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ON EXISTENCE OF BEST PROXIMITY PAIR THEOREMS FOR RELATIVELY NONEXPANSIVE MAPPINGS

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Abstract. In this paper we study the existence of best proximity pair of points for relatively nonexpansive mappings on a pair A, B of subsets of a Banach space X. A recent paper by Espínola (*A new approach to relatively nonexpansive mappings*, Proc. Amer. Math. Soc. **136** (2008), no. 6, 1987–1995) contains best proximity pair theorems for relatively nonexpansive mappings on a proximial parallel pair in X. In this paper we obtain similar results for a more general class of pairs.

1. Introduction

Let X be a normed linear space and $D \subset X$. Recall that a mapping $T : D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in D. In this work we consider mappings that are relatively nonexpansive, in the sense that they are defined on the union of a pair of subsets A, B of X satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in A$, $y \in B$. In [2], Eldred *et al.,* introduced a notion of *proximal normal structure* to prove best proximity pair theorems for relatively nonexpansive mappings of the types (i) $TA \subset A$, $TB \subset B$ and (ii) $TA \subset B$, $TB \subset A$. Espínola recently introduced a new approach in [1] to establish the same result for relatively nonexpansive mappings in a pair of proximinal parellel sets. Espínola has also proved that every convex proximinal pair (A, B) in a strictly convex Banach space is a proximinal parallel pair. In general there are proximinal pairs (A, B) in X in which neither X is a strictly convex Banach space nor the pair (A, B) is a proximinal parallel pair. Motivated by this, we study best proximity pair theorems for relatively nonexpansive mappings on $A \cup B$. It might be noted that the best proximity pair theorems proved in [1] can also be proved by the approach given in [2]. Our aim here is to follow the approach of [1] to prove best proximity pair theorems for proximinal pairs which are not necessarily proximinal parallel.

Definition 1.1. Let M be a nonempty set. A function $d : M \times M \to [0, \infty)$ is said to be a semimetric on M, if it satisfies:

1. $d(x, y) = 0$ if and only if $x = y \in M$.
2. $d(x, y) = d(y, x)$ for any $x, y \in M$.

In this work we adopt the following notations.
For any subsets A, B of a semimetric space (M, d),

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$$\begin{aligned} d(A,B) &= \inf\{d(x,y) : x \in A, y \in B\}; \\ \delta(x,B) &= \sup\{d(x,y) : y \in B\}, \text{ for } x \in M; \\ \delta(A,B) &= \sup\{d(x,y) : x \in A,\ y \in B\}, \\ \delta(A) &= \delta(A,A), \text{the diameter of } A. \end{aligned}$$

Given a semimetric d on a set M a B-set will be a set like

$$B_d(x,r) = \{y \in M :\ d(x,y) \le r\}.$$

Consider the family of admissible subsets of M which are intersections of B-sets. We denote this collection by $\mathcal{A}(M)$. For a subset $D \subset M$, we say that a point $x \in D$ is a nondiametral point if $\delta(x,D) < \delta(D)$.

A family $\mathcal{B}$ of subsets of a semimetric space (M,d) is said to be a *convexity structure* if $\mathcal{B}$ contains the $B-$sets of M and if $\mathcal{B}$ is closed under intersection. $\mathcal{B}$ is *compact* if every subfamily of $\mathcal{B}$ which has the finite intersection property has nonempty intersection, and *normal* if every member of $\mathcal{B}$ containing more than one point has a nondiametral point. For more on these concepts the reader may check the references [3, 4, 5, 6].

Nonexpansiveness with respect to a semimetric is defined in the natural way.

Definition 1.2 ([1])**.** A mapping $T : M \to M$ is said to be nonexpansive with respect to the semimetric d, if $d(Tx,Ty) \le d(x,y)$ for any $x, y \in M$.

An abstract version of Kirk's Fixed point Theorem is stated in [3]. The following theorem is a particular case for semimetrics of this abstract version.

Theorem 1.3 ([1])**.** *Let (M,d) be a bounded semimetric space such that $\mathcal{A}(M)$ is compact and has normal structure. Then every nonexpansive mapping $T : M \to M$ has a fixed point.*

2. Preliminaries

Let X be a Banach space and A, B be subsets of X. We shall say that a pair (A,B) of sets of X satisfies a property p if each of the sets A and B has the same property p. A pair (A,B) of subsets of a Banach space X is said to be a *proximinal pair* if for each $(x,y) \in A \times B$ there exists $(x^{'}, y^{'}) \in A \times B$ such that $\|x - y^{'}\| = \|x^{'} - y\| = d(A,B)$. If, additionally, we impose the condition that the pair of points $(x^{'}, y^{'}) \in A \times B$ is unique for each $(x,y) \in A \times B$, then we say that the pair (A,B) is a sharp proximinal pair. In this case, such a $y' \in B$ is said to be the best proximity point for $x \in A$ and vice-versa.

Lemma 2.1. *Let (A,B) be a proximinal pair in a Banach space X. Then (A,B) is a sharp proximinal pair if and only if for every $x \in A$ there exists an unique $y \in B$ such that $\|x - y\| = d(A,B)$ and vice-versa.*

In [2], Eldred *et al.,* introduced the notion of *proximal normal structure* for a pair of subsets of a Banach space, as follows:

Definition 2.2 ([2])**.** A convex pair (A, B) in a Banach space X is said to have *proximal normal structure* if for any closed, bounded and convex proximinal pair $(H_1, H_2) \subseteq (A, B)$ for which $d(H_1, H_2) = d(A, B)$ and $d(H_1, H_2) < \delta(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta(x_1, H_2) < \delta(H_1, H_2) \text{ and } \delta(x_2, H_1) < \delta(H_1, H_2).$$

In [2] the reader can find different conditions (like (A, B) is a compact convex pair of a Banach space or a closed bounded convex pair of a uniformly convex Banach space) which guarantee the existence of proximal normal structure for a given pair of sets.

In [1], Espínola established best proximity pair results, using a notion called proximinal parallel pair, in a Banach space.

Definition 2.3 ([1])**.** Let A, B be nonempty subsets of a Banach space X. We say that (A, B) is a *proximinal parallel pair* if the following conditions are fulfilled
(1.) (A, B) is a sharp proximinal pair.
(2.) $B = A + h$ for some $h \in X$.

Remark 2.4. For any proximinal parallel pair (A, B) in a Banach space X there exists $h \in X$ such that $B = A + h$ and so for any $x, y \in A$, $\|x - y\| = \|(x + h) - (y + h)\|$. Hence $\delta(A) = \delta(B)$

Though in [1], Espínola has proved that every convex proximinal pair in a strictly convex Banach space is a proximinal parallel pair, we give a simple proof for the same result.

Lemma 2.5. *Let (A, B) be a convex proximinal pair in a strictly convex Banach space X. Then (A, B) is a proximinal parallel pair.*

Proof. The fact that (A, B) is a sharp proximinal pair immediately follows from the strict convexity of X. Suppose for x, y in A, $x^{'}, y^{'}$ in B are the corresponding best proximity points. Now if $x - x^{'} \neq y - y^{'}$ then by strict convexity of X, we have $\|\frac{x-x^{'}}{2} + \frac{y-y^{'}}{2}\| < d(A, B)$. Thus $\|\frac{x+y}{2} - \frac{x^{'}+y^{'}}{2}\| < d(A, B)$, a contradiction. □

3. Main Results

Let X be a Banach space and (A, B) be a sharp proximinal pair of subsets of X. Then for each $x \in B$ there is a unique $y \in A$ such that $\|x - y\| = d(A, B)$. In this case, such a y is denoted by $x^{'}$. From the definition of sharp proximinal pair, it is easy to see that $(x^{'})^{'} = x$. Throughout this work we denote $d(A, B)$ by d.

Now we define a semimetric on B and use this semimetric to prove the main results.

Lemma 3.1. *Let (A, B) be a sharp proximinal pair in a Banach space X. Let $d_1 : B \times B \to [0, \infty)$ be defined by*

$$d_1(x, y) = \max\{\|x^{'} - y\| - d, \|x - y^{'}\| - d\}.$$

Then d_1 defines a semimetric on B.

Proof. For $x = y \in B$, by uniqueness of best proximity points, $d_1(x,y) = 0$. If $d_1(x,y) = 0$ then $\|x^{'} - y\| = \|x - y^{'}\| = d = d(A,B)$. So by uniqueness of best proximity point of x, $x^{'} = y^{'}$ and hence $x = y$. By the definition of d_1, $d_1(x,y) = d_1(y,x)$ for all $x, y \in B$. □

We denote the collection of all admissible subsets of B with respect to the semimetric d_1 by $\mathcal{A}_1(B)$. Also if $B = A$ then $d_1(x,y) = \|x - y\|$ for all x, y in B. It is easy to verify that each $B-$set in B is a convex subset of X. Now we show that each $B-$set in B is a weakly closed subset of the given Banach space X.

Lemma 3.2. *Let (A,B) be a weakly compact, convex, sharp proximinal pair in a Banach space X and d_1 be the semimetric on B defined as above. Then each $B-$set in B is a weakly closed subset of X.*

Proof. Since each $B-$set in B is a convex subset of X, it is enough to prove that each $B-$set in B is closed in the Banach space X. Let y_n be a sequence in $B_{d_1}(x,r)$ such that $y_n \to y$ in X. By weak compactness of A, we get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y^{'}_{n_k} \to \ z$ weakly for some $z \in A$. Norm is weakly lower semi continuous with respect to the weak topology on X implies $\|y - z\| \le \liminf_k \|y_{n_k} - y^{'}_{n_k}\| = d$. Therefore $z = y^{'}$. Also $\|x - y^{'}\| \le \liminf_k \|x - y^{'}_{n_k}\| \le r + d$, $\|x^{'} - y\| \le \liminf_k \|x^{'} - y_k\| \le r + d$ and hence $y \in B_{d_1}(x,r)$. □

Now we see the relation between the proximal normal structure on (A,B) and d_1-normal structure on $\mathcal{A}_1(B)$.

Proposition 3.3. *Let (A,B) be a nonempty sharp proximinal convex pair in Banach space X. Then (A,B) has proximal normal structure if and only if each closed, bounded and convex subset of B having more than one point has a d_1-nondiametral point.*

Proof. $\Rightarrow$: Let (A,B) have proximal normal structure and let $D \subseteq B$ be a nonempty closed, bounded and convex set having more than one point. Then the pair $(D^{'}, D)$ is a closed, bounded and convex proximinal pair of (A,B) satisfying $d(A,B) = d(D^{'}, D)$ by setting

$$D^{'} = \{x^{'} \in A : x \in D\}.$$

Let x, y in D be such that $x \neq y$ then $\|y^{'} - x\| > d$ and hence $d(D^{'}, D) < \delta(D^{'}, D)$. By proximal normal structure, there exists $(x^{'}, y) \in D^{'} \times D$ such that $\delta(x^{'}, D) < \delta(D^{'}, D)$ and $\delta(y, D^{'}) < \delta(D^{'}, D)$. Get $(x^{'}_2, x_3), (x^{'}_4, x_5) \in D^{'} \times D$ such that

$$\delta(x^{'}, D) \ < \ d(x^{'}_2, x_3) \tag{3.1}$$

$$\delta(y, D^{'}) \ < \ d(x^{'}_4, x_5) \tag{3.2}$$

Now we claim that $\delta(x^{'}, D) - d < \delta_1(D)$. Suppose not $d_1(x_2, x_3) \le \delta(x^{'}, D) - d$, which gives $\|x^{'}_2 - x_3\| \le \delta(x^{'}, D)$, a contradiction to 3.1. By a similar argument it is easy to see that $\delta(y, D^{'}) - d < \delta_1(D)$. Now by the convexity of D, $\frac{x+y}{2} \in D$ and for any $z \in D$, $\|(\frac{x+y}{2})^{'} - z\| - d \le \frac{1}{2}\big[(\|x^{'} - z\| - d) + (\|y^{'} + z\| - d)\big] \le \frac{1}{2}\big[(\delta(x^{'}, D) - d) + d_1(y,z)\big] < \frac{1}{2}\big(\delta_1(D) + \delta_1(D)\big) < \delta_1(D)$. By the similar argument

$\|(\frac{x+y}{2}) - z^{'}\| - d < \delta_1(D)$ and hence $\frac{x+y}{2}$ is a nondiametrical point D with respect to d_1.
$\Leftarrow$: Let (H_1, H_2) be a closed, bounded and convex proximinal subset of (A, B) with $d(H_1, H_2) = d(A, B)$ and $d(H_1, H_2) < \delta(H_1, H_2)$. Let $H_2^{'} = \{x^{'} \in A : x \in H_2\}$. Then $= H_2^{'} = H_1$. Now B has d_1-normal structure, so $\delta_1(H_2) > 0$, implies there exists $x \in H_2$ such that $\delta_1(x, H_2) < \delta_1(H_2)$. Then for the pair $(x^{'}, x) \in H_1 \times H_2$ and for any $z \in H_2$, $\|x^{'} - z\| \leq d_1(x, z) + d \leq \delta_1(x, H_2) + d < \delta_1(H_2) + d = \delta(H_1, H_2)$. Hence $\delta(x^{'}, H_2) < \delta(H_1, H_2)$. By a similar argument we get $\delta(x, H_1) < \delta(H_1, H_2)$. □

Remark 3.4. Using Lemma 3.2 one can easily prove that if (A, B) is a weakly compact convex sharp proximinal pair in a Banach space X then $\mathcal{A}_1(B)$ is compact.

Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping with $TA \subseteq A$ and $TB \subseteq B$.

Remark 3.5. Let (A, B) and T be as above. For $x \in B$, $d(A, B) \leq \|Tx - Tx^{'}\| \leq \|x - x^{'}\| = d(A, B)$ and hence $(Tx)^{'} = Tx^{'}$.

Lemma 3.6. *Let (A, B) be a sharp proximinal pair in a Banach space X. If $T : A \cup B \to A \cup B$ is a relatively nonexpansive mapping with $TA \subseteq A$ and $TB \subseteq B$. Then T is d_1-nonexpansive on B.*

Proof. For $x, y \in B$, $\|Ty - Tx^{'}\| - d \leq \|y - x^{'}\| - d \leq d_1(x, y)$ and similarly $\|Ty^{'} - Tx\| - d \leq \|y^{'} - x\| - d \leq d_1(x, y)$. Hence $d_1(Tx, Ty) \leq d_1(x, y)$ □

The following theorem is an immediate consequence of Proposition 3.3.

Theorem 3.7. *Let (A, B) be a nonempty weakly compact convex sharp proximinal pair in a Banach space X having proximal normal structure. If $T : A \cup B \to A \cup B$ is a relatively nonexpansive mapping with $TA \subseteq A$ and $TB \subseteq B$, then there exists $(a, b) \in A \times B$ such that $Ta = a$, $Tb = b$ and $\|a - b\| = d(A, B)$.*

As particular case of Theorem 3.7, we get the following Corollary.

Corollary 3.8. ([1],[2]) *Let (A, B) be a nonempty, weakly compact convex pair in strictly convex space X, and suppose (A, B) has proximal normal structure. Further suppose $T : A \cup B \to A \cup B$ is a relatively nonexpansive mapping with $TA \subseteq A$ and $TB \subseteq B$, then there exist $x_0 \in A$, $y_0 \in B$ such that $Tx_0 = x_0$, $Ty_0 = y_0$, and $\|x_0 - y_0\| = d(A, B)$.*

Remark 3.9. If the pair (A, B) is not a sharp proximinal pair, the conclusions of the Theorem 3.7 need not be satisfied.

Example 3.10. For $n \geq 2$, let $X = \mathbb{R}^n$ with supremum norm, $A = \{x = (0, x_2, x_3, \ldots x_n) : 0 \leq x_i \leq 1,\ 2 \leq i \leq n\}$ and $B = \{y = (1, y_2, y_3, \ldots y_n) : 0 \leq y_i \leq 1,\ 2 \leq i \leq n\}$. Then for any $x \in A$, $y \in B$, $\|x - y\| = 1$. Hence $d(A, B) = 1$. Then the pair (A, B) is a compact, convex pair having proximal normal structure and any $T : A \cup B \to A \cup B$ with $TA \subset A, TB \subset B$ is a relatively nonexpansive mapping. Hence there exist fixed point free relatively nonexpansive mappings.

Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping with $TA \subseteq B$ and $TB \subseteq A$. Define a map $T^{'} : A \cup B \to A \cup B$ as, $T^{'}x = Tx^{'}$ for $x \in A$ and $T^{'}y = Ty^{'}$ for $y \in B$. Then it is easy to see that $T^{'}A \subseteq A$, $T^{'}B \subseteq B$ and $(T^{'}x)^{'} = Tx$.

Lemma 3.11. *Let T and $T^{'}$ be as above. Then $T^{'}$ is d_1-nonexpansive on B.*

Proof. Let $x, y \in B$. $\|(T^{'}x)^{'} - T^{'}y\| - d = \|Tx - Ty^{'}\| - d \leq \|x - y^{'}\| - d \leq d_1(x, y)$. Similarly $\|(T^{'}y)^{'} - T^{'}x\| - d \leq d_1(x, y)$. Hence $T^{'}$ is d_1-nonexpansive on B. □

The following theorem is an immediate consequence of Lemma 3.11,Proposition 3.3 and Theorem 1.3.

Theorem 3.12 ([2]). *Let (A, B) be a nonempty bounded closed convex pair in a uniformly convex Banach space X. If $T : A \cup B \to A \cup B$ is a relatively nonexpansive mapping with $TA \subseteq B$ and $TB \subseteq A$, then there exists $(a, b) \in A \times B$ such that $\|a - Ta\| = \|b - Tb\| = d(A, B)$.*

Finally we give an example of a sharp proximinal pair (A, B) of subsets of an infinite dimensional Banach space X, which is not a proximinal parallel pair, to illustrate Theorem 3.7.

Example 3.13. Consider the space X of all complex valued continuous functions on $[0, 1]$ with supremum norm, *i.e.*, $X = (\mathcal{C}[0, 1], \|.\|_\infty)$.
$A := \{f_\alpha : \alpha \in [0, 1]\}$ and $B := \{g_\alpha : \alpha \in [0, 1]\}$, where

$$f_\alpha(t) := \begin{cases} 2i\alpha t, & \text{if } t \in [0, \frac{1}{2}] \\ 2i\alpha(1 - t), & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

$$g_\alpha(t) := \begin{cases} 1 + \alpha(t - \frac{1}{2}) + 2i\alpha t, & \text{if } t \in [0, \frac{1}{2}] \\ 1 - \alpha(t - \frac{1}{2}) + 2i\alpha(1 - t), & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

For any fixed $\alpha \in [0, 1]$ and for any $t \in [0, \frac{1}{2}]$, $|f_\alpha(t) - g_\alpha(t)| = |2i\alpha t - (1 + \alpha(t - \frac{1}{2}) + 2i\alpha t)| = |1 + \alpha(t - \frac{1}{2})| \leq 1$ and for any $t \in [\frac{1}{2}, 1]$, $|f_\alpha(t) - g_\alpha(t)| = |2i\alpha(1 - t) - (1 - \alpha(t - \frac{1}{2}) + 2i\alpha(1 - t))| = |1 - \alpha(t - \frac{1}{2})| \leq 1$. Also $|f_\alpha(\frac{1}{2}) - g_\alpha(\frac{1}{2})| = |i\alpha - (1 + i\alpha)| = 1$. Therefore $\|f_\alpha - g_\alpha\| = 1$. Now for any $\alpha \neq \beta \in [0, 1]$, $\|f_\alpha - g_\beta\| \geq |f_\alpha(\frac{1}{2}) - g_\beta(\frac{1}{2})| = |i\alpha - (1 + i\beta)| = |1 - i(\beta - \alpha)| > 1$. Hence for any $(f_\alpha, g_\beta) \in A \times B$, $(g_\alpha, f_\beta) \in B \times A$ is the unique pair satisfying $\|f_\alpha - g_\alpha\| = \|f_\beta - g_\beta\| = 1 = d(A, B)$ *i.e.*, the pair (A, B) is a sharp proximinal pair in X. Also $f_0(0) - g_0(0) = 1 \neq \frac{1}{2} = f_1(0) - g_1(0)$. Hence the pair (A, B) is not a proximinal parallel pair. Now for any sequence $f_{\alpha_n} \in A$, the sequence α_n has a convergent subsequence, again denote by α_n, which converges to some $\alpha \in [0, 1]$. It is easy to see that the sequence f_{α_n} converges to f_α. Therefore (A, B) is a compact subset of X. Now (A, B) is compact convex pair and and hence has proximal normal structure (c.f. [2]). Let $T : A \cup B \to A \cup B$ be defined as

$$T(f_\alpha) = f_{\frac{\alpha}{2}},\ T(g_\alpha) = g_{\frac{\alpha}{2}} \text{ for all } \alpha \in [0, 1],$$

Then T is a relatively nonexpansive map on $A \cup B$, T satisfies all the conditions of Theorem 3.7 and (f_0, g_0) is the required best proximity pair satisfying the conclusions of Theorem 3.7.

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