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STRONG CONVERGENCE THEOREMS FOR MAXIMAL MONOTONE OPERATORS AND GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove strong convergence theorems by two hybrid methods for finding a common element of the set of zero points of a maximal monotone operator and the set of fixed points of a generalized nonexpansive mapping in a Banach space. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and for generalized nonexpansive mappings in a Banach space.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. Let B be a maximal monotone operator from E to E^*. It is interesting to study the problem of finding a point $u \in E$ satisfying

$$0 \in Bu.$$

Such $u \in E$ is called a *zero point* (or a *zero*) of B. A well-known method to solve this problem is called the proximal point algorithm: $x_1 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \ n = 1, 2, ...,$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} is the resolvent of B for all $n \in \mathbb{N}$. This algorithm was first introduced by Martinet [15]. In 1976, Rockafellar [21] proved the following in the Hilbert space setting: If the solution set $B^{-1}0$ is nonempty and $\liminf_n r_n > 0$, then $\{x_n\}$ converges weakly to an element of $B^{-1}0$; see also Brézis and Lions [2] and Lions [13]. It was shown by Güler [3] that the sequence $\{x_n\}$ generated by this algorithm does not converge strongly in general. In 2000, motivated by Mann's type iteration [14, 20] and Halpern's type iteration [4, 24] for nonexpansive mappings, Kamimura and Takahashi [10] modified the proximal point algorithm and obtained weak and strong convergence theorems for maximal monotone operators in a Hilbert space. Solodov and Svaiter [25] also obtained a modification of the proximal point algorithm with *metric projections*. Ohsawa and Takahashi [19], and Kamimura and Takahashi [11] generalized Solodov and Svaiter's result to maximal monotone operators defined in a Banach space; see also Kohsaka and Takahashi [12], and Ibaraki and Takahashi [5, 6, 7].

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A mapping T of C into E is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T.

Many researchers have studied several methods for approximation of fixed points of a nonexpansive mapping; see [4, 14, 20, 24, 29] for instance. In 2003, Nakajo and Takahashi [18] proved the following theorem by using the hybrid method:

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n \leq a < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Recently, Takahashi-Takeuchi-Kubota [28] used another hybrid method called the shrinking projection method to prove the following theorem.

Theorem 1.2 ([28])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0 \end{cases}$$

for every $n \in \mathbb{N}$, where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Very recently, by using the normal hybrid method and the shrinking projection method, Inoue, Takahashi, and Zembayashi [9] proved strong convergence theorems for finding a common element of the set of zero points of a maximal monotone operator and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Theorem 1.3 ([9])**.** *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator and let $J_r = (J + rA)^{-1} J$ for all $r > 0$. Let T be a relatively nonexpansive mapping from C into itself such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T J_{r_n} x_n), \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, *where* $\{\alpha_n\} \subset [0, 1)$ *satisfies* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap A^{-1}0}x$, *where* $\Pi_{F(T)\cap A^{-1}0}$ *is the generalized projection from* E *onto* $F(T) \cap A^{-1}0$.

Theorem 1.4 ([9]). *Let* E *be a uniformly smooth and uniformly convex Banach space and let* C *be a nonempty closed convex subset of* E. *Let* $A \subset E \times E^*$ *be a maximal monotone operator and let* $J_r = (J + rA)^{-1}J$ *for all* $r > 0$. *Let* T *be a relatively nonexpansive mapping from* C *into itself such that* $F(T) \cap A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C, H_0 = C$ *and*

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTJ_{r_n}x_n), \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, *where* $\{\alpha_n\} \subset [0, 1)$ *satisfies* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap A^{-1}0}x$, *where* $\Pi_{F(T)\cap A^{-1}0}$ *is the generalized projection from* E *onto* $F(T) \cap A^{-1}0$.

The purpose of this paper, motivated by [9], is to obtain strong convergence theorems for finding a common element of the set of zero points of a maximal monotone operator defined in a dual Banach space and the set of fixed points of a generalized nonexpansive mapping introduced by Ibaraki and Takahashi [5, 6, 7]. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and for generalized nonexpansive mappings in Banach spaces.

2. Preliminaries

Let E be a Banach space with $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of x^* at x by $\langle x, x^*\rangle$. Then the duality mapping J on E defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [26] for more details. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth*. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in U$.

We also know the following properties; see [26, 27] for more details:
(1) $J(x) \neq \emptyset$ for each $x \in E$;
(2) J is a monotone operator;
(3) if E is strictly convex, then J is one-to-one;

(4) if E is reflexive, then J is a mapping of E onto E^*;
(5) if E is smooth, then J is single-valued;
(6) E is uniformly convex if and only if E^* is uniformly smooth;
(7) if E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded sets of E.

Let E be a smooth Banach space and consider the following function $\phi : E \times E \to [0, \infty)$ studied in Alber [1] and Kamimura and Takahashi [11]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $(x, y) \in E \times E$. We know that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for each $x, y, z \in E$. By the fact that $(\|x\| - \|y\|) \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$ for all $x, y \in E$. Let $\phi_* : E^* \times E^* \to [0, \infty)$ be the mapping defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for all $(x^*, y^*) \in E^* \times E^*$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx)$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y.$$

The following lemma is well known:

Lemma 2.1 ([11]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let C be a nonempty closed convex subset of a smooth Banach space E, let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

A mapping $T : C \to C$ is called *generalized nonexpansive* ([5, 6, 7]) if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \ \forall (x, y) \in C \times F(T).$$

A point p in C is said to be a *generalized asymptotic fixed point* of T [8] if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $\lim_{n\to\infty}(Jx_n - JTx_n) = 0$. We denote the set of generalized asymptotic fixed points of T by $\check{F}(T)$. Let D be a nonempty closed subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \ \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty closed subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [5, 6, 7] for more details. Let E be a reflexive, strictly convex and smooth Banach space and let $B \subset E \times E^*$

be a set-valued mapping with graph $G(B) = \{(x, x^*) : x^* \in Bx\}$ and domain $D(B) = \{z \in E : Bz \neq \emptyset\}$. Then the mapping B is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \ \forall (x, x^*), (y, y^*) \in B \subset E \times E^*.$$

It is also said to be *maximal monotone* if B is monotone and its graph is not properly contained in the graph of any other monotone operator. It is known that if $B \subset E \times E^*$ is maximal monotone, then $B^{-1}0$ is closed and convex.

Let E be as above and let $B \subset E^* \times E$ be a maximal monotone operator. For each $r > 0$ and $x \in E$, consider the set

$$J_r x = \{z \in E : x \in z + rBJz\}.$$

Then $J_r x$ consists of one point. We also denote the domain and the range of J_r by $D(J_r) = R(I + rBJ)$ and $R(J_r) = D(BJ)$, respectively. Such J_r is called the *generalized resolvent* of B and is denoted by

$$J_r = (I + rBJ)^{-1}.$$

The *Yosida approximation* of B is also denoted by $B_r = (I - J_r)/r$. It is shown in [7] that $(JJ_r x, B_r x) \in B$ for $x \in E$; see Ibaraki and Takahashi [7] for more details.

Ibaraki and Takahashi [7] also proved some properties of J_r and $(BJ)^{-1}0$.

Proposition 2.2. *Let E be a reflexive and strictly convex Banach space with a Fréchet differntiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*
(1) *$D(J_r) = E$ for each $r > 0$;*
(2) *$(BJ)^{-1}0 = F(J_r)$ for each $r > 0$;*
(3) *$(BJ)^{-1}0$ is closed;*
(4) *J_r is generalized nonexpansive for each $r > 0$.*

Remark 2.3. From the proof of (4), we can conclude that for all $x \in C$ and $y \in (BJ)^{-1}0$,

$$\phi(x, J_r x) + \phi(J_r x, y) \leq \phi(x, y). \tag{2.3}$$

They also proved the following lemmas:

Lemma 2.4 ([7])**.** *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([7])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*
(1) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(2) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [12] proved the following results:

Theorem 2.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*
(1) *C is a sunny generalized nonexpansive retract of E;*
(2) *C is a generalized nonexpansive retract of E;*

(3) *JC is closed and convex.*

Proposition 2.7. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(1) $z = Rx$;
(2) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

Very recently, Ibaraki and Takahashi [8] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Theorem 2.8. *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Theorem 2.6 and Theorem 2.8.

Theorem 2.9 ([8])**.** *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

3. Convergence theorem by the normal hybrid method

In this section, we prove a strong convergence theorem by the normal hybrid method [18] for generalized nonexpansive mappings with resolvents of maximal monotone operators in a Banach space. Before proving it, we prove the following lemma by using the techniques developed by Matsushita and Takahashi [17]; see also [12]. Compare this lemma with Theorem 2.8.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. If $T : C \to C$ is a generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Proof. We first prove that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ with $x_n \to x$. Since T is generalized nonexpansive,

$$\phi(Tx, x_n) \leq \phi(x, x_n)$$

for each $n \in \mathbb{N}$. This implies

$$\phi(Tx, x) = \lim_{n\to\infty} \phi(Tx, x_n) \leq \lim_{n\to\infty} \phi(x, x_n) = \phi(x, x) = 0.$$

Therefore, we have $\phi(Tx, x) = 0$ and hence $x \in F(T)$.

We next show that $JF(T)$ is closed. Let $\{x_n^*\} \subset JF(T)$ such that $x_n^* \to x^*$ for some $x^* \in E^*$. Note that since JC is closed and convex, we have $x^* \in JC$. Then, there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} \phi(Tx, x_n) &\leq \phi(x, x_n) \\ &= \|x\|^2 - 2\langle x, x_n^* \rangle + \|x_n^*\|^2 \\ &\to \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 = 0. \end{aligned}$$

Hence $\lim_{n\to\infty} \phi(Tx, x_n) = 0$. Since

$$\begin{aligned} 0 = \lim_{n\to\infty} \phi(Tx, x_n) &= \lim_{n\to\infty} (\|Tx\|^2 - 2\langle Tx, x_n^*\rangle + \|x_n^*\|^2) \\ &= \|Tx\|^2 - 2\langle Tx, x^*\rangle + \|x^*\|^2 = \phi(Tx, x), \end{aligned}$$

we have $\phi(Tx, x) = 0$ and hence $x = Tx$. This implies $x^* = Jx \in JF(T)$.

We finally show that $JF(T)$ is convex. Let $x^*, y^* \in JF(T)$ and let $\alpha \in (0, 1)$ and $\beta = 1 - \alpha$. Then we have $x, y \in F(T)$ such that $x^* = Jx$ and $y^* = Jy$. Thus, we have

$$\begin{aligned} &\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\ &\quad= \|TJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle TJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx \\ &\qquad+\beta Jy\rangle + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &\quad= \alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) \\ &\qquad+\|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2). \end{aligned}$$

Since $x, y \in F(T)$ and T is generalized nonexpansive, we have

$$\begin{aligned} &\alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) \\ &\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &\quad\le \alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) \\ &\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &\quad= \alpha\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\big\} \\ &\qquad + \beta\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\big\} \\ &\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &\quad= 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\ &\quad= 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0. \end{aligned}$$

Then we have $TJ^{-1}(\alpha Jx+\beta Jy) = J^{-1}(\alpha Jx+\beta Jy)$ and hence $\alpha Jx+\beta Jy \in JF(T)$. Therefore $JF(T)$ is convex and the proof is complete. □

As a direct consequence of Theorem 2.6 and Lemma 3.1, we obtain the following result.

Proposition 3.2. *Let E be a smooth, strictly convex and reflexive Banach space and C be a closed subset of E such that JC is closed and convex. If $T : C \to C$ is a generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $B \subset E^* \times E$ be a maximal monotone operator with $JC \supset D(B)$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \cap J^{-1}B^{-1}0 \neq \emptyset$ and assume that $\check{F}(T) = F(T)$. Let $\{x_n\}$ be a* *sequence generated by* $x_0 = x \in C$ *and*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n) T J_{r_n} x_n, \\ H_n = \{z \in C : \phi(u_n, z) \le \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle x - x_n, Jz - Jx_n\rangle \le 0\}, \\ x_{n+1} = R_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$*, where* J *is the duality mapping on* E*, and* $\{\alpha_n\} \subset [0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\liminf_{n\to\infty}(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} r_n > 0$*, respectively. Then* $\{x_n\}$ *converges strongly to* $R_{F(T)\cap J^{-1}B^{-1}0}x$*, where* $R_{F(T)\cap J^{-1}B^{-1}0}$ *is the sunny generalized nonexpansive retraction from* E *onto* $F(T) \cap J^{-1}B^{-1}0$.

Proof. We first show that $F(T) \cap J^{-1}B^{-1}0$ is a sunny generalized nonexpansive retract of E. From Proposition 2.2 and Lemma 3.1, we have $J^{-1}B^{-1}0$ and $F(T)$ are closed, respectively. By using Lemma 3.1 again, we have $JF(T)$ is closed and convex. From the maximal monotonicity of B, we have $B^{-1}0$ is closed and convex. Since E is uniformly convex, J is injective and hence

$$J(F(T) \cap J^{-1}B^{-1}0) = JF(T) \cap B^{-1}0$$

which is also closed and convex. Using Theorem 2.6, we have that $F(T)\cap J^{-1}B^{-1}0$ is a sunny generalized nonexpansive retract of E.

For each $n \in \mathbb{N} \cup \{0\}$, it is easy to see that H_n is closed. Since J is norm-to-weak* continuous, W_n is closed for all $n \in \mathbb{N} \cup \{0\}$. Hence $H_n \cap W_n$ is closed. Since E is reflexive, J is surjective and hence

$$JW_n = \{z^* \in JC : \langle x - x_n, z^* - Jx_n\rangle \le 0\}$$

and

$$JH_n = \{z^* \in JC : \phi_*(z^*, Ju_n) \le \phi_*(z^*, Jx_n)\}$$

for all $n \in \mathbb{N} \cup \{0\}$. We can see that JH_n is convex since

$$\begin{aligned} \phi(u_n, z) \le \phi(x_n, z) &\Leftrightarrow \ \|u_n\|^2 - 2\langle u_n, Jz\rangle - \|x_n\|^2 + 2\langle x_n, Jz\rangle \le 0 \\ &\Leftrightarrow \ \|u_n\|^2 - \|x_n\|^2 - 2\langle x_n - u_n, Jz\rangle \le 0. \end{aligned}$$

Since J is injective,

$$J(H_n \cap W_n) = JH_n \cap JW_n.$$

Thus JH_n, JW_n and $J(H_n \cap W_n)$ are closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

We next show that $H_n \cap W_n$ is nonempty. Let $w \in F(T) \cap J^{-1}B^{-1}0$. Put $y_n = J_{r_n} x_n$. Since J_{r_n} and T are generalized nonexpansive, from the convexity of

$\|\cdot\|^2$ we have

$$\begin{aligned}\phi(u_n, w) &= \phi(\alpha_n x_n + (1-\alpha_n)Ty_n, w)\\ &= \|\alpha_n x_n + (1-\alpha_n)Ty_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Ty_n, Jw\rangle + \|w\|^2\\ &\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Ty_n\|^2\\ &\quad -2\alpha_n\langle x_n, Jw\rangle - 2(1-\alpha_n)\langle Ty_n, Jw\rangle + \|w\|^2\\ &= \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(Ty_n, w)\\ &\leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(y_n, w) \quad (3.1)\\ &= \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(J_{r_n}x_n, w) \quad (3.2)\\ &\leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(x_n, w)\\ &= \phi(x_n, w).\end{aligned}$$

So, we have $w \in H_n$ and hence $F(T) \cap J^{-1}B^{-1}0 \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Next we show by induction that $F(T)\cap J^{-1}B^{-1}0 \subset H_n \cap W_n$ for all $n \in \mathbb{N}\cup\{0\}$. From $W_0 = C$, we have $F(T)\cap J^{-1}B^{-1}0 \subset H_0\cap W_0$. This implies that $H_0\cap W_0 \neq \emptyset$. By Theorem 2.6, $H_0\cap W_0$ is a sunny generalized nonexpansive retract of E. Thus we can define $x_1 = R_{H_0\cap W_0}x$ and $y_1 = J_{r_1}x_1$. Suppose that $F(T)\cap J^{-1}B^{-1}0 \subset H_k\cap W_k$ for some $k \in \mathbb{N}$. If $w \in F(T)\cap J^{-1}B^{-1}0 \subset H_k \cap W_k$ and $x_{k+1} = R_{H_k\cap W_k}x$, then we have from Lemma 2.5 that

$$\langle x - x_{k+1}, Jw - Jx_{k+1}\rangle \leq 0,$$

which implies $w \in W_{k+1}$. Hence $w \in H_{k+1} \cap W_{k+1}$. Thus we obtain $F(T) \cap J^{-1}B^{-1}0 \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{x_n\}$ and $\{y_n\}$ are well defined.

We next show that $\lim\limits_{n\to\infty}\phi(x, x_n)$ exists. Note that for each $n \in \mathbb{N}\cup\{0\}$, $x_n \in W_n$ and

$$\langle x - x_n, Jz - Jx_n\rangle \leq 0, \quad \forall z \in W_n.$$

So by Lemma 2.5, we have $x_n = R_{W_n}x$. Using Lemma 2.5 again, we have

$$\phi(x, x_n) = \phi(x, R_{W_n}x) \leq \phi(x, z) - \phi(R_{W_n}x, z) \leq \phi(x, z), \ \ \forall z \in F(T)\cap J^{-1}B^{-1}0.$$

Thus $\{\phi(x, x_n)\}$ is bounded, and hence $\{x_n\}$ and $\{y_n\}$ are bounded. Since $x_{n+1} = R_{H_n\cap W_n}x \in H_n \cap W_n \subset W_n$ and $x_n = R_{W_n}x$, it follows from Proposition 2.7 that

$$\phi(x, x_n) \leq \phi(x, x_{n+1}), \ \ \forall n \in \mathbb{N}\cup\{0\}.$$

Thus $\{\phi(x, x_n)\}$ is nondecreasing and hence $\lim_{n\to\infty}\phi(x, x_n)$ exists.

We next show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Consider

$$\begin{aligned}\phi(x_n, x_{n+1}) &= \phi(R_{W_n}x, x_{n+1})\\ &\leq \phi(x, x_{n+1}) - \phi(x, R_{W_n}x)\\ &= \phi(x, x_{n+1}) - \phi(x, x_n).\end{aligned}$$

Since $\lim_{n\to\infty}\phi(x, x_n)$ exists, we have $\lim_{n\to\infty}\phi(x_n, x_{n+1}) = 0$. From $x_{n+1} = R_{H_n\cap W_n}x \in H_n$, we have

$$\phi(u_n, x_{n+1}) \leq \phi(x_n, x_{n+1}), \ \ \forall n \in \mathbb{N}\cup\{0\}.$$

Therefore, $\lim_{n\to\infty} \phi(u_n, x_{n+1}) = 0$. From Lemma 2.1, we have

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

Since $\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Next, we show that $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim\limits_{n\to\infty} \|JTy_n - Jy_n\| = 0$. From

$$\begin{aligned} \|u_n - x_{n+1}\| &= \|\alpha_n x_n + (1-\alpha_n)Ty_n - x_{n+1}\| \\ &= \|\alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(Ty_n - x_{n+1})\| \\ &\geq (1-\alpha_n)\|Ty_n - x_{n+1}\| - \alpha_n\|x_n - x_{n+1}\|, \end{aligned}$$

we have

$$\begin{aligned} \|Ty_n - x_{n+1}\| &\leq \frac{1}{1-\alpha_n}(\|u_n - x_{n+1}\| + \alpha_n\|x_n - x_{n+1}\|) \\ &\leq \frac{1}{1-\alpha_n}(\|u_n - x_{n+1}\| + \|x_n - x_{n+1}\|). \end{aligned}$$

From (3.3) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we have $\lim_{n\to\infty} \|Ty_n - x_{n+1}\| = 0$. From $\|Ty_n - x_n\| \leq \|Ty_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, we have

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0. \tag{3.4}$$

Let $w \in F(T) \cap J^{-1}B^{-1}0$. Using $y_n = J_{r_n}x_n$, from (2.3) we have

$$\begin{aligned} \phi(x_n, w) &\geq \phi(x_n, J_{r_n}x_n) + \phi(J_{r_n}x_n, w) \\ &= \phi(x_n, y_n) + \phi(y_n, w). \end{aligned}$$

Hence $\phi(x_n, y_n) \leq \phi(x_n, w) - \phi(y_n, w)$. From (3.1), we have $\phi(u_n, w) \leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(y_n, w)$ and hence

$$\phi(y_n, w) \geq \frac{\phi(u_n, w) - \alpha_n\phi(x_n, w)}{(1-\alpha_n)}.$$

Therefore, we have

$$\begin{aligned} \phi(x_n, y_n) &\leq \phi(x_n, w) - \frac{\phi(u_n, w) - \alpha_n\phi(x_n, w)}{(1-\alpha_n)} \\ &= \frac{\phi(x_n, w) - \phi(u_n, w)}{(1-\alpha_n)}. \end{aligned} \tag{3.5}$$

We also have

$$\begin{aligned} \phi(x_n, w) - \phi(u_n, w) &= \|x_n\|^2 - 2\langle x_n, Jw\rangle + \|w\|^2 - \|u_n\|^2 + 2\langle u_n, Jw\rangle - \|w\|^2 \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jw\rangle \\ &\leq |\|x_n\|^2 - \|u_n\|^2| + 2|\langle x_n - u_n, Jw\rangle| \\ &\leq (\|x_n\| + \|u_n\|)\|x_n - u_n\| + 2\|x_n - u_n\|\|Jw\|. \end{aligned}$$

From $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have $\lim_{n\to\infty}(\phi(x_n, w) - \phi(u_n, w)) = 0$. Since $\liminf_{n\to\infty}(1-\alpha_n) > 0$, from (3.5) we have $\lim_{n\to\infty}\phi(x_n, y_n) = 0$. From Lemma 2.1, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.6}$$

Since $\|Ty_n - y_n\| \le \|Ty_n - x_n\| + \|x_n - y_n\|$, from (3.4) and (3.6) we have

$$\lim_{n\to\infty}\|Ty_n - y_n\| = 0. \tag{3.7}$$

Since E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded sets. So, from (3.7) we have

$$\lim_{n\to\infty}\|JTy_n - Jy_n\| = 0. \tag{3.8}$$

Since $\{Jx_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $Jx_{n_i} \rightharpoonup z^*$. Since J is uniformly norm-to-norm continuous on bounded sets, we have from (3.6) that

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0.$$

This implies $Jy_{n_i} \rightharpoonup z^*$ and hence from (3.8), $J^{-1}z^* \in \check{F}(T)$. Putting $z = J^{-1}z^*$, we have $z \in \check{F}(T)$.

We next show that $z \in F(T) \cap J^{-1}B^{-1}0$. By the assumption, we have $z \in F(T)$. Since $B_{r_n} = (I - J_{r_n})/r_n$ and $\liminf_{n\to\infty} r_n > 0$, we also have

$$\lim_{n\to\infty}\|B_{r_n}x_n\| = \lim_{n\to\infty}\frac{\|x_n - y_n\|}{r_n} = 0.$$

If $(w^*, w) \in B$, then it follows from the monotonicity of B and $(Jy_n, B_{r_n}x_n) \in B$ that

$$\langle w - B_{r_n}x_n, w^* - Jy_n\rangle \ge 0, \quad \forall n \in \mathbb{N}\cup\{0\}.$$

Hence

$$\langle w - B_{r_{n_i}}x_{n_i}, w^* - Jy_{n_i}\rangle \ge 0.$$

Letting $i \to \infty$, we have $\langle w, w^* - z^*\rangle \ge 0$. By the maximality of B, we have $z^* \in B^{-1}0$ and hence $z \in J^{-1}B^{-1}0$.

We next show that $z = R_{F(T)\cap J^{-1}B^{-1}0}x$. Let $u = R_{F(T)\cap J^{-1}B^{-1}0}x$. From $x_{n+1} = R_{H_n\cap W_n}x$ and $u \in F(T)\cap J^{-1}B^{-1}0 \subset H_n \cap W_n$, we have

$$\phi(x, x_{n+1}) \le \phi(x, u).$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $Jx_{n_i} \rightharpoonup Jz$ we have

$$\begin{aligned}
\phi(x,z) &= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 \\
&\le \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, Jx_{n_i}\rangle + \|x_{n_i}\|^2) \\
&= \liminf_{i\to\infty}\phi(x, x_{n_i}) \\
&\le \limsup_{i\to\infty}\phi(x, x_{n_i}) \\
&\le \phi(x,u).
\end{aligned}$$

From the definition of u, we have $u = z$. Thus we obtain $z^* = Jz = Ju$.

Furthermore, we can conclude that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Jx_{n_k} \rightharpoonup z^*$, $z^* = Ju$. Hence $Jx_n \rightharpoonup z^* = Ju$.

We finally show that $x_n \to z$. From (2.2), we have

$$\phi(z, x_n) = \phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle, \ \ \forall n \in \mathbb{N} \cup \{0\}.$$

Since $x_n = R_{W_n}x$ and $z \in F(T) \cap J^{-1}B^{-1}0 \subset W_n$, we have $\phi(x, x_n) \leq \phi(x, z)$ and hence

$$\begin{aligned} \limsup_{n\to\infty} \phi(z, x_n) &= \limsup_{n\to\infty}\{\phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle\} \\ &\leq \limsup_{n\to\infty}\{\phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jx_n\rangle\} \\ &= \phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jz\rangle \\ &= \phi(z, z) = 0. \end{aligned}$$

Thus $\lim_{n\to\infty} \phi(z, x_n) = 0$ and hence $\lim_{n\to\infty} \|z - x_n\| = 0$. This completes the proof. □

As consequences of Theorem 3.3, we can obtain the following corollaries.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$ and*

$$\begin{cases} u_n = J_{r_n}x_n, \\ H_n = \{z \in E : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in E : \langle x - x_n, Jz - Jx_n\rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $R_{J^{-1}B^{-1}0}x$, where $R_{J^{-1}B^{-1}0}$ is the sunny generalized nonexpansive retraction from E onto $J^{-1}B^{-1}0$.

Proof. Putting $T = I, C = E$ and $\alpha_n = 0$ in Theorem 3.3, we can complete the proof. □

Let E be a reflexive Banach space and let $f : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. By Rockafellar's theorem [22, 23], the subdifferential $\partial f \subset E^* \times E$ of f defined by

$$\partial f(x^*) = \{x \in E : f(x^*) + \langle x, y^* - x^*\rangle \leq f(y^*), \ \forall y^* \in E^*\}$$

for all $x^* \in E^*$ is maximal monotone.

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$ and assume that $\check{F}(T) = F(T)$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ H_n = \{z \in C : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle x - x_n, Jz - Jx_n\rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$*, where* J *is the duality mapping on* E *and* $\{\alpha_n\} \subset [0,1)$ *satisfies* $\liminf_{n\to\infty}(1-\alpha_n) > 0$*. Then* $\{x_n\}$ *converges strongly to* $R_{F(T)}x$*, where* $R_{F(T)}$ *is the sunny generalized nonexpansive retraction from* E *onto* $F(T)$*.*

Proof. Set $B = \partial i_{JC}$ in Theorem 3.3, where i_{JC} is the indicator function of JC, i.e,

$$i_{JC} = \begin{cases} 0, & x^* \in JC, \\ \infty, & \text{otherwise.} \end{cases}$$

Then, we have that B is a maximal monotone operator. Let J_r be the resolvent of B. Then $J_r = R_C$ for $r > 0$, where R_C is the sunny generalized nonexpansive retraction of E onto C. Indeed, for any $x \in E$ and $r > 0$, we have from Lemma 2.5 that

$$\begin{aligned} z = J_r x \quad &\Leftrightarrow \quad x \in z + r\partial i_{JC}(Jz) \\ &\Leftrightarrow \quad x - z \in r\partial i_{JC}(Jz) \\ &\Leftrightarrow \quad i_{JC}(Jz) + \langle \frac{x-z}{r}, y^* - Jz \rangle \le i_{JC}(y^*), \ \forall y^* \in E^* \\ &\Leftrightarrow \quad 0 \ge \langle x - z, Jy - Jz \rangle, \ \forall y \in C \\ &\Leftrightarrow \quad z = R_C x. \end{aligned}$$

So, from Theorem 3.3, we obtain this corollary. □

4. Convergence theorem by the shrinking method

In this section, we prove a strong convergence theorem by the shrinking projection method [28] for generalized nonexpansive mappings with resolvents of maximal monotone operators in a Banach space.

Theorem 4.1. *Let* E *be a uniformly convex and uniformly smooth Banach space and let* C *be a nonempty closed subset of* E *such that* JC *is closed and convex. Let* $B \subset E^* \times E$ *be a maximal monotone operator with* $JC \supset D(B)$ *and let* $J_r = (I + rBJ)^{-1}$ *for all* $r > 0$*. Let* $T : C \to C$ *be a generalized nonexpansive mapping such that* $F(T) \cap J^{-1}B^{-1}0 \neq \emptyset$ *and assume that* $\check{F}(T) = F(T)$*. Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C, H_0 = C$ *and*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)TJ_{r_n}x_n, \\ H_{n+1} = \{z \in H_n : \phi(u_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$*, where* $\{\alpha_n\} \subset [0,1)$ *satisfies* $\liminf_{n\to\infty}(1-\alpha_n) > 0$*,* $\{r_n\} \subset (0,\infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* J *is the duality mapping on* E*. Then* $\{x_n\}$ *converges strongly to* $R_{F(T)\cap J^{-1}B^{-1}0}x$*, where* $R_{F(T)\cap J^{-1}B^{-1}0}$ *is the sunny generalized nonexpansive retraction from* E *onto* $F(T) \cap J^{-1}B^{-1}0$*.*

Proof. As in the proof of Theorem 3.3, we have that $F(T) \cap J^{-1}B^{-1}0$ is a sunny generalized nonexpansive retract of E.

For each $n \in \mathbb{N} \cup \{0\}$, it is easy to see that H_n is closed. Further, $JH_n = \{z^* \in H_n : \phi_*(z^*, Ju_n) \le \phi_*(z^*, Jx_n)\}$ is also closed and convex. Indeed, since $JH_0 = JC$

and JC is closed and convex, JH_0 is closed and convex. Suppose that JH_k is closed and convex for some $k \in \mathbb{N} \cup \{0\}$. Since

$$\begin{aligned}\phi_*(z^*, Ju_n) \leq \phi_*(z^*, Jx_n) &\Leftrightarrow \|Ju_n\|^2 - 2\langle u_n, z^*\rangle - \|Jx_n\|^2 + 2\langle x_n, z^*\rangle \leq 0 \\ &\Leftrightarrow \|Ju_n\|^2 - \|Jx_n\|^2 - 2\langle x_n - u_n, z^*\rangle \leq 0,\end{aligned}$$

we have JH_{k+1} is closed and convex. So, JH_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. If we can show that H_n is nonempty, then Theorem 2.6 ensures that H_n is a sunny generalized nonexpansive retract of E for all $n \in \mathbb{N} \cup \{0\}$. We will show that $F(T) \cap J^{-1}B^{-1}0 \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Put $y_n = J_{r_n}x_n$. From $H_0 = C$, we have $F(T) \cap J^{-1}B^{-1}0 \subset H_0$. Suppose that $F(T) \cap J^{-1}B^{-1}0 \subset H_k$ for some $k \in \mathbb{N}$. Let $w \in F(T) \cap J^{-1}B^{-1}0 \subset H_k$. Since J_{r_n} and T are generalized nonexpansive, from the convexity of $\|\cdot\|^2$ we have

$$\begin{aligned}\phi(u_n, w) &= \phi(\alpha_n x_n + (1-\alpha_n)Ty_n, w) \\ &= \|\alpha_n x_n + (1-\alpha_n)Ty_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Ty_n, Jw\rangle + \|w\|^2 \\ &\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Ty_n\|^2 \\ &\quad -2\alpha_n\langle x_n, Jw\rangle - 2(1-\alpha_n)\langle Ty_n, Jw\rangle + \|w\|^2 \\ &= \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(Ty_n, w) \\ &\leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(y_n, w) \qquad (4.1) \\ &\leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(x_n, w) \\ &= \phi(x_n, w).\end{aligned}$$

So, we have $w \in H_{k+1}$ and hence $F(T) \cap J^{-1}B^{-1}0 \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\{x_n\}$ and $\{y_n\}$ are well-defined.

We next prove that $\lim_{n\to\infty} \phi(x, x_n)$ exists. From $x_n = R_{H_n}x$ and Lemma 2.5, we have

$$\phi(x, x_n) = \phi(x, R_{H_n}x) \leq \phi(x, z) - \phi(R_{H_n}x, z) \leq \phi(x, z), \quad \forall z \in F(T) \cap J^{-1}B^{-1}0.$$

Thus $\{\phi(x, x_n)\}$ is bounded, and hence $\{x_n\}$ and $\{y_n\}$ are bounded. Since $H_{n+1} \subset H_n$ and $x_n = R_{H_n}x$, it follows from Proposition 2.7 that

$$\phi(x, x_n) \leq \phi(x, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Thus $\{\phi(x, x_n)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x, x_n)$ exists.

We next show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. From Lemma 2.5,

$$\begin{aligned}\phi(x_n, x_{n+1}) &= \phi(R_{H_n}x, x_{n+1}) \\ &\leq \phi(x, x_{n+1}) - \phi(x, R_{H_n}x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n).\end{aligned}$$

Therefore, $\lim_{n\to\infty} \phi(x_n, x_{n+1}) = 0$. From $x_{n+1} = R_{H_{n+1}}x \in H_{n+1}$, we have

$$\phi(u_n, x_{n+1}) \leq \phi(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\lim_{n\to\infty} \phi(u_n, x_{n+1}) = 0$. From Lemma 2.1, we have

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \qquad (4.2)$$

Since $\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, we have $\lim_{n\to\infty} \|u_n - x_n\| = 0$.

Next, we show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|JTy_n - Jy_n\| = 0$. From

$$\begin{aligned}\|u_n - x_{n+1}\| &= \|\alpha_n x_n + (1-\alpha_n)Ty_n - x_{n+1}\| \\ &= \|\alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(Ty_n - x_{n+1})\| \\ &\geq (1-\alpha_n)\|Ty_n - x_{n+1}\| - \alpha_n\|x_n - x_{n+1}\|,\end{aligned}$$

we have

$$\begin{aligned}\|Ty_n - x_{n+1}\| &\leq \frac{1}{1-\alpha_n}(\|u_n - x_{n+1}\| + \alpha_n\|x_n - x_{n+1}\|) \\ &\leq \frac{1}{1-\alpha_n}(\|u_n - x_{n+1}\| + \|x_n - x_{n+1}\|).\end{aligned}$$

From (4.2) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we have $\lim_{n\to\infty}\|Ty_n - x_{n+1}\| = 0$. From $\|Ty_n - x_n\| \leq \|Ty_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, we have

$$\lim_{n\to\infty}\|Ty_n - x_n\| = 0. \tag{4.3}$$

Let $w \in F(T) \cap J^{-1}B^{-1}0$. Using $y_n = J_{r_n}x_n$, from (2.3) we have

$$\begin{aligned}\phi(x_n, w) &\geq \phi(x_n, J_{r_n}x_n) + \phi(J_{r_n}x_n, w) \\ &= \phi(x_n, y_n) + \phi(y_n, w).\end{aligned}$$

Hence

$$\phi(x_n, y_n) \leq \phi(x_n, w) - \phi(y_n, w).$$

From (4.1), we have $\phi(u_n, w) \leq \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(y_n, w)$ and hence

$$\phi(y_n, w) \geq \frac{\phi(u_n, w) - \alpha_n\phi(x_n, w)}{1-\alpha_n}.$$

Therefore, we have

$$\begin{aligned}\phi(x_n, y_n) &\leq \phi(x_n, w) - \frac{\phi(u_n, w) - \alpha_n\phi(x_n, w)}{1-\alpha_n} \\ &= \frac{\phi(x_n, w) - \phi(u_n, w)}{1-\alpha_n}. \qquad (4.4)\end{aligned}$$

Since

$$\begin{aligned}\phi(x_n, w) - \phi(u_n, w) &= \|x_n\|^2 - 2\langle x_n, Jw\rangle + \|w\|^2 - \|u_n\|^2 + 2\langle u_n, Jw\rangle - \|w\|^2 \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jw\rangle \\ &\leq |\|x_n\|^2 - \|u_n\|^2| + 2|\langle x_n - u_n, Jw\rangle| \\ &\leq (\|x_n\| + \|u_n\|)\|x_n - u_n\| + 2\|x_n - u_n\|\|Jw\|\end{aligned}$$

and $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have $\lim_{n\to\infty}(\phi(x_n, w) - \phi(u_n, w)) = 0$. Since $\liminf_{n\to\infty}(1-\alpha_n) > 0$, from (4.4) we have $\lim_{n\to\infty}\phi(x_n, y_n) = 0$. From Lemma 2.1, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{4.5}$$

From $\|Ty_n - y_n\| \leq \|Ty_n - x_n\| + \|x_n - y_n\|$, (4.3) and (4.5), we have

$$\lim_{n\to\infty}\|Ty_n - y_n\| = 0. \tag{4.6}$$

Since E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded sets. And from (4.6), we have

$$\lim_{n\to\infty}\|JTy_n - Jy_n\| = 0. \tag{4.7}$$

Since $\{Jx_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $Jx_{n_i} \rightharpoonup z^*$. Since J is uniformly norm-to-norm continuous on bounded sets, we have from (4.5) that

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0.$$

This implies $Jy_{n_i} \rightharpoonup z^*$ and hence from (4.7), $J^{-1}z^* \in \check{F}(T)$. Putting $z = J^{-1}z^*$, we have $z \in \check{F}(T)$.

We next show that $z \in F(T) \cap J^{-1}B^{-1}0$. By the assumption, we have $z \in F(T)$. Since $B_{r_n} = (I - J_{r_n})/r_n$ and $\liminf_{n\to\infty} r_n > 0$, we also have

$$\lim_{n\to\infty}\|B_{r_n}x_n\| = \lim_{n\to\infty}\frac{\|x_n - y_n\|}{r_n} = 0.$$

If $(w^*, w) \in B$, then it follows from the monotonicity of B and $(Jy_n, B_{r_n}x_n) \in B$ that

$$\langle w - B_{r_n}x_n, w^* - Jy_n\rangle \geq 0, \ \ \forall n \in \mathbb{N}\cup\{0\}.$$

Hence

$$\langle w - B_{r_{n_i}}x_{n_i}, w^* - Jy_{n_i}\rangle \geq 0.$$

Letting $i \to \infty$, we have $\langle w, w^* - z^*\rangle \geq 0$. By the maximality of B, we have $z^* \in B^{-1}0$ and hence $z \in J^{-1}B^{-1}0$.

We next show that $z = R_{F(T)\cap J^{-1}B^{-1}0}x$. Let $u = R_{F(T)\cap J^{-1}B^{-1}0}x$. From $x_{n+1} = R_{H_{n+1}}x$ and $u \in F(T) \cap J^{-1}B^{-1}0 \subset H_{n+1}$, we have

$$\phi(x, x_{n+1}) \leq \phi(x, u).$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $Jx_{n_i} \rightharpoonup Jz$ we have

$$\begin{aligned}
\phi(x,z) &= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 \\
&\leq \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, Jx_{n_i}\rangle + \|x_{n_i}\|^2) \\
&= \liminf_{i\to\infty}\phi(x, x_{n_i}) \\
&\leq \limsup_{i\to\infty}\phi(x, x_{n_i}) \\
&\leq \phi(x,u).
\end{aligned}$$

From the definition of u, we have $u = z$. Thus we obtain $z^* = Jz = Ju$.

Furthermore, we can conclude that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Jx_{n_k} \rightharpoonup z^*$, $z^* = Ju$. Hence $Jx_n \rightharpoonup Ju$.

We finally show that $x_n \to z$. From (2.2), we have

$$\phi(z, x_n) = \phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle, \ \ \forall n \in \mathbb{N}\cup\{0\}.$$

Since $x_n = R_{H_n}x$ and $z \in F(T) \cap J^{-1}B^{-1}0 \subset H_n$, we have $\phi(x, x_n) \leq \phi(x, z)$ and hence

$$\begin{aligned}
\limsup_{n\to\infty}\phi(z, x_n) &= \limsup_{n\to\infty}\{\phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle\} \\
&\leq \limsup_{n\to\infty}\{\phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jx_n\rangle\} \\
&= \phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jz\rangle \\
&= \phi(z, z) = 0.
\end{aligned}$$

Thus $\lim_{n\to\infty}\phi(z, x_n) = 0$ and hence $\lim_{n\to\infty}\|z - x_n\| = 0$. This completes the proof. □

As consequences of Theorem 4.1, we can obtain the following corollaries.

Corollary 4.2. *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$, $H_0 = E$ and*

$$\begin{cases} u_n = J_{r_n}x_n, \\ H_{n+1} = \{z \in H_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $R_{J^{-1}B^{-1}0}x$, where $R_{J^{-1}B^{-1}0}$ is the sunny generalized nonexpansive retraction from E onto $J^{-1}B^{-1}0$.

Proof. Putting $T = I, C = E$ and $\alpha_n = 0$ in Theorem 4.1, we can complete the proof. □

Corollary 4.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$ and assume that $\check{F}(T) = F(T)$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ H_{n+1} = \{z \in H_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n\to\infty}(1 - \alpha_n) > 0$. Then $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Proof. Set $B = \partial i_{JC}$ in Theorem 4.1, where i_{JC} is the indicator function of JC. So, we obtain this corollary. □

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