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FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS AND STRICT CONVEXITY OF BANACH SPACES

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Abstract. In this paper, we first prove a fixed point theorem for generalized nonexpansive type mappings in a Banach space by using Kohsaka and Takahashi's fixed point theorem [10] for nonspreading mappings. Then using Takahashi, Yao and Kohsaka's result [21], we obtain a necessary and sufficient condition for the existence of fixed points of generalized nonexpansive type mappings. Further, we prove a fixed point theorem for nonspreading mappings with compact domains in a Banach space. Using this result, we give a necessary and sufficient condition for strict convexity of Banach spaces.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then a mapping T from C into itself is said to be *firmly nonexpansive* [2] if

$$\|Tx - Ty\| \leq \|r(x-y) + (1-r)(Tx-Ty)\|$$

for all $r > 0$ and $x, y \in C$. It is known that T is firmly nonexpansive if and only if there exists an accretive operator $A \subset E \times E$ such that $D(A) \subset C \subset R(I + A)$ and $Tx = (I + A)^{-1}x$ for all $x \in C$. In this case, $F(T) = A^{-1}0$ holds. It is also known that T is firmly nonexpansive if and only if for all $x, y \in C$, there exists $j \in J(Tx - Ty)$ such that

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle,$$

where J is the normalized duality mapping from E into 2^{E^*}.

Recently, Kohsaka and Takahashi [9] introduced a new class of firmly nonexpansive type mappings in a Banach space and then they showed in [10] that the class coincides with that of resolvents of monotone operators in a Banach space. Further, they introduced the class of nonspreading mappings in [10] which contains the class of firmly nonexpansive type mappings and then showed that every nonspreading mapping in a Banach space with a fixed point is relatively nonexpansive in the sense of Matsushita and Takahashi [11, 12]. Moreover, they proved a fixed point theorem for a single nonspreading mapping and a common fixed point theorem for a commutative family of nonspreading mappings in a Banach space. Very recently, Takahashi, Yao and Kohsaka [21] studied the fixed point property for nonspreading

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mappings and unbounded sets in a Banach space and they extended Ray's theorem [14] in a Hilbert space to that of a Banach space. On the other hand, motivated by Kohsaka and Takahashi [9, 10], Ibaraki and Takahashi [6, 7] defined the class of generalized nonexpansive type mappings in a Banach space which is connected with the class of nonspreading mappings and then they obtained some results for generalized nonexpansive type mappings.

In this paper, we first prove a fixed point theorem for generalized nonexpansive type mappings in a Banach space by using Kohsaka and Takahashi's fixed point theorem [10] for nonspreading mappings. Then using Takahashi, Yao and Kohsaka's result [21], we obtain a necessary and sufficient condition for the existence of fixed points of generalized nonexpansive type mappings. Further, we prove a fixed point theorem for nonspreading mappings with compact domains in a Banach space. Using this result, we give a necessary and sufficient condition for strict convexity of Banach spaces.

2. Preliminaries

Throughout this paper the ground field for all Banach spaces is the real field $\mathbb{R}$. Let E be a Banach space and let E^* be the dual space of E. Then the *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$.

Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere centered at the origin of E. Then the Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. The norm of E is also said to be *Gâteaux differentiable*. A Banach space E is *reflexive* if $E = E^{**}$. A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$.

We know the following; see, for instance, [4] and [18]:

(1) If E is smooth, then J is single-valued;
(2) if E is reflexive, then J is onto;
(3) if E is strictly convex, then J is one-to-one, that is, $Jx \bigcap Jy \neq \emptyset$ implies that $x = y$.

Let E be a Banach space and let T be a mapping from a subset C of E into itself. We denote by

$$F(T) = \{x \in C : Tx = x\}$$

the set of fixed points of T. We say that a mapping T from a subset C of a smooth Banach space E into itself is of *firmly nonexpansive type* [9] if

$$\langle Tx - Ty, JTx - JTy \rangle \leq \langle Tx - Ty, Jx - Jy \rangle$$

for all $x, y \in C$, where J is the duality mapping of E into E^*. Let us consider the function ϕ from $E \times E$ into $\mathbb{R}$ defined by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2$$

for all $u, v \in E$. We know that

$$0 \leq (\|u\| - \|v\|)^2 \leq \phi(u, v) \tag{2.2}$$

for all $u, v \in E$. Further, we have that for any $u, v, w \in E$,

$$\phi(u, v) = \phi(u, w) + \phi(w, v) + 2\langle u - w, Jw - Jv\rangle. \tag{2.3}$$

It is also known that

$$2\langle u - v, Jw - Jz\rangle = \phi(u, z) + \phi(v, w) - \phi(u, w) - \phi(v, z) \tag{2.4}$$

for all $u, v, w, z \in E$. Let $\phi_*\colon E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.5}$$

for $x, y \in E$. A mapping $T : C \to C$ is called *nonspreading* [10] if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. A mapping $T : C \to C$ is of generalized nonexpansive type [6, 7] or skew-nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. Using (2.4), Kohsaka and Takahashi showed in [9] that a mapping $T : C \to C$ is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$$

for each $x, y \in C$. So, we have that a firmly nonexpansive type mapping is nonspreading. Further, Kohsaka and Takahashi [10] proved the following theorem.

Theorem 2.1. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then the following are equivalent:*

(i) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(ii) *$F(T)$ is nonempty.*

Very recently, Takahashi, Yao and Kohsaka [21] proved the following theorem which extends Ray's theorem in a Hilbert space to that of a Banach space.

Theorem 2.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex suvset of E. Then, the following are equivalent:*

(i) *Every nonspreading mapping of C into itself has a fixed point in C;*
(ii) *Every firmly nonexpansive type mapping of C into itself has a fixed point in C;*
(iii) *C is bounded.*

3. Fixed point theorems

In this section, we first prove a fixed point theorem for skew-nonspreading mappings in a Banach space by using Kohsaka and Takahashi's fixed point theorem [10]. Before proving it, we show the following lemma.

Lemma 3.1. *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to C$ be a skew-nonspeading mapping. Then, the following hold:*

(i) $JTJ^{-1} : J(C) \to J(C)$ *is a nonspeading mapping;*
(ii) $F(T) = \emptyset$ *if and only if* $F(JTJ^{-1}) = \emptyset$;
(iii) $\|T^n x\| = \|(JTJ^{-1})^n Jx\|$ *for each* $x \in C$ *and* $n \in \mathbb{N}$.

Proof. Since E is smooth and strictly convex, E^* is also smooth and strictly convex. Thus the duality mapping J^* from E^* into $2^{E^{**}}$ is a single-valued injection. It follows from the reflexivity of E that J^* is a surjection from E^* into 2^E and $J^* = J^{-1}$. Thus $JTJ^{-1} : J(C) \to J(C)$ is well-defined. In fact, for $x^* \in J(C)$, we have a unique $x \in C$ with $x^* = Jx$. So, we have $JTJ^{-1}x^* = JTx \in J(C)$.

(i) Write $T^* = JTJ^{-1}$. If $x, y \in C$, $x^* = Jx$ and $y^* = Jy$, then we have

$$\begin{aligned}\phi_*(T^*x^*, T^*y^*) &+ \phi_*(T^*y^*, T^*x^*)\\ &= \phi_*(JTJ^{-1}Jx, JTJ^{-1}Jy) + \phi_*(JTJ^{-1}Jy, JTJ^{-1}Jx)\\ &= \phi_*(JTx, JTy) + \phi_*(JTy, JTx)\\ &= \phi(Ty, Tx) + \phi(Tx, Ty)\end{aligned}$$

and

$$\begin{aligned}\phi_*(T^*x^*, y^*) + \phi_*(T^*y^*, x^*) &= \phi_*(JTJ^{-1}Jx, Jy) + \phi_*(JTJ^{-1}Jy, Jx)\\ &= \phi_*(JTx, Jy) + \phi_*(JTy, Jx)\\ &= \phi(y, Tx) + \phi(x, Ty).\end{aligned}$$

Since T is skew-nonspreading, we have

$$\begin{aligned}\phi_*(T^*x^*, T^*y^*) + \phi_*(T^*y^*, T^*x^*) &= \phi(Ty, Tx) + \phi(Tx, Ty)\\ &\le \phi(y, Tx) + \phi(x, Ty)\\ &= \phi_*(T^*x^*, y^*) + \phi_*(T^*y^*, x^*).\end{aligned}$$

Hence JTJ^{-1} is nonspreading.

(ii) We also have that for each $x \in C$,

$$\begin{aligned}x \in F(T) &\Leftrightarrow x = Tx\\ &\Leftrightarrow Jx = JTx\\ &\Leftrightarrow Jx = JTJ^{-1}Jx\\ &\Leftrightarrow Jx \in F(JTJ^{-1})\\ &\Leftrightarrow x \in J^{-1}F(JTJ^{-1}).\end{aligned}$$

Thus $F(T) = J^{-1}F(JTJ^{-1})$. So, $F(T) = \emptyset$ if and only if $F(JTJ^{-1}) = \emptyset$.

(iii) We show by induction that

$$(JTJ^{-1})^n Jx = JT^n x$$

for each $x \in C$ and $n \in \mathbb{N}$. In fact, for any $x \in C$, we have $JTx = JTJ^{-1}Jx$. So, the equality is true in the case of $k = 1$. Suppose that

$$(JTJ^{-1})^k Tx = JT^k x$$

for some $k \in \mathbb{N}$. Then, we have

$$\begin{aligned}(JTJ^{-1})^{k+1}Tx &= (JTJ^{-1})(JTJ^{-1})^k Tx \\ &= JTJ^{-1}JT^k x \\ &= JTT^k x \\ &= JTT^{k+1}x.\end{aligned}$$

So, the equality is true in the case of $k + 1$. Hence

$$\|T^n x\| = \|JT^n x\| = \|(JTJ^{-1})^n Jx\|$$

for each $x \in C$ and $n \in \mathbb{N}$. □

Theorem 3.2. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed subset of E such that $J(C)$ is closed and convex, and let $T : C \to C$ be a skew-nonspreading mapping. Then the following are equivalent:*
(i) There is an element $x \in C$ such that $\{T^n x\}$ is bounded;
(ii) $F(T)$ is nonempty.

Proof. From Lemma 3.1 (i), $JTJ^{-1} : JC \to JC$ is nonspreading. From Theorem 2.1, it follows that $F(JTJ^{-1})$ is nonempty if and only if there is an element $x \in C$ such that $\{(JTJ^{-1})^n Jx\}$ is bounded. So, we have from Lemma 3.1 (ii) and (iii) that $F(T)$ is nonempty if and only if there is an element $x \in C$ such that $\{T^n x\}$ is bounded. □

Next, using Theorems 3.2 and 2.2, we obtain a necessary and sufficient condition for the existence of fixed points of generalized nonexpansive type mappings. This is connected with Ray's theorem [14] and Takahashi's theorem [19] in a Hilbert space.

Lemma 3.3. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a nonspreading mapping of JC into itself. Then $J^{-1}TJ$ is a skew-nonspreading mapping of C into itself.*

Proof. Put $S = J^{-1}TJ$. Then, we have that for any $x \in C$, $Sx = J^{-1}TJx \in C$. So, S is a mapping of C into itself. Further, we have that for $x, y \in C$, $x^* = Jx$ and $y^* = Jy$,

$$\begin{aligned}\phi(Sx, Sy)&+\phi(Sy, Sx) \\ &= \phi(J^{-1}TJx, J^{-1}TJy) + \phi(J^{-1}TJy, J^{-1}TJx) \\ &= \phi_*(TJy, TJx) + \phi_*(TJx, TJy) \\ &= \phi_*(Ty^*, Tx^*) + \phi_*(Tx^*, Ty^*)\end{aligned}$$

and

$$\begin{aligned}\phi(Sx,y)+&\phi(Sy,x)\\ &= \phi(J^{-1}TJx,y)+\phi(J^{-1}TJy,x)\\ &= \phi_*(Jy,TJx)+\phi_*(Jx,TJy)\\ &= \phi_*(y^*,Tx^*)+\phi_*(x^*,Ty^*).\end{aligned}$$

Since T is a nonspreading mapping, we have

$$\begin{aligned}\phi(Sx,Sy)+&\phi(Sy,Sx)\\ &= \phi_*(Ty^*,Tx^*)+\phi_*(Tx^*,Ty^*)\\ &\le \phi_*(y^*,Tx^*)+\phi_*(x^*,Ty^*)\\ &= \phi(Sx,y)+\phi(Sy,x).\end{aligned}$$

So, S is a skew-nonspreading mapping. □

Using Lemma 3.3, we obtain the following theorem.

Theorem 3.4. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed subset of E such that $J(C)$ is closed and convex. Then the following are equivalent:*
(i) Every skew-nonspreading mapping of C into itself has a fixed point in C;
(ii) C is bounded.

Proof. We know by Theorem 3.2 that if C is bounded, then every skew-nonspreading mapping of C into itself has a fixed point in C. So, we have that (ii) implies (i). Let us prove (ii) ⇒ (i). If C is unbounded, then $J(C)$ is unbounded. We know from Theorem 2.2 that there exists a nonspreading mapping T of $J(C)$ into itself such that T has no fixed points in $J(C)$. Using Lemma 3.3, we have that $J^{-1}TJ : C \to C$ is a skew-nonspreading mapping of C into itself, which has no fixed points in C; see also [20]. This means that (ii) ⇒ (i). □

4. Strict convexity of Banach spaces

In this section, we first prove a fixed point theorem for nonspreading mappings with compact domains in a smooth and strictly convex Banach space.

Theorem 4.1. *Let E be a smooth and strictly convex Banach space and let C be a nonempty compact convex subset of E. Let T be a nonspreading mapping of C into itself. Then, $F(T)$ is nonempty.*

Proof. Take $x \in C$. Let $y \in C$, $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ be given. Since T is nonspreading and (2.3) holds, we have

$$\begin{aligned}&\phi(T^{k+1}x,Ty)+\phi(Ty,T^{k+1}x)\\ &\quad= \phi(T^{k+1}x,y)+\phi(Ty,T^kx)\\ &\quad\le \phi(T^{k+1}x,Ty)+\phi(Ty,y)+2\langle T^{k+1}x-Ty, JTy-Jy\rangle+\phi(Ty,T^kx).\end{aligned}$$

This implies that

$$0 \le \phi(Ty,y)+\phi(Ty,T^kx)-\phi(Ty,T^{k+1}x)+2\langle T^{k+1}x-Ty, JTy-Jy\rangle.$$

Summing these inequalities with respect to $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned} 0 \leq & n\phi(Ty, y) \\ & + \phi(Ty, x) - \phi(Ty, T^n x) + 2\langle \sum_{k=0}^{n-1} T^{k+1}x - nTy, JTy - Jy \rangle. \end{aligned}$$

Dividing this inequality by n, we have

$$0 \leq \phi(Ty, y) + \frac{1}{n}\{\phi(Ty, x) - \phi(Ty, T^n x)\} + 2\langle S_n(Tx) - Ty, JTy - Jy \rangle, \tag{4.1}$$

where $S_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} T^k z$ for all $z \in C$. Since $\{S_n(Tx)\} \subset C$ and C is compact, we have a subsequence $\{S_{n_i}(Tx)\}$ of $\{S_n(Tx)\}$ such that $S_{n_i}(Tx) \to u \in C$. Letting $n_i \to \infty$ in (4.1), we obtain

$$0 \leq \phi(Ty, y) + 2\langle u - Ty, JTy - Jy \rangle. \tag{4.2}$$

Putting $y = u$ in (4.2), we have from (2.4) that

$$\begin{aligned} 0 &\leq \phi(Tu, u) + 2\langle u - Tu, JTu - Ju \rangle \\ &= \phi(Tu, u) + \phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u) \\ &= -\phi(u, Tu). \end{aligned}$$

Hence we have $\phi(u, Tu) \leq 0$ and hence $\phi(u, Tu) = 0$. Since E is strictly convex, we have $u = Tu$. Therefore $F(T)$ is nonempty. This completes the proof. □

Next, we show that the strict convexity of E in Theorems 2.1 and 3.2 can not be omitted. Before showing it, we prove the following lemma.

Lemma 4.2. *Let E be a smooth Banach space and let J be the duality mapping of E into E^*. If E is not strictly convex, then there are $u, v \in S(E)$ with $u \neq v$ such that $\phi(x, y) = 0$ for each $x, y \in [u, v]$, where*

$$[u, v] = \{(1-\alpha)u + \alpha v : \alpha \in [0, 1]\}.$$

Moreover, $J([u, v])$ consists of one point.

Proof. If E is not strictly convex, then there exist $u, v \in S(E)$ such that $u \neq v$ and $J(u) = J(v)$. In fact, if E is a smooth Banach space, then E is strictly convex if and only if for any $x, y \in E$ with $x \neq y$, $Jx \neq Jy$. So, if E is not strictly convex, then there exist $x, y \in E$ such that $x \neq y$ and $Jx = Jy$. Such $x, y \in E$ satisfy $\|x\| = \|y\| \neq 0$. Putting $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$, we obtain that $\|u\| = \|v\| = 1$, $u \neq v$ and

$$J(u) = \frac{1}{\|x\|}J(x) = \frac{1}{\|y\|}J(y) = J(v).$$

Put $x^* = J(u) = J(v)$. Then we have $x^* \in S(E^*)$. Further, we have

$$\langle u, x^* \rangle = \langle v, x^* \rangle = 1.$$

If $c \in [u, v]$, then $c = (1-\alpha)u + \alpha v$ for some $\alpha \in [0, 1]$. So, we have that

$$\langle c, x^* \rangle = \langle (1-\alpha)u + \alpha v, x^* \rangle = (1-\alpha)\langle u, x^* \rangle + \alpha\langle v, x^* \rangle = 1$$

and

$$1 = \langle c, x^* \rangle \leq \|c\| \leq (1-\alpha)\|u\| + \alpha\|v\| = 1.$$

Hence $\langle c, x^* \rangle = 1 = \|c\|$. Thus we have that $\langle c, x^* \rangle = \|c\|^2 = \|x^*\|^2$ for each $c \in [u, v]$. Therefore, $J(c) = \{x^*\}$ for all $c \in [u, v]$. Further, we have that for each $x, y \in [u, v]$,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 = 1 - 2\langle x, x^* \rangle + 1 = 0.$$

This completes the proof. □

Now, we can prove the following theorem.

Theorem 4.3. *Let E be a smooth Banach space. Then, the following are equivalent:*

(i) *E is strictly convex;*
(ii) *For every $u, v \in S(E)$ with $u \neq v$ and every nonspreading mapping T of $[u, v]$ into itself, T has a fixed point in $[u, v]$.*

Proof. Let us assume (i). Take $u, v \in S(E)$ with $u \neq v$. Then $[u, v]$ is a nonempty compact convex subset of a smooth and strictly convex Banach space E. So, we have from Theorem 4.1 that every nonspreading mapping T of $[u, v]$ into itself has a fixed point in $[u, v]$. This means that (i) implies (ii). Let us show (ii) $\Rightarrow$ (i). Let E be a smooth Banach space. If E is not strictly convex, then it follows from Lemma 4.2 that there exist $u, v \in S(E)$ with $u \neq v$ such that $\phi(x, y) = 0$ for each $x, y \in [u, v]$, where

$$[u, v] = \{(1 - \alpha)u + \alpha v : \alpha \in [0, 1]\}.$$

Define $T : [u, v] \to [u, v]$ by

$$T((1 - \alpha)u + \alpha v) = \begin{cases} \alpha u + (1 - \alpha)v, & \text{if} \quad \alpha \neq \frac{1}{2}, \\ v, & \text{if} \quad \alpha = \frac{1}{2}. \end{cases}$$

Then we can see that $F(T) = \emptyset$. Since $\phi(x, y) = 0$ for each $x, y \in C$, T satisfies that

$$\phi(Tx, Ty) + \phi(Ty, Tx) = 0 = \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in [u, v]$. So, T is a nonspreading mapping of $[u, v]$ into itself and this contadicts (ii). Therefore, (ii) $\Rightarrow$ (i). □

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