathcal{S} = \{S\left(t\right)\}_{t\geq 0}$ is weakly almost convergent to a fixed point of $\mathcal{S}$.

Corollary 4.2. *Let H be a real Hilbert space and let $\mathcal{S} = \{S\left(t\right)\}_{t\geq 0}$ be a nonexpansive continuous semigroup on H. Assume that $Fix\mathcal{S} \neq \emptyset$. Then there exists a point $v \in Fix\mathcal{S}$ such that, for each $y \in H$,*

$$\lim_{T\to\infty}\left\langle y, \frac{1}{T}\int_{c}^{c+T} S\left(t\right)xdt\right\rangle = \langle y, v\rangle\,,$$

uniformly in $c \geq 0$.

We do not know if Theorem 4.1 remains valid when the semigroup $\mathcal{S}$ is defined only on a closed and convex subset of the Banach space E. However, Corollary 4.2 is known to hold in this case too (*cf.* [6, p. 327], [7, p. 270] and [8, p. 547]).

References

[1] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

[2] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind,* Fund. Math. **20** (1933), 262–276.

[3] R. E. Bruck, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific J. Math. **53** (1974), 59–71.

[4] R. E. Bruck and S. Reich, *Accretive operators, Banach limits, and dual ergodic theorems*, Bull. Acad. Polon. Sci. **29** (1981), 585–589.

[5] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta. Math. **80** (1948), 167–190.

[6] S. Reich, *Nonlinear evolution equations and nonlinear ergodic theorems*, Nonlinear Anal. **1** (1977), 319–330.

[7] S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory **24** (1978), 269–272.

[8] S. Reich, *A note on the mean ergodic theorem for nonlinear semigroups*, J. Math. Anal. Appl. **91** (1983), 547–551.

[9] S. Reich, *Convergence, resolvent consistencey, and the fixed point property for nonexpansive mappings*, Contemporary Math. **18** (1983), 167–174.

[10] I. Shafrir, Personal communication, 2009.

[11] T. Suzuki, *Common fixed points of one parameter nonexpansive semigroups*, Bull. London Math. Soc. **38** (2006), 1009–1018.

[12] A. Wallwater, *Almost Convergence and a Dual Ergodic Theorem for Nonlinear Semigroups in Banach Spaces*, M. Sc. Thesis, The Technion - Israel Institute of Technology, Haifa, 2009.

Manuscript received August 13, 2009
revised September 28, 2009

Simeon Reich
Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il

Aya Wallwater
Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: aya.wallwater@gmail.com