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A RIGIDITY THEOREM FOR COMMUTING HOLOMORPHIC FUNCTIONS

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Abstract. We first prove a rigidity theorem for commuting holomorphic self-mappings of the open unit disk Δ in the complex plane $\mathbb{C}$ and then use it to improve upon a recent result regarding commuting parabolic semigroups on Δ. This result provides sufficient conditions for two semigroups of holomorphic functions to commute.

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $\mathrm{Hol}(D, E)$ the set of all holomorphic functions which map a domain $D \subset \mathbb{C}$ into a set $E \subset \mathbb{C}$, and by $\mathrm{Hol}(D)$ the set of all holomorphic self-mappings of D, *i. e.*, $\mathrm{Hol}(D) := \mathrm{Hol}(D, D)$.

The classical Denjoy–Wolff theorem asserts that if $F \in \mathrm{Hol}(\Delta)$ is neither the identity mapping nor an elliptic automorphism of Δ, then there exists a unique point $\tau \in \overline{\Delta}$, the closed unit disk, which is an attractive point of the discrete semigroup $\{F_n\}_{n=0}^{\infty}$, where $F_0 = I$, $F_n = F \circ F_{n-1}$, that is,

$$\lim_{n\to\infty} F_n(z) = \tau \quad \text{for all} \quad z \in \Delta.$$

This point τ is called the **Denjoy-Wolff point** of F.

All holomorphic self-mappings of Δ which are neither the identity mapping nor elliptic automorphisms of Δ fall into three different classes. Let $F \in \mathrm{Hol}(\Delta)$ and let $\tau \in \overline{\Delta}$ be its Denjoy–Wolff point. Then F is of

– **dilation type** if $\tau \in \Delta$,

– **hyperbolic type** if $\tau \in \partial\Delta$ and $0 < F'(\tau) < 1$,

– **parabolic type** if $\tau \in \partial\Delta$ and $F'(\tau) = 1$.

Note that an analogous classification is used for holomorphic self-mappings of the open right half-plane $\Pi^+ := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$.

The self-mappings of parabolic type fall into two subclasses:

– **automorphic type**, if all orbits $\{F_n(z)\}_{n=0}^{\infty}$ are separated in the (hyperbolic) Poincaré metric, *i. e.*, $\lim_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) > 0$ for all $z \in \Delta$;

– **nonautomorphic type**, if no orbit $\{F_n(z)\}_{n=0}^{\infty}$ is hyperbolically separated, *i. e.*, $\lim_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) = 0$ for all $z \in \Delta$.

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Holomorphic self-mappings of Δ are known to have some rigidity properties. For example, let $F \in \mathrm{Hol}(\Delta)$ have an interior fixed point $\tau \in \Delta$. In this case, if F coincides with the identity mapping I up to the first derivative at τ, *i. e.*, $F(\tau) = \tau$ and $F'(\tau) = 1$, then $F \equiv I$ on Δ by the classical Schwarz lemma.

In the boundary case ($\tau \in \partial\Delta$) a rigidity theorem in the spirit of the uniqueness part of the classical Schwarz lemma was established by D. M. Burns and S. G. Krantz [3]. We formulate their result as follows:

Theorem 1.1 ([3, Theorem 2.1])**.** *Let $F \in \mathrm{Hol}(\Delta)$. If the unrestricted limit*

$$\lim_{z\to 1} \frac{F(z) - z}{(z-1)^3} = 0,$$

then $F \equiv I$ on Δ.

This means that if $F \in \mathrm{Hol}(\Delta)$ coincides with the identity mapping I up to the third order at its boundary Denjoy–Wolff point τ, then $F \equiv I$.

Suppose now that $F \in \mathrm{Hol}(\Delta)$ coincides with I only up to the second order at its boundary Denjoy–Wolff point. We prove that if F commutes with a function $G \in \mathrm{Hol}(\Delta)$ such that the second (unrestricted) derivative $G''(\tau) \neq 0$, then $F = I$.

Theorem 1.2. *Let $F \in \mathrm{Hol}(\Delta)$ be such that*

$$\lim_{z\to 1} \frac{F(z) - z}{(z-1)^2} = 0. \tag{1.1}$$

Suppose that $F \circ G = G \circ F$ for some $G \in \mathrm{Hol}(\Delta)$ with

$$\lim_{z\to 1} \frac{G(z) - z}{(z-1)^2} = \alpha \neq 0 \, (\neq \infty). \tag{1.2}$$

Then $F \equiv I$ on Δ.

In our proof of Theorem 1.2 we use the following result due to C. C. Cowen [5, Proposition 3.1]:

Proposition 1.3 ([5, Proposition 3.1])**.** *Let $\psi \in \mathrm{Hol}(\Pi^+)$ and let ∞ be its Denjoy–Wolff point. Then there exists an open, connected, and simply connected set $V \subset \Pi^+$, called a **fundamental set** for ψ, such that:*

1. *$\psi(V) \subset V$;*
2. *for each compact set $K \subset \Pi^+$, the sequence of iterates $\{\psi_n(K)\}_{n=1}^{\infty}$ is eventually contained in V;*
3. *ψ is univalent in V.*

2. Rigidity

We begin this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. Consider $\varphi := C \circ F \circ C^{-1}$ and $\psi := C \circ G \circ C^{-1}$, where $C : \Delta \mapsto \Pi^+$ is the Cayley transform given by $C(z) = \dfrac{1+z}{1-z}$, $z \in \Delta$. Then φ and ψ are commuting parabolic self-mappings of Π^+ having ∞ as their common Denjoy-Wolff point.

Denote

$$w_0^0 := 1, w_n^0 := \psi(w_{n-1}^0), \quad n = 1, 2, ...,$$

$$w_0 := w, w_n := \psi(w_{n-1}), \quad w \in \Pi^+, \quad n = 1, 2, ...,$$

and

$$h_n(w) := \frac{w_n - w_n^0}{w_{n+1}^0 - w_n^0}, \quad n = 1, 2, ...$$

Then $h_n \in \mathrm{Hol}(\Pi^+, \mathbb{C})$ and the sequence $\{h_n\}_{n=1}^{\infty}$ converges in the compact open topology to a holomorphic function $h \in \mathrm{Hol}(\Pi^+, \mathbb{C})$ such that $h \circ \psi = h + 1$ (*cf.* [4], Theorem 2.2).

If V is a fundamental set for ψ, then the functions h_n are univalent in V and, consequently, h is univalent in V by Hurwitz's theorem.

It follows from (1.1) and (1.2) that the functions φ and ψ admit the following expansions at ∞ (see [2], p. 51):

$$\varphi(w) = w + \gamma_\varphi(w), \qquad \lim_{w\to\infty} \gamma_\varphi(w) = 0, \tag{2.1}$$

$$\psi(w) = w + 2\alpha + \gamma_\psi(w), \quad \lim_{w\to\infty} \gamma_\psi(w) = 0.$$

Therefore

$$h_n(\varphi(w)) = \frac{\psi_n(\varphi(w)) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{\varphi(\psi_n(w)) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{w_n + \gamma_\varphi(w_n) - w_n^0}{w_{n+1}^0 - w_n^0}$$

$$= \frac{w_n - w_n^0}{w_{n+1}^0 - w_n^0} + \frac{\gamma_\varphi(w_n)}{w_{n+1}^0 - w_n^0} = h_n(w) + \frac{\gamma_\varphi(w_n)}{2\alpha + \gamma_\psi(w_n)} \cdot \frac{2\alpha + \gamma_\psi(w_n)}{w_{n+1}^0 - w_n^0},$$

and letting n tend to ∞, we get

$$h(\varphi(w)) = h(w) + \lim_{n\to\infty} \frac{\gamma_\varphi(w_n)}{2\alpha + \gamma_\psi(w_n)} \cdot \lim_{n\to\infty} \frac{2\alpha + \gamma_\psi(w_n)}{w_{n+1}^0 - w_n^0}, \quad w \in \Pi^+. \tag{2.2}$$

Since $\alpha \neq 0$, the first limit equals 0. The second limit equals 1. Indeed, repeating the above calculations with ψ instead of φ, we find that

$$h(\psi(w)) = h(w) + \lim_{n\to\infty} \frac{2\alpha + \gamma_\psi(w_n)}{w_{n+1}^0 - w_n^0}, \quad w \in \Pi^+.$$

At the same time, $h \circ \psi = h + 1$.

It follows that $h(\varphi(w)) = h(w)$ for all $w \in \Pi^+$.

If $K \subset \varphi(V) \subset \Pi^+$ is a compact set, then

$$\psi_n(K) \subset \psi_n(\varphi(V)) = \varphi(\psi_n(V)) \subset \varphi(V), \quad n = 1, 2, ...$$

On the other hand, for n large enough, $\psi_n(K) \subset V$ and so $V \cap \varphi(V) \neq \varnothing$. Hence there exists a domain $W \subset V$ such that $\varphi(W) \subset V$. For each $w \in W$, $h(\varphi(w)) = h(w)$. Since h is univalent in V, φ is the identity mapping on W and, by the uniqueness principle, $\varphi(w) = w$ for all $w \in \Pi^+$. Therefore $F = C^{-1} \circ \varphi \circ C$ coincides with the identity mapping on Δ, as claimed. □

The following theorem asserts that if $\mathrm{Re}\, \alpha > 0$, then the unrestricted limit (1.1) can be replaced by the angular one. We recall that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has angular limit $L = \angle \lim_{z \to \tau} f(z)$ at a boundary point $\tau \in \partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$ if $f(z) \to L$ as $z \to \tau$ in each nontangential approach region

$$\Gamma(\tau, k) = \left\{ z \in \Delta : \frac{|z - \tau|}{1 - |z|} < k \right\}, \quad k > 1.$$

Theorem 2.1. *Let $F \in \mathrm{Hol}(\Delta)$ be such that*

$$\angle \lim_{z \to 1} \frac{F(z) - z}{(z-1)^2} = 0. \tag{2.3}$$

Suppose that $F \circ G = G \circ F$ for some $G \in \mathrm{Hol}(\Delta)$ with

$$\lim_{z \to 1} \frac{G(z) - z}{(z-1)^2} = \alpha \neq \infty, \ \mathrm{Re}\, \alpha > 0. \tag{2.4}$$

Then $F \equiv I$ on Δ.

The scheme of the proof of Theorem 2.1 is the same as that of Theorem 1.2. We only have to use the expansion

$$\varphi(w) = w + \gamma_\varphi(w), \qquad \angle \lim_{w \to \infty} \gamma_\varphi(w) = 0,$$

instead of (2.1) and the fact that w_n in the first limit of equality (2.2) converges to ∞ nontangentially because $\mathrm{Re}\, \alpha > 0$ (see [2]). This expansion can be proved similarly to expansion (2.1), taking into account that (1.1) is replaced by (2.3) (*cf.* [7], p. 947).

3. Commuting semigroups

It turns out that Theorem 1.2 allows us to improve upon a recent result [6] concerning commuting semigroups of holomorphic self-mappings of Δ which are of parabolic type. To explain this, we first recall that a family $S = \{F_t\}_{t \geq 0} \subset \mathrm{Hol}(\Delta)$ is **a one-parameter continuous semigroup on** Δ (a semigroup, for short) if

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$ and $z \in \Delta$,

and

(ii) $\lim_{t \to 0^+} F_t(z) = z$ for all $z \in \Delta$.

It follows from a result of E. Berkson and H. Porta [1] that each semigroup is differentiable with respect to $t \in \mathbb{R}^+ = [0, \infty)$. So, for each one-parameter continuous semigroup $S = \{F_t\}_{t \geq 0} \subset \mathrm{Hol}(\Delta)$, the limit

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in \Delta,$$

exists and defines a holomorphic function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. This function is called the **(infinitesimal) generator of** $S = \{F_t\}_{t \geq 0}$.

We say that two semigroups $S_1 = \{F_t\}_{t \geq 0}$ and $S_2 = \{G_s\}_{s \geq 0}$ on Δ commute if

$$F_t \circ G_s = G_s \circ F_t \quad \text{for all} \quad s, t \geq 0.$$

We now present an improvement of Theorem 5 in [6]. This improvement is a consequence of Theorem 1.2. The difference between our Theorem 3.1 and [6, Theorem 5] is that in item (ii) we no longer assume that α and $\widetilde{\alpha}$ are different from zero.

Theorem 3.1. *Let* $S_1 = \{F_t\}_{t\geq 0}$ *and* $S_2 = \{G_t\}_{t\geq 0}$ *be two non-trivial continuous semigroups on* Δ *generated by* f *and* g*, respectively, and let* $F_1 \circ G_1 = G_1 \circ F_1$.

Suppose that $\tau = 1$ *is the boundary null point of* f *such that* $f'(1) = 0$. *If either one of the following conditions:*

(i) *the semigroups* S_1 *and* S_2 *are of non-automorphic type;*

(ii) $S_1, S_2 \subset C^0(1)$ *and the unrestricted limits* $\alpha := \lim\limits_{z\to 1} f''(z)$, $\widetilde{\alpha} := \lim\limits_{z\to 1} g''(z)$ *exist;*

holds, then the semigroups commute.

Proof. The conditions $f'(1) = 0$, $\alpha := \lim\limits_{z\to 1} f''(z)$, $\widetilde{\alpha} := \lim\limits_{z\to 1} g''(z)$ and the commutativity of S_1 and S_2 imply that $\tau = 1$ is the common Denjoy–Wolff point of all the functions F_t and G_t $(t > 0)$ belonging to the semigroups S_1 and S_2, respectively, and that

$$\lim_{z\to 1} \frac{F_t(z) - z}{(z-1)^2} = -\frac{\alpha t}{2} \quad \text{and} \quad \lim_{z\to 1} \frac{G_t(z) - z}{(z-1)^2} = -\frac{\widetilde{\alpha} t}{2}$$

for all $t \geq 0$ (see Remark 3 on p. 304 of [6]).

If α and $\widetilde{\alpha}$ are different from zero, then the semigroups commute by [6, Theorem 5(ii)]. If $\alpha = \widetilde{\alpha} = 0$, then the semigroups are of non-automorphic type (see [2]) and they commute by [6, Theorem 5(i)]. Otherwise, if one of the limits α and $\widetilde{\alpha}$ is different from zero and the other one equals zero, then by Theorem 1.2, one of the semigroups is the trivial semigroup consisting of identity mappings on Δ and, consequently, the semigroups commute in this case too. □

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