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FIXED POINT THEOREMS FOR NEW NONLINEAR MAPPINGS IN A HILBERT SPACE

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Abstract. In this paper, we first consider nonlinear mappings which are deduced from an equilibrium problem in a Hilbert space. Further, we deal with fixed point theorems for the nonlinear mappings in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping F is also said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [2], Goebel and Kirk [4], Goebel and Reich [5], Reich and Shoikhet [11] and Takahashi [14]. It is known that a mapping $F : C \to H$ is firmly nonexpansive if and only if

$$\|Fx - Fy\|^2 + \|(I - F)x - (I - F)y\|^2 \le \|x - y\|^2$$

for all $x, y \in C$, where I is the identity mapping on H. It is also known that a firmly nonexpansive mapping F is deduced from an equilibrium problem in a Hilbert space as follows: Let C be a nonempty closed convex subset of H and let us assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditons:

(A1) $f(x, x) = 0, \quad \forall x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0, \quad \forall x, y \in C$;
(A3) $\lim_{t\downarrow 0} f(tz + (1 - t)x, y) \le f(x, y), \quad \forall x, y, z \in C$;
(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [1] and [3].

Lemma 1.1. *Let C be a nonempty closed convex subset of H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C\}$ for all $r > 0$ and $x \in H$, then the following hold:

(1) *T_r is single-valued;*

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(2) *T_r is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H.$$

Recently, Kohsaka and Takahashi [8] introduced the following nonlinear mapping: Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to E$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping S in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$.

In this paper, we first consider nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. Further, we deal with fixed point theorems for the nonlinear mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \tag{1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see, for instance, [16]. Further, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2}$$

for all $x, y, z, w \in H$. Indeed, we have that

$$\begin{aligned} 2\langle x - y, z - w\rangle &= 2\langle x, z\rangle - 2\langle x, w\rangle - 2\langle y, z\rangle + 2\langle y, w\rangle \\ &= (-\|x\|^2 + 2\langle x, z\rangle - \|z\|^2) + (\|x\|^2 - 2\langle x, w\rangle + \|w\|^2) \\ &\quad + (\|y\|^2 - 2\langle y, z\rangle + \|z\|^2) + (-\|y\|^2 + 2\langle y, w\rangle - \|w\|^2) \\ &= \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2. \end{aligned}$$

Let C be a closed convex subset of H and let T be a mapping of C into H. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of x_n to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A mapping $T : C \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We can prove from (1) that $F(T)$ is closed and convex. We also know that if C is a bounded closed convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. A mapping $F : C \to H$ is firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$. A mapping $S : C \to H$ is nonspreading if

$$2\,\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. From Kohsaka and Takahashi [8], we know the following fixed point theorems.

Theorem 2.1 ([8])**.** *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then the following are equivalent:*

(i) *There exists $x \in C$ such that $\{S^n x\}$ is bounded;*
(ii) *$F(S)$ is nonempty.*

Theorem 2.2 ([8])**.** *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

From Iemoto and Takahashi [6], we know the following lemma.

Lemma 2.3. *Let C be a nonempty closed convex subset of H. Then a mapping $S : C \to H$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\,\langle x - Sx, y - Sy\rangle$$

for all $x, y \in C$.

3. Nonlinear mappings

Let H be a Hilbert space. Let C be a nonempty closed convex subset of H and let T be a mapping of C into H. Then, we have the following equality:

$$\|Tx - Ty\|^2 = \|x - y - (Tx - Ty)\|^2 - \|x - y\|^2 + 2\langle x - y, Tx - Ty\rangle \tag{3}$$

for all $x, y \in C$. We have also from (2) that

$$2\,\langle x - y, Tx - Ty\rangle = \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2. \tag{4}$$

Further, we have that

$$\|x - y - (Tx - Ty)\|^2 = \|x - Tx\|^2 + \|y - Ty\|^2 - 2\langle x - Tx, y - Ty\rangle. \tag{5}$$

If $T : C \to H$ is firmly nonexpansive, then for any $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

So, we have from (3) that

$$\begin{aligned} 2\|Tx - Ty\|^2 &\leq 2\langle x - y, Tx - Ty\rangle \\ &= \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 + \|x - y\|^2 \\ &\leq \|Tx - Ty\|^2 + \|x - y\|^2. \end{aligned}$$

Then, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2$$

and hence

$$\|Tx - Ty\| \leq \|x - y\|.$$

Such a mapping is nonexpansive. Thus, we can obtain other nonlinear operators from a firmly nonexpansive mapping in a Hilbert space. Kohsaka and Takahahi [8] obtained a nonspreading mapping from a firmly nonexpansive mapping. Let $T : C \to H$ be a firmly nonexpansive mapping. Then, we have, for any $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty\rangle.$$

From (4), we obtain

$$\begin{aligned} 2\|Tx - Ty\|^2 &\leq \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 \\ &\leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned}$$

So, we have

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

This is a nonspreading mapping. Further, we define a new nonlinear operator from a firmly nonexpansive mapping. We have that for any $x, y \in C$,

$$\begin{aligned} &2\|Tx-Ty\|^2 \leq 2\langle x - y, Tx - Ty\rangle \\ &\quad \iff \|Tx - Ty\|^2 + \|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, Ty\rangle \leq 2\langle x - y, Tx - Ty\rangle \\ &\quad \implies \|Tx - Ty\|^2 - 2\langle Tx, Ty\rangle \leq 2\langle x - y, Tx - Ty\rangle \\ &\quad \iff \|Tx - Ty\|^2 \leq 2\langle Tx, Ty\rangle + 2\langle x - y, Tx - Ty\rangle. \end{aligned}$$

So, we can define a new mapping called a metric mapping, i.e.,

$$\|Tx - Ty\|^2 \leq 2\langle Tx, Ty\rangle + 2\langle x - y, Tx - Ty\rangle$$

for all $x, y \in C$. Finally, we obtain another new nonlinear mapping from a firmly nonexpansive mapping. We have from (3) and (5) that for any $x, y \in C$,

$$\begin{aligned} &4\|Tx-Ty\|^2 \leq 4\langle x - y, Tx - Ty\rangle \\ &\quad \iff 4\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty\rangle + 2\langle x - y, Tx - Ty\rangle \\ &\quad \iff 4\|Tx - Ty\|^2 \leq \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 + \|x - y\|^2 \\ &\quad + \|Tx - Ty\|^2 + \|x - y\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 + 2\langle x - Tx, y - Ty\rangle \\ &\quad \implies 4\|Tx - Ty\|^2 \leq 2\|Tx - Ty\|^2 + 2\|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \\ &\quad \iff 2\|Tx - Ty\|^2 \leq 2\|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \\ &\quad \iff \|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle. \end{aligned}$$

So, if $T : C \to H$ is firmly nonexpansive, then T satislies that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. We call such a mapping a hybrid mapping. A hybrid mapping $T : C \to H$ is different from a nonspreading mapping. In fact, from Lemma 2.3, we know that for any $x, y \in C$,

$$2\|Tx - Ty\|^2 \le \|y - Tx\|^2 + \|x - Ty\|^2 \\ \iff \|Tx - Ty\|^2 \le \|x - y\| + 2\langle x - Tx, y - Ty\rangle.$$

So, a hybrid mapping $T : C \to H$ is different from a nonspreading mapping.

Let $T : C \to H$ be a nonexpansive mapping and put $A = I - T$. Then, we have from [16] that A is 1/2-inverse strongly monotone, i.e.,

$$\frac{1}{2}\|Ax - Ay\|^2 \le \langle x - y, Ax - Ay\rangle$$

for all $x, y \in C$. Let $T : C \to H$ be a nonspreading mapping and put $A = I - T$. Then, we have from Lemma 2.3 and (3) that for any $x, y \in C$,

$$\begin{aligned}\|Ax - Ay\|^2 &= \|x - y - (Ax - Ay)\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\ &= \|Tx - Ty\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\ &\le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\ &= 2\langle Ax, Ay\rangle + 2\langle x - y, Ax - Ay\rangle.\end{aligned}$$

This implies that A is a metric mapping.

4. Fixed point theorems for hybrid mappings

In this section, we start with the following lemma.

Lemma 4.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to H$ is hybrid if and only if*

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$.

Proof. We have from (5) and (4) that for any $x, y \in C$,

$$\begin{aligned}&\|Tx - Ty\|^2 \le \|x - y\|^2 + \langle x - Tx, y - Ty\rangle \\ &\iff 2\|Tx - Ty\|^2 \le 2\|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \\ &\iff 2\|Tx - Ty\|^2 \le 2\|x - y\|^2 + \|x - Tx\|^2 \\ &+ \|y - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 \\ &\iff 2\|Tx - Ty\|^2 \le 2\|x - y\|^2 + \|x - Tx\|^2 \\ &+ \|y - Ty\|^2 - \|x - y\|^2 - \|Tx - Ty\|^2 + 2\langle x - y, Tx - Ty\rangle \\ &\iff 3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|x - Tx\|^2 + \|y - Ty\|^2 \\ &+ \|x - Ty\|^2 + \|x - Ty\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 \\ &\iff 3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2.\end{aligned}$$

□

Using Lemma 4.1, we can show an example of hybrid mappings which is not nonexpansive.

Example 4.2. Let H be a Hilbert space. Let A, B and C be subsets of H which are defined by

$$A = \{x \in H : \|x\| \leq 1\};$$
$$B = \{x \in H : \|x\| \leq 3\};$$
$$C = \{x \in H : \|x\| \leq 4\}.$$

Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 0, & \text{if } x \in B; \\ P_A x, & \text{if } x \in C \setminus B. \end{cases}$$

Then, T is a hybrid mapping of C into itself.

Proof. Checking three cases, we can prove that T is a hybrid mapping. In the case of $x, y \in B$, we have

$$3\|Tx - Ty\|^2 = 0 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2. \tag{6}$$

So, from Lemma 4.1, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle.$$

Similarly, in the case of $x \in B$ and $y \in C \setminus B$, we have

$$\begin{aligned} 3\|Tx - Ty\|^2 = 3\|Ty\|^2 = 3 & \\ &\leq \|x - y\|^2 + \|y\|^2 + \|x - Ty\|^2 \\ &= \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2. \end{aligned}$$

In the case of $x, y \in C \setminus B$, we have

$$\|P_A x - P_A y\|^2 \leq \langle x - y, P_A x - P_A y\rangle.$$

As in Section 3, we have

$$\|P_A x - P_A y\|^2 \leq \|x - y\|^2 + \langle x - P_A x, y - P_A y\rangle$$

and hence

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle.$$

So, $T : C \to C$ is a hybrid mapping. Since T is not continuous, $T : C \to C$ is not nonexpansive. □

Using the technique developed by Takahashi [13], we prove a fixed point theorem for hybrid mappings in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a hybrid mapping of C into itself. Then the following are equivalent:*

(i) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(ii) *$F(T)$ is nonempty.*

Proof. Fix $x \in C$. Then, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}2\|T^{k+1}x - Ty\|^2 &\le 2\|T^k x - y\|^2 + 2\langle T^k x - T^{k+1}x, y - Ty\rangle \\ &= 2\|T^k x - y\|^2 + \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 \\ &- \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2 \\ &= 2\|T^k x - Ty\|^2 + 4\langle T^k x - Ty, Ty - y\rangle + 2\|Ty - y\|^2 \\ &+ \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2.\end{aligned}$$

So, we obtain that

$$\begin{aligned}3\|T^{k+1}x - Ty\|^2 \le 3\|T^k x - Ty\|^2 &+ 4\langle T^k x - Ty, Ty - y\rangle \\ &+ 2\|Ty - y\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2.\end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, 2, \dots, n-1$, we have

$$\begin{aligned}3\|T^n x - Ty\|^2 \le 3\|x - Ty\|^2 &+ 4\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y\rangle \\ &+ 2n\|Ty - y\|^2 + \|T^n x - y\|^2 - \|x - y\|^2.\end{aligned}$$

Deviding this inequality by n, we have

$$\begin{aligned}\frac{3}{n}\|T^n x - Ty\|^2 \le \frac{3}{n}\|x - Ty\|^2 &+ 4\langle S_n(x) - Ty, Ty - y\rangle \\ &+ 2\|Ty - y\|^2 + \frac{1}{n}\|T^n x - y\|^2 - \frac{1}{n}\|x - y\|^2,\end{aligned}$$

where $S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Since $\{T^n x\}$ is bounded by assumption, $\{S_n(x)\}$ is also bounded. Thus we have a subsequence $\{S_{n_i}(x)\}$ of $\{S_n(x)\}$ such that $S_{n_i}(x)$ converges weakly to $u \in C$. Replacing n by n_i and letting $n_i \to \infty$, we obtain

$$0 \le 2\|Ty - y\|^2 + 4\langle u - Ty, Ty - y\rangle.$$

Putting $y = u$, we have

$$0 \le 2\|Tu - u\|^2 + 4\langle u - Tu, Tu - u\rangle.$$

So, we have $0 \le -2\|Tu - u\|^2$ and hence $Tu = u$. This completes the proof. □

Next, we show the demiclosedness of a hybrid mapping in a Hilbert space.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a hybrid mapping of C into itself. Then T is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.*

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Suppose that $u \neq Tu$. From Opial's theorem [9], we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - u\|^2 &< \liminf_{n\to\infty} \|x_n - Tu\|^2 \\ &= \liminf_{n\to\infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\ &= \liminf_{n\to\infty} (\|x_n - Tx_n\|^2 + \|Tx_n - Tu\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\langle x_n - Tx_n, Tx_n - Tu\rangle) \\
&\leq \liminf_{n\to\infty}(\|x_n - Tx_n\|^2 + \|x_n - u\|^2 + \langle x_n - Tx_n, u - Tu\rangle \\
&\quad + 2\langle x_n - Tx_n, Tx_n - Tu\rangle) \\
&= \liminf_{n\to\infty}\|x_n - u\|^2 .
\end{aligned}
$$

This is a contradiction. Hence we get the conclusion. □

We have also the following result concerning the set of fixed points of a hybrid mapping in a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a hybrid mapping of C into itself. Then $F(T)$ is closed and convex.*

Proof. It follows from Theorem 4.4 that $F(T)$ is closed. In fact, Let $\{x_n\} \subset F(T)$ and $x_n \to z$. Then, we have $x_n \rightharpoonup z$ and $x_n - Tx_n = 0$. So, from Theorem 4.4 we have $z = Tz$. Let us show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0, 1]$ and put $z = \alpha x + (1-\alpha)y$. Then, we have from (1) that

$$
\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha\|Tx - Tz\|^2 + (1-\alpha)\|Ty - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(\|x - z\|^2 + \langle x - Tx, z - Tz\rangle) \\
&\quad + (1-\alpha)(\|y - z\|^2 + \langle y - Ty, z - Tz\rangle) - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\
&= 0.
\end{aligned}
$$

So, we have $Tz = z$. This completes the proof. □

5. The fixed point property and unbounded sets

Ray [10] proved the following theorem.

Theorem 5.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following are equivalent:*

(i) *Every nonexpansive mapping of C into itself has a fixed point in C;*
(ii) *C is bounded.*

Sine [12] gave a simple proof. Using Ray's theorem, we prove the following theorem.

Theorem 5.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following are equivalent:*

(i) *Every hybrid mapping of C into itself has a fixed point in C;*
(ii) *C is bounded.*

Proof. From Theorem 4.3, we know that (ii) implies (i). Let us show that (i) implies (ii). We know that every firmly nonexpansive mapping is a hybrid mapping. So, the class of hybrid mappings of C into itself contains the class of firmly nonexpansive mappings of C into itself. To show (i) $\Longrightarrow$ (ii), it is sufficient to show that if every firmly nonexpansive mapping of C into itself has a fixed point in C, then every nonexpansive mapping of C into itself has a fixed point in C. Let T be a nonexpansive mapping of C into itself. Then, $S = \frac{1}{2}I + \frac{1}{2}T$ is a firmly nonexpansive mapping; see [4, p. 128]. In fact, we have that for any $x, y \in C$,

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|\frac{1}{2}x + \frac{1}{2}Tx - (\frac{1}{2}y + \frac{1}{2}Ty)\|^2 \\
&= \|\frac{1}{2}(x-y) + \frac{1}{2}(Tx - Ty)\|^2 \\
&= \frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|Tx - Ty\|^2 - \frac{1}{4}\|x - y - (Tx - Ty)\|^2 \\
&= \frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|Tx - Ty\|^2 \\
&\quad - \frac{1}{4}(\|x-y\|^2 + \|Tx - Ty\|^2 - 2\langle x - y, Tx - Ty\rangle) \\
&= \frac{1}{4}(\|x-y\|^2 + \|Tx - Ty\|^2) + \frac{1}{2}\langle x - y, Tx - Ty\rangle \\
&\leq (\frac{1}{4} + \frac{1}{4})\|x-y\|^2 + \frac{1}{2}\langle x - y, Tx - Ty\rangle \\
&= \frac{1}{2}\|x-y\|^2 + \frac{1}{2}\langle x - y, Tx - Ty\rangle \\
&= \langle x - y, \frac{1}{2}(x-y) + \frac{1}{2}(Tx - Ty)\rangle \\
&= \langle x - y, \frac{1}{2}(x + Tx) - \frac{1}{2}(y + Ty)\rangle \\
&= \langle x - y, \frac{1}{2}x + \frac{1}{2}Tx - (\frac{1}{2}y + \frac{1}{2}Ty)\rangle \\
&= \langle x - y, Sx - Sy\rangle.
\end{aligned}$$

This implies that S is a firmly nonexpansive mapping. Further, it is not difficult to show $F(T) = F(S)$. So, every firmly nonexpansive mapping of C into itself has a fixed point in C if and only if every nonexpansive mapping of C into itself has a fixed point in C. This completes the proof. □

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