Journal of Nonlinear and Convex Analysis
Volume 11, Number 1, 2010, 35–43

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

# ALGORITHMS CONSTRUCTION FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

MENGQIN LI, YONGHONG YAO*, YEONG-CHENG LIOU, AND SHIN MIN KANG

Abstract. In this paper, we construct two new algorithms for nonexpansive mappings in Hilbert spaces. We show that the proposed algorithms converge strongly to fixed points of nonexpansive mappings in Hilbert spaces.

## 1. Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

We use $Fix(T)$ to denote the set of fixed points of $T$.

Construction of fixed points of nonlinear mappings is an important and active research area. In particular, iterative algorithms for finding fixed points of nonexpansive mappings have received vast investigations since these algorithms find applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing (see, e.g., [7], [10]-[11], [21], [37]).

It is well-known that the Picard iteration $x_{n+1} = Tx_n = \cdots = T^{n+1}x$ of the mapping $T$ at a point $x \in C$ may, in general, not behave well. This means that it may not converge even in the weak topology. One way to overcome this difficulty is to use Mann's iteration algorithm that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\} \subset [0, 1]$ and the initial value $x_0 \in C$ is chosen arbitrarily. For example, Reich [23] proved that if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of $T$. However, this scheme has only weak convergence even in a Hilbert space.

Some attempts to construct iteration algorithm so that strong convergence is guaranteed have recently been made (see, e.g., [1]-[9], [12]-[20], [22]-[36], [38]).

It is our purpose in this paper to introduce two new algorithms for nonexpansive mappings in Hilbert spaces. We show that the proposed methods converge strongly to fixed points of nonexpansive mappings in Hilbert spaces.

2000 *Mathematics Subject Classification.* 49H05, 47H10.
*Key words and phrases.* Nonexpansive mapping, fixed point, implicit method, explicit method.
The third author was supported in part by NSC 98-2622-E-230-006-CC3 and NSC 98-2923-E-110-003-MY3.
*Corresponding author.

## 2. Preliminaries

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, respectively. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C.$$

The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H.$$

Moreover, $P_C$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \leq 0,$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H, y \in C.$$

In order to prove our main results, we need the following well-known lemmas.

**Lemma 2.1** ([33]). *Let $C$ be a nonempty closed convex of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demi-closed at $0$, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

**Lemma 2.2** ([29]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)x_n + \beta_n y_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.3** ([31]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, n \geq 0$ where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $R$ such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

## 3. Main results

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping. Let $\beta$ be a constant in $(0, 1)$. For each $t \in (0, 1)$, define a mapping $T_t : C \to C$ by

$$T_t x = P_C[t(\beta x) + (1 - t)Tx], \ \forall x \in C.$$

For $x, y \in C$, we have

$$\begin{aligned} \|T_t x - T_t y\| &= \|P_C[t(\beta x) + (1 - t)Tx] - P_C[t(\beta y) + (1 - t)Ty]\| \\ &\leq [1 - (1 - \beta)t]\|x - y\|, \end{aligned}$$

which implies that $T_t$ is a contraction. Using the Banach contraction principle, there exists a unique fixed point $x_t$ of $T_t$ in $C$, i.e.,

$$x_t = P_C[t(\beta x_t) + (1 - t)Tx_t]. \tag{3.1}$$

Now we show the strong convergence of this implicit algorithm.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. For each $t \in (0,1)$, let the net $\{x_t\}$ be generated by (3.1). Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point of $T$.*

*Proof.* First, we prove that $\{x_t\}$ is bounded. Take $u \in Fix(T)$. From (3.1), we have

$$\begin{aligned} \|x_t - u\| &= \|P_C[t(\beta x_t) + (1-t)Tx_t] - P_C u\| \\ &\leq \beta t\|x_t - u\| + (1-t)\|Tx_t - u\| + (1-\beta)t\|u\| \\ &\leq [1-(1-\beta)t]\|x_t - u\| + (1-\beta)t\|u\|, \end{aligned}$$

that is,

$$\|x_t - u\| \leq \|u\|.$$

Hence, $\{x_t\}$ is bounded.

Again from (3.1), we obtain

$$\begin{aligned} \|x_t - Tx_t\| &= \|P_C[t(\beta x_t) + (1-t)Tx_t] - P_C Tx_t\| \\ &\leq t\|\beta x_t - Tx_t\| \\ &\to 0 \ \ as \ \ t \to 0. \end{aligned} \tag{3.2}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.2), we have

$$\|x_n - Tx_n\| \to 0. \tag{3.3}$$

From (3.1) and (3.2), we get

$$\begin{aligned} \|x_t - u\|^2 &= \langle P_C[t(\beta x_t) + (1-t)Tx_t] - [t(\beta x_t) + (1-t)Tx_t], x_t - u\rangle \\ &\quad + \langle [t(\beta x_t) + (1-t)Tx_t] - u, x_t - u\rangle \\ &\leq \langle [t(\beta x_t) + (1-t)Tx_t] - u, x_t - u\rangle \\ &= \beta t\langle x_t - u, x_t - u\rangle + (1-t)\langle Tx_t - x_t + x_t - u, x_t - u\rangle \\ &\quad -(1-\beta)t\langle u, x_t - u\rangle \\ &\leq [1-(1-\beta)t]\|x_t - u\|^2 - (1-\beta)t\langle u, x_t - u\rangle. \end{aligned}$$

Hence,

$$\|x_t - u\|^2 \leq \langle u, u - x_t\rangle.$$

In particular,

$$\|x_n - u\|^2 \leq \langle u, u - x_n\rangle, \quad u \in Fix(T). \tag{3.4}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Noticing (3.3) we can use Lemma 2.1 to get $x^* \in Fix(T)$. Therefore we can substitute $x^*$ for $u$ in (3.4) to get

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - x_n\rangle.$$

Hence, the weak convergence of $\{x_n\}$ to $x^*$ actually implies that $x_n \to x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

To show that the entire net $\{x_t\}$ converges to $x^*$, assume $x_{t_m} \to \tilde{x} \in Fix(T)$, where $t_m \to 0$. Put $x_m = x_{t_m}$. Similarly we have

$$\|x_m - x^*\|^2 \le \langle x^*, x^* - x_m\rangle$$

Therefore,

$$\|\tilde{x} - x^*\|^2 \le \langle x^*, x^* - \tilde{x}\rangle.$$

Interchange $x^*$ and $\tilde{x}$ to obtain

$$\|x^* - \tilde{x}\|^2 \le \langle \tilde{x}, \tilde{x} - x^*\rangle.$$

Adding up the last two inequalities yields

$$2\|x^* - \tilde{x}\|^2 \le \|x^* - \tilde{x}\|^2,$$

which implies that $\tilde{x} = x^*$. This completes the proof.

□

Next we construct an explicit algorithm and prove this algorithm has strong convergence under some mild conditions on control parameters.

**Theorem 3.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0, 1]$ and $\lambda$ be a constant in $(0, 1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (1-\lambda)x_n + \lambda P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n], n \ge 0. \tag{3.5}$$

*Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \beta_n < 1$.

*Then the sequence $\{x_n\}$ generated by (3.5) strongly converges to a fixed point of $T$.*

*Proof.* First, we prove that the sequence $\{x_n\}$ is bounded. Take $u \in Fix(T)$. From (3.5), we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|(1-\lambda)x_n + \lambda P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n] - u\| \\
&\le (1-\lambda)\|x_n - u\| + \lambda\|P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n] - P_C u\| \\
&\le (1-\lambda)\|x_n - u\| + \lambda\|\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n - u\| \\
&\le (1-\lambda)\|x_n - u\| + \lambda[\alpha_n\beta_n\|x_n - u\| + (1-\beta_n)\alpha_n\|u\| \\
&\quad +(1-\alpha_n)\|x_n - u\|] \\
&= [1-(1-\beta_n)\lambda\alpha_n]\|x_n - u\| + (1-\beta_n)\lambda\alpha_n\|u\| \\
&\le \max\{\|x_n - u\|, \|u\|\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

Set $y_n = P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n]$ for all $n \geq 0$. It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n] \\ &\quad -P_C[\alpha_{n-1}\beta_{n-1}x_{n-1}) + (1-\alpha_{n-1})Tx_{n-1}]\| \\ &\leq \|[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n] \\ &\quad -[\alpha_{n-1}\beta_{n-1}x_{n-1}) + (1-\alpha_{n-1})Tx_{n-1}]\| \\ &\leq \|Tx_n - Tx_{n-1}\| + \alpha_n\|\beta_n x_n - Tx_n\| \\ &\quad +\alpha_{n-1}\|\beta_{n-1}x_{n-1} - Tx_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + M(\alpha_n + \alpha_{n-1}), \end{aligned}$$

where $M$ is a constant such that $\sup_n\{\|\beta_n x_n - Tx_n\|\} \leq M$.

Therefore, we have

$$\limsup_{n\to\infty}(\|y_n - y_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

This together with Lemma 2.2 implies that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\lambda\|y_n - x_n\| = 0.$$

Note that

$$\begin{aligned} \|x_{n+1} - Tx_n\| &\leq (1-\lambda)\|x_n - Tx_n\| + \lambda\|P_C[\alpha_n\beta_n x_n + (1-\alpha_n)Tx_n] - P_C Tx_n\| \\ &\leq (1-\lambda)\|x_n - Tx_n\| + \lambda\alpha_n\|\beta_n x_n - Tx_n\|. \end{aligned}$$

Then, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + (1-\lambda)\|x_n - Tx_n\| + \lambda\alpha_n\|\beta_n x_n - Tx_n\|, \end{aligned}$$

which implies that

$$\|x_n - Tx_n\| \leq \frac{1}{\lambda}\|x_n - x_{n+1}\| + \alpha_n\|\beta_n x_n - Tx_n\| \to 0.$$

Let the net $\{x_t\}$ be defined by (3.1). By Theorem 3.1, we have $x_t \to x^*$ as $t \to 0$. Next we prove $\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle \leq 0$. Set $y_t = t(\beta x_t) + (1-t)Tx_t$.

$$\begin{aligned} \|x_t - x_n\|^2 &= \langle P_C y_t - x_n, P_C y_t - x_n\rangle \\ &= \langle P_C y_t - y_t, P_C y_t - x_n\rangle + \langle y_t - x_n, x_t - x_n\rangle \\ &\leq \langle y_t - x_n, x_t - x_n\rangle \\ &= t\langle x_t - x_n, x_t - x_n\rangle - (1-\beta)t\langle x_t, x_t - x_n\rangle \\ &\quad +(1-t)\langle Tx_t - Tx_n, x_t - x_n\rangle + (1-t)\langle Tx_n - x_n, x_t - x_n\rangle \\ &\leq \|x_t - x_n\|^2 - (1-\beta)t\langle x_t, x_t - x_n\rangle + (1-t)\langle Tx_n - x_n, x_t - x_n\rangle \\ &\leq \|x_t - x_n\|^2 - (1-\beta)t\langle x_t, x_t - x_n\rangle + M\|Tx_n - x_n\|, \end{aligned}$$

where $M > 0$ such that $\sup\{\|x_t\|^2, 2\|x_t - Tx_n\|, \|x_t - x_n\|, t \in (0,1), n \geq 0\} \leq M$. It follows that

$$\langle x_t, x_t - x_n\rangle \leq \frac{M}{(1-\beta)t}\|Tx_n - x_n\|.$$

Therefore,

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t, x_t - x_n\rangle \leq 0. \tag{3.6}$$

We note that

$$\begin{aligned}\langle x^*, x^* - x_n\rangle &= \langle x^*, x^* - x_t\rangle + \langle x^* - x_t, x_t - x_n\rangle + \langle x_t, x_t - x_n\rangle \\ &\leq \langle x^*, x^* - x_t\rangle + \|x^* - x_t\|\|x_t - x_n\| + \langle x_t, x_t - x_n\rangle \\ &\leq \langle x^*, x^* - x_t\rangle + \|x^* - x_t\|M + \langle x_t, x_t - x_n\rangle.\end{aligned}$$

This together with $x_t \to x^*$ and (3.6) imply that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \leq 0.$$

Hence, we have

$$\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle \leq 0.$$

Finally we show that $x_n \to x^*$. First, we set $u_n = \alpha_n\beta_n x_n + (1-\alpha_n)Tx_n, n \geq 0$. Note that

$$\langle P_C[u_n] - u_n, P_C[u_n] - x^*\rangle \leq 0.$$

From (3.5), we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \langle P_C[u_n] - u_n, P_C[u_n] - x^*\rangle + \langle u_n - x^*, y_n - x^*\rangle \\ &\leq \langle u_n - x^*, y_n - x^*\rangle \\ &= \alpha_n\beta_n\langle x_n - x^*, y_n - x^*\rangle + (1-\beta_n)\alpha_n\langle x^*, x^* - y_n\rangle \\ &\quad +(1-\alpha_n)\langle Tx_n - x^*, y_n - x^*\rangle \\ &\leq [\alpha_n\beta_n + (1-\alpha_n)]\|x_n - x^*\|\|y_n - x^*\| + (1-\beta_n)\alpha_n\langle x^*, x^* - y_n\rangle \\ &\leq \frac{1-(1-\beta_n)\alpha_n}{2}(\|x_n - x^*\|^2 + \|y_n - x^*\|^2) + (1-\beta_n)\alpha_n\langle x^*, x^* - y_n\rangle \\ &\leq \frac{1-(1-\beta_n)\alpha_n}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 + (1-\beta_n)\alpha_n\langle x^*, x^* - y_n\rangle.\end{aligned}$$

It follows that

$$\|y_n - x^*\|^2 \leq [1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\langle x^*, x^* - y_n\rangle. \tag{3.7}$$

From (3.5) and (3.7), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1-\lambda)\|x_n - x^*\|^2 + \lambda\|y_n - x^*\|^2 \\ &\leq (1-\lambda)\|x_n - x^*\|^2 + \lambda[1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 \\ &\quad +2\lambda\alpha_n(1-\beta_n)\langle x^*, x^* - y_n\rangle \\ &= [1-\lambda(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\lambda\alpha_n(1-\beta_n)\langle x^*, x^* - y_n\rangle.\end{aligned}$$

We can check that all assumptions of Lemma 2.3 are satisfied. Therefore, $x_n \to x^*$. This completes the proof. □

**Corollary 3.3.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ and $\lambda$ be a constant in $(0, 1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda P_C[(1 - \alpha_n)Tx_n], \ n \geq 0.$$

*Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.*

## References


[1] A. Aleyner and S. Reich, *Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach spaces*, *J. Math. Anal. Appl.* **343** (2008), 427–435.

[2] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties*, *Journal of Nonlinear and Convex Analysis* **10** (2009), 131–147.

[3] H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996), 150–159.

[4] F. E. Browder, *Convergence of approximation to fixed points of nonexpansive nonlinear mappings in Hilbert spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] R. E. Bruck, W. A. Kirk and S. Reich, *Strong and weak convergence theorems for locally nonexpansive mappings in Banach spaces*, Nonlinear Analysis **6** (1982), 151–155.

[7] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20** (2004), 103–120.

[8] L. C. Ceng, S. M. Guu and J. C. Yao, *Hybrid viscosity-like approximation methods for nonexpansive mappings in Hilbert spaces*, *Comput. Math. Appl.* **58** (2009), 605–617.

[9] C. E. Chidume and C. O. Chidume, *Iterative approximation of fixed points of nonexpansive mappings*, J. Math. Anal. Appl. **318** (2006), 288–295.

[10] P. L. Combettes, *The convex feasibility problem in image recovery*, in Advances in Imaging and Electron Physics, P. Hawkes (ed.),Academic Press, New York, 1996, pp. 155–270.

[11] H. W. Engl and A. Leitao, *A Mann iterative regularization for elliptic Cauchy problems*, Numer. Funct. Anal. Optim. **22** (2001), 861–884.

[12] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[13] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Am. Math. Soc. **73** (1967), 957–961.

[14] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, Journal of Nonlinear and Convex Analysis **10** (2009), 21–32.

[15] Y. Kimura, K. Nakajo and W. Takahashi, *Strongly convergent iterative schemes for a sequence of nonlinear mappings*, Journal of Nonlinear and Convex Analysis **9** (2008), 407–416.

[16] P. L. Lions, *Approximation de points fixes de contractions*, C.R. Acad. Sci. Sĕr. A-B Paris **284** (1977), 1357–1359.

[17] P. E. Mainge, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, *J. Math. Anal. Appl.* **325** (2007), 469–479.

[18] A. Moudafi, *Krasnoselski-Mann iteration for hierarchical fixed-point problems*, Inverse Problems **23** (2007), 1635–1640.

[19] Z. Opial, *Weak convergence of the sequence of successive approximations of nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 595–597.

[20] S. Plubtieng and R. Wangkeeree, *Strong convergence of modified Mann iterations for a countable family of nonexpansive mappings*, *Nonlinear Analysis* **70** (2009), 3110–3118.

[21] C. I. Podilchuk and R. J. Mammone, *Image recovery by convex projections using a least-squares constraint*, J. Optim. Soc. Amer. **7** (1990), 517–521.

[22] X. Qin, Y. J. Cho, J. I. Kang and S. M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math. **230** (2009), 121–127.

[23] S. Reich, *Weak convergence theorems for non-expansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[24] S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approximation Theory **24** (1978), 269–272.

[25] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[26] S. Reich and A. J. Zaslavski, *Convergence of generic infinite products of nonexpansive and uniformly continuous operators*, Nonlinear Analysis **36** (1999), 1049–1065.

[27] S. Reich and A. J. Zaslavski, *Convergence of Krasnoselskii-Mann iterations of nonexpansive operators*, Mathematical and Computer Modelling **32** (2000), 1423–1431.

[28] N. Shioji and W. Takahashi, *Strong convergence of approximated sequence for non-expansive mappings*, Proc. Amer. Math. Soc. **125** (1997), 3641–3645.

[29] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory Appl. **2005** (2005), 103–123.

[30] R. Wittmann, *Approximation of fixed points of non-expansive mappings*, Arch. Math. **58** (1992), 486–491.

[31] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256.

[32] H. K. Xu, *An iterative approach to quadratic optimization*, J. Optimiz. Theory Appl. **116** (2003), 659–678.

[33] H. K. Xu, *Viscosity approximation methods for non-expansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[34] I. Yamada and N. Ogura, *Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. **25** (2004), 619–55.

[35] Y. Yao, R. Chen and J. C Yao, *Strong convergence and certain control conditions of modified Mann iteration*, Nonlinear Analysis **68** (2008), 1687–1693.

[36] Y. Yao and Y. C. Liou, *Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed point problems*, Inverse Problems **24** (2008), 501–508.

[37] D. Youla, *Mathematical theory of image restoration by the method of convex projection*, in Image Recovery Theory and Applications, H.Stark (ed.), Academic Press, Orlando, 1987, pp. 29–77.

[38] H. Zegeye and N. Shahzad, *Viscosity approximation methods for a common fixed point of finite family of nonexpansive mappings*, Appl. Math. Comput. **191** (2007), 155–163.

*Manuscript received September 14, 1009*
*revised December 19, 2009*

Mengqin Li
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
*E-mail address*: limengqin@tjpu.edu.cn

Yonghong Yao
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
*E-mail address*: yaoyonghong@yahoo.cn

Yeong-Cheng Liou
Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan
*E-mail address*: simplex_liou@hotmail.com

Shin Min Kang
Department of Mathematics and the RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea
*E-mail address*: smkang@gnu.ac.kr