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STOCHASTIC TRIGONOMETRY AND STOCHASTIC INVARIANTS

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Abstract. We study a geometry where each point is described by stochastic coordinates. In the first part we deal with a stochastic analog of trigonometry and provide some applications in the stochastic framework. In the second part we introduce the concept of stochastic transforms which are transforms that commute with the expectation operator, and study their properties. In the last part we prove that these transforms are harmonic and discuss the geometry induced by them.

1. Introduction

When measuring a length for instance, one introduces inadvertently some errors of measurement. Since these errors are due to multiple independent causes, by the central limit theorem it makes sense to consider them normally distributed with mean zero. For instance, if a square has the side equal to ℓ, and the error of measurement is denoted by ϵ, with $\epsilon \sim N(0, k)$ (normally distributed with zero mean and standard deviation k), then the measured length is a random variable equal to $\hat{\ell} = \ell + \epsilon$. Then the estimated area of the square computed from the measured length is given by

$$E(\hat{\ell}^2) = E\big((\ell + \epsilon)^2\big) = \ell^2 + E(\epsilon^2) = \ell^2 + k^2,$$

which is with an amount k^2 larger than the real area of the square. This might be a significant error, especially if the errors tend to cumulate as new measurements are made. If one continues with estimating the volume of a cube of side ℓ, then we obtain

$$E(\hat{\ell}^3) = \ell^3 + 3\ell k^2 + O(\epsilon^3),$$

which implies an error which cannot be neglected since increases linearly with respect to the cube side ℓ.

It is important in our analysis to distinguish between the properties of the elements we are measuring. Some of them are *stochastic elements*, which means that one or more underlying parameters are random variables, and the other are *fixed elements* that are deterministic elements which can be measured exactly. We shall discuss next the case of a few stochastic elements such as points, lines, and circles.

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A *stochastic point* $\hat{M}$ in the plane is a point with at least one of the coordinates stochastic. (The hat will always be used to denote a stochastic element). If $(\hat{X}_M, \hat{Y}_M)$ are the Cartesian coordinates of $\hat{M}$, then either $\hat{X}_M$, or $\hat{Y}_M$, or both coordinates are random variables. The expected position of the stochastic point $\hat{M}$ is $E(\hat{M}) = \big(E(\hat{X}_M), E(\hat{Y}_M)\big) = (X_M, Y_M) = M$. A fixed point M is a point with both coordinates (X_M, Y_M) deterministic.

For instance, due to vibration, a molecule contained in a piece of paper does not have fixed coordinates; its coordinates are stochastic, so that one can think of it as a stochastic point in the plane. If three such molecules are considered, the area of the triangle defined by them is a random variable. One problem is whether one can recover the expected area of this triangle from the expected positions of the molecules. If the answer is positive we say that we have obtained a *stochastic invariant*; in general this concept stands for some measurable concept which is not lost in the stochasticity and can be recovered. The same mechanism can be applied for the center of mass of the molecules, for instance. More stochastic invariants will be investigated in section 4. These stochastic invariants will be used to define the stochastic transforms of the plane in the section 5.

A *stochastic line* in the standard form $y = \hat{m}x + \hat{b}$ has at least one of the parameters $\hat{m}$ or $\hat{b}$ stochastic. If only the slope is stochastic, then the sample space consists of lines passing through the fixed point $(0, b)$. If just the parameter $\hat{b}$ is stochastic, then the sample space consists of a family of parallel lines of slope m.

A *stochastic circle* $\mathcal{C}(\hat{O}, \hat{r})$ might have either a stochastic radius $\hat{r}$ or a stochastic center $\hat{O}$, or both. One may define any type of *stochastic polygon* if at least one of the vertices is a stochastic point.

In the first part of this paper we introduce a new type of trigonometry with stochastic elements. Here we shall discuss the stochastic sine and stochastic cosine functions and their properties. A few applications given in section 3 will show how different the new geometry with stochastic elements can be from the Euclidean one.

In section 4 we shall investigate those properties which are stochastically invariant. This leads to the definition of the stochastic invariants. We provide examples of stochastic invariants on the stochastic line (the line where the coordinate is stochastic) and on the stochastic plane (the plane where the cartesian coordinates are stochastic).

In section 5 we introduce the concept of stochastic transform and study its properties. In section 6 we provide the theorems of characterization of stochastic transforms on the stochastic line and stochastic plane as linear and harmonic functions, respectively. In the fifth section we deal with the geometry introduced by the invertible stochastic transforms and discuss its relationship with the metrical and affine geometries. In the last section we assume the coordinates of the point depend on time and are defined as Brownian motions and show that their images through a stochastic transform are also Brownian motions with a different time scale.

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2. Stochastic trigonometric functions

This section defines two stochastic functions, called *stochastic sine* and *stochastic cosine*, which play a similar role to the functions sine and cosine from the usual trigonometry.

Consider a fixed point M on the fixed trigonometric circle $C(O, 1)$. Let t be its argument, so the fixed coordinates of M are $X_M = \cos t$ and $Y_M = \sin t$. Now we shall consider that due to some measurement error the angle t becomes stochastic and is replaced by the random variable $\tau_t = \hat{t} = t + \epsilon_t$, where the error $\epsilon_t \sim N(0, k_t)$. The standard deviation k_t is considered smooth with respect to the parameter t. The probability density function of ϵ_t will be denoted by φ_t.

The corresponding stochastic point $\hat{M}$ will have the stochastic coordinates $\hat{X}_M = \cos \tau_t$ and $\hat{Y}_M = \sin \tau_t$. The expected position of the stochastic point $\hat{X}_M$ is $E(\hat{M}) = \overline{M} = (\overline{X}_M, \overline{Y}_M)$. In order to find the expected coordinates $\overline{X}_M$, $\overline{Y}_M$ we shall compute the terms $E(\sin \epsilon_t)$ and $E(\cos \epsilon_t)$ first. We have

$$E(\sin \epsilon_t) = \int_{\mathbb{R}} \sin x \, \varphi_t(x) \, dx = \int_{\mathbb{R}} \sin x \, \frac{1}{\sqrt{2\pi} k_t} e^{-\frac{x^2}{2k_t^2}} \, dx = 0,$$

as the integral of an odd function over a symmetric interval. Since

$$E(\cos \epsilon_t) = \int \cos x \, \varphi_t(x) \, dx, \tag{2.1}$$

the density function $\varphi_t(x)$ determines uniquely the value of $E(\cos \epsilon_t)$. Then it will suffice to compute $E(\cos \epsilon_t)$ in the case of a particular random variable with the same density function as ϵ_t. Choosing this random variable to be $\frac{k_t}{\sqrt{t}} W_t$, $t > 0$, where W_t is the 1-dimensional Brownian motion, we note that

$$\frac{k_t}{\sqrt{t}} W_t \sim N(0, k_t).$$

Then

$$E(\cos \epsilon_t) = E\big(\cos(\frac{k_t}{\sqrt{t}} W_t)\big) = E\big(\cos(\sigma_t W_t)\big), \tag{2.2}$$

with $\sigma_t = \frac{k_t}{\sqrt{t}}$. Using formulas of [2], p. 56

$$E(W_t^{2n+1}) = 0, \qquad E(W_t^{2n}) = \frac{(2n)!}{2^n n!} t^n, \tag{2.3}$$

taking the expectation operator in the series expansion yields

$$\begin{aligned} E\big(\cos(\sigma_t W_t)\big) &= \sum (-1)^n \frac{1}{(2n)!} \sigma_t^{2n} E(W_t^{2n}) = \sum (-1)^n \frac{1}{(2n)!} \sigma_t^{2n} \frac{(2n)!}{2^n n!} t^n \\ &= \sum (-1)^n \frac{1}{n!} \Big(\frac{\sigma_t^2 t}{2}\Big)^n = e^{-\sigma_t^2 t/2}. \end{aligned}$$

Substituting in (2.2) we get

$$E(\cos\epsilon_t) = e^{-k_t^2/2}. \tag{2.4}$$

Then the value of the expected coordinates $\overline{X}_M$ and $\overline{Y}_M$ are

$$\begin{aligned}
\overline{X}_M &= E(\hat{X}_M) = E(\cos\tau_t) = E\big(\cos(t+\epsilon_t)\big) = E(\cos t\cos\epsilon_t - \sin t\sin\epsilon_t) \\
&= \cos t\, E(\cos\epsilon_t) - \sin t\, E(\sin\epsilon_t) = \cos t\, E(\cos\epsilon_t) \\
&= e^{-k_t^2/2}\cos t.
\end{aligned}$$

$$\begin{aligned}
\overline{Y}_M &= E(\hat{Y}_M) = E(\sin\tau_t) = E\big(\sin(t+\epsilon_t)\big) = E\big(\sin t\cos\epsilon_t + \cos t\sin\epsilon_t\big) \\
&= \sin t\, E(\cos\epsilon_t) + \cos t\, E(\sin\epsilon_t) \\
&= e^{-k_t^2/2}\sin t.
\end{aligned}$$

Proposition 2.1. *The expected coordinates of the stochastic point $\hat{M}(\cos\epsilon_t, \sin\epsilon_t)$ are given by $(e^{-k_t^2/2}\cos t, e^{-k_t^2/2}\sin t)$.*

The distance between the origin and the expected position of the stochastic point $\hat{M}$ is $|O\overline{M}| = e^{-k_t^2/2} \le 1$. The trade-off between the radius length and the angle accuracy can be stated by saying that *the larger the error, the shorter the radius.* Since the largest error in measuring the argument angle t is π (an error of $\pi + \delta$ counts as δ), the shrinking factor has the lower and upper bounds

$$1 \ge e^{-k_t^2/2} \ge e^{-\pi^2/2}.$$

In the next definition we shall consider the standard deviation $k_t = k$, constant.

Definition 2.2. Let t be an angle measured within the error $\epsilon \sim N(0,k)$, $k > 0$ constant. Define the stochastic functions

$$\begin{aligned}
\mathrm{Ssin}(t,k) &= e^{k^2/2}\sin(t+\epsilon) \\
\mathrm{Scos}(t,k) &= e^{k^2/2}\cos(t+\epsilon),
\end{aligned}$$

called the stochastic sine and the stochastic cosine with parameter k.

By Proposition 2.1 we have

$$\begin{aligned}
E\big(\mathrm{Ssin}(t,k)\big) &= \sin t \\
E\big(\mathrm{Scos}(t,k)\big) &= \cos t,
\end{aligned}$$

We also have

$$\mathrm{Ssin}^2(t,k) + \mathrm{Scos}^2(t,k) = e^{k^2} > 1.$$

We notice the following asymptotic relations for $k \to 0$

$$\mathrm{Ssin}(t,k) \to \sin t, \qquad \mathrm{Scos}(t,k) \to \cos t.$$

Since if $\epsilon \sim N(0,k)$ then also $-\epsilon \sim N(0,k)$, we have

$$\mathrm{Scos}(-t,k) = e^{k^2/2}\cos(-t+\epsilon) = e^{k^2/2}\cos(t-\epsilon) = \mathrm{Scos}(t,k),$$

i.e. $t \to \mathrm{Scos}(t,k)$ is an even function. Similarly we can show that $t \to \mathrm{Ssin}(t,k)$ is an odd function.

Next we shall deal with some trigonometric formulas for the stochastic sine and cosine.

Proposition 2.3.

$$\begin{aligned}\mathrm{Ssin}(t+u,k) &= \mathrm{Scos}(u,k)\,\sin t + \mathrm{Ssin}(u,k)\,\cos t\\ &= \mathrm{Scos}(t,k)\,\sin u + \mathrm{Ssin}(t,k)\,\cos u,\end{aligned}$$

$$\begin{aligned}\mathrm{Scos}(t+u,k) &= \cos t\,\mathrm{Scos}(u,k) - \sin t\,\mathrm{Ssin}(u,k)\\ &= \cos u\,\mathrm{Scos}(t,k) - \sin u\,\mathrm{Ssin}(t,k).\end{aligned}$$

Proof. Stating from the definition we have

$$\begin{aligned}\mathrm{Ssin}(t+u,k) &= e^{k^2/2}\sin(t+u+\epsilon) = e^{k^2/2}\sin\big(t+(u+\epsilon)\big)\\ &= e^{k^2/2}\big(\sin t\cos(u+\epsilon)+\cos t\sin(u+\epsilon)\big)\\ &= \sin t\,\mathrm{Scos}(u,k)+\cos t\,\mathrm{Ssin}(u,k).\end{aligned}$$

In a similar way we can prove the other formulas. □

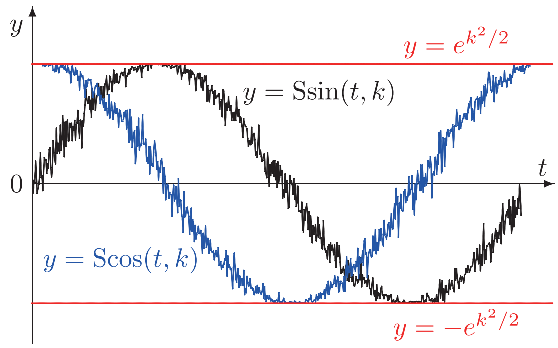

FIGURE 1. The graph of the stochastic sine $\mathrm{Ssin}(t,k)$ and the stochastic cosine $\mathrm{Scos}(t,k)$.

The graphs of the stochastic sine $\mathrm{Ssin}(t,k)$ and the stochastic cosine $\mathrm{Scos}(t,k)$ are depicted in Figure 1 for $t \in [0, 2\pi]$. They look like graphs of the usual sine and cosine which are loaded with some noise and are oscillating between $\pm e^{k^2/2}$. This noise is controlled by the parameter k. The next result states that the variance of the stochastic sine and cosine are equal and they are independent of t.

Proposition 2.4. *We have*

$$Var\big(\mathrm{Ssin}(t,k)\big) = Var\big(\mathrm{Scos}(t,k)\big) = \frac{1}{2}(e^{k^2}-1).$$

Proof. Let $X_t = \mathrm{Ssin}(t,k)$. Then

$$\begin{aligned}E(X_t^2) &= e^{k^2}E\big(\sin^2(t+\epsilon)\big) = \frac{1}{2}e^{k^2}E\big(1-\cos(2t+2\epsilon)\big)\\ &= \frac{1}{2}e^{k^2}\big(1-\cos(2t)\,E(\cos 2\epsilon)+\sin(2t)\,\underbrace{E(\sin 2\epsilon)}_{=0}\big)\end{aligned}$$

$$= \frac{1}{2}e^{k^2} - \frac{1}{2}\cos(2t),$$

where we used $E(\cos 2\epsilon) = e^{-k^2}$, see (2.4). The variance is

$$Var(X_t) = E(X_t^2) - E(X_t)^2 = \frac{1}{2}e^{k^2} - \frac{1}{2}\cos(2t) - \sin^2 t = \frac{1}{2}(e^{k^2} - 1).$$

The proof for the variance of the stochastic cosine is similar.

□

3. Applications

1. The Pythagorean Theorem Consider a right triangle ABC with stochastic angle $\angle BAC = \pi/2 + \epsilon$. The hypotenuse $\hat{a}$ is also stochastic and the lengths of the sides b and c are considered fixed. Taking the expectation in the law of cosines

$$\begin{aligned} \hat{a}^2 &= b^2 + c^2 - 2bc\cos(\pi/2 + \epsilon) \\ &= b^2 + c^2 + 2bc\sin\epsilon, \end{aligned}$$

yields

$$E(\hat{a}^2) = b^2 + c^2$$

since $E(\sin\epsilon) = 0$. Using the inequality $E(\hat{a}^2) \geq E(\hat{a})^2$ we obtain

$$\bar{a}^2 \leq b^2 + c^2, \tag{3.1}$$

where $\bar{a} = E(\hat{a})$ is the expected length of the hypothenuse. The inequality (3.1) shows that the expected length of the hypothenuse in the stochastic case is smaller than in the deterministic case.

2. The estimation of one side opposite to a stochastic angle. Consider the triangle ABC with stochastic angle A and fixed sides b, c and angles B and C. If $\hat{A} = A + \epsilon$, with $\epsilon \sim N(0, k)$, then taking the expectation in the relation

$$\hat{a}^2 = b^2 + c^2 - 2bc\cos(A + \epsilon)$$

yields

$$\begin{aligned} \bar{a}^2 = E(\hat{a}^2) &= b^2 + c^2 - 2bc\,E\big(\cos(A + \epsilon)\big) \\ &= b^2 + c^2 - 2e^{-k^2/2}bc\cos A \\ &= e^{-k^2/2}(b^2 + c^2 - 2bc\cos A) + b^2 + c^2 - e^{-k^2/2}(b^2 + c^2) \\ &= e^{-k^2/2}a^2 + (1 - e^{-k^2/2})(b^2 + c^2). \end{aligned}$$

Hence the expected length $\bar{a}$ of the opposite side to the stochastic angle A satisfies

$$\bar{a}^2 = e^{-k^2/2}a^2 + (1 - e^{-k^2/2})(b^2 + c^2).$$

3. Stochastic triangle inscribed in a fixed circle. Let ABC be a triangle inscribed in a circle of radius R, with sides lengths a, b, c and angle measures α, β, γ, respectively. Suppose now that the points B and C become stochastic but they are still under the constraint that belong to the fixed circle. The problem has now the following given elements

Fixed elements: the circle and the point A on the circle;

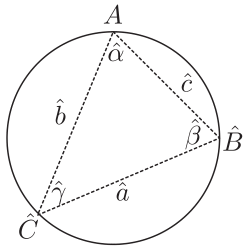

FIGURE 2. Stochastic points B and C on a fixed circle; A fixed point on the circle.

Stochastic elements: points $\hat{B}$ and $\hat{C}$ on the circle.

As a consequence, the sides lengths and the angle measures are also stochastic. Let $\hat{a}, \hat{b}, \hat{c}$ be the stochastic estimations of the lengths of the sides, and let $\bar{a} = E(\hat{a})$, $\bar{b} = E(\hat{b})$, $\bar{c} = E(\hat{c})$ be the expected lengths, see Figure 2. We have the following result.

Proposition 3.1.

$$\frac{\bar{b}\bar{c}}{\bar{a}} = \frac{bc}{a}.$$

Proof. Consider the stochastic measures of the angles

$$\hat{\alpha} = \alpha + \epsilon_\alpha, \quad \hat{\beta} = \beta + \epsilon_\beta, \quad \hat{\gamma} = \gamma + \epsilon_\gamma,$$

where $\epsilon_\beta, \epsilon_\gamma \sim N(0, k)$ are independent random variables. From the law of sines we have

$$\hat{b} = 2R \sin \hat{\beta} = 2R \sin(\beta + \epsilon_\beta),$$

with the radius R fixed. Taking the expectation operator and using the law of sines yields

$$\bar{b} = E(\hat{b}) = 2R\, E\big(\sin(\beta + \epsilon_\beta)\big) = 2R \sin \beta\, e^{-k^2/2} = e^{-k^2/2} b,$$

where we used the law of sines for the fixed elements b, β and R. In a similar way, one can show that $\bar{c} = e^{-k^2/2} c$. Since

$$\alpha + \beta + \gamma = \hat{\alpha} + \hat{\beta} + \hat{\gamma} = \pi,$$

the random variables ϵ_α, ϵ_β, ϵ_γ are related by

$$\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma = 0,$$

and hence $\epsilon_\alpha = -(\epsilon_\beta + \epsilon_\gamma) \sim N(0, k\sqrt{2})$. Then by the law of sines

$$\hat{a} = 2R \sin \hat{\alpha} = 2R \sin(\alpha + \epsilon_\alpha),$$

and taking the expectation we get

$$\bar{a} = E(\hat{a}) = E\big(2R \sin(\alpha + \epsilon_\alpha)\big) = 2R \sin \alpha e^{-(k\sqrt{2})^2/2} = a e^{-k^2}.$$

Then

$$\frac{\bar{b}\bar{c}}{\bar{a}} = \frac{b e^{-k^2/2}\, c e^{-k^2/2}}{a e^{-k^2}} = \frac{bc}{a}.$$

□

4. Stochastic invariants

Let $\hat{A}, \hat{B}, \dots$ be a finite or infinite number of arbitrary stochastic points in the plane with the expected positions $E(\hat{A}) = A$, $E(\hat{B}) = B$, $\dots$ A real valued function f is called a *stochastic invariant* of the plane if

$$E\big(f(\hat{A}, \hat{B}, \dots)\big) = f(A, B, \dots).$$

In other words, stochastic invariants are real-valued functions which admit as arguments stochastic points and commute with the expectation operator

$$E\big(f(\hat{A}, \hat{B}, \dots)\big) = f\big(E(\hat{A}), E(\hat{B}), \dots\big).$$

We note that the sum of two stochastic invariants is still a stochastic invariant, while the product is not. Then it makes sense to define the variance of the stochastic invariant f by

$$E\big(f^2(\hat{A}, \hat{B}, \dots)\big) - \Big(E\big(f(\hat{A}, \hat{B}, \dots)\big)\Big)^2 = E\big(f^2(\hat{A}, \hat{B}, \dots)\big) - f^2(A, B, \dots).$$

The study of stochastic invariants may help with understanding the geometry of the stochastic plane. We are interested with those geometric concepts which can be expressed in terms of stochastic invariants. Next we shall present a few examples of stochastic invariants.

The distance on the stochastic line. The *stochastic line* is a line where the coordinates of points are stochastic. This line will be denoted by $\hat{\mathbb{R}}$. If on the real line a point M has the coordinate X_M, on the stochastic line this corresponds to a stochastic point $\hat{M}$ with stochastic coordinate $\hat{X}_M = X_M + \epsilon$, with ϵ random variable normally distributed with mean zero and standard deviation k, independent on the point. This way each point on the line corresponds to a normal distribution centered at the coordinate of that point, see Fig. 3. It worth noting that when $k \to 0$ the distribution tends to the Dirac delta function δ_{X_M} and the stochastic line becomes in this case the usual real line.

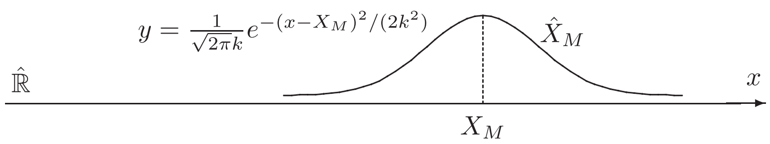

Figure 3. Each point on the stochastic line corresponds to a normal distribution centered at the coordinate of that point.

Consider two stochastic points $\hat{M}$, $\hat{N}$ on the stochastic line with the coordinates

$$\hat{X}_M = X_M + \epsilon_M, \qquad \hat{X}_N = X_N + \epsilon_N$$

with $\epsilon_M, \epsilon_N \sim N(0, k)$ independent random variables. Let M and N denote the expected positions of the aforementioned stochastic points. The distance between the points is the random variable

$$|\hat{M}\hat{N}| = |\hat{X}_M - \hat{X}_N| = |(X_M - X_N) + (\epsilon_M - \epsilon_N)| \sim N(|X_M - X_N|, k\sqrt{2}),$$

and hence

$$E(|\hat{M}\hat{N}|) = |X_M - X_N| = |MN|,$$

which shows that the distance $f(\hat{M}, \hat{N}) = |\hat{M}\hat{N}|$ is a stochastic invariant with the variance

$$E(|\hat{M}\hat{N}|^2) - \big(E(|\hat{M}\hat{N}|)\big)^2 = |MN|^2 + 2k^2 - |MN|^2 = 2k^2.$$

This stochastic invariant explains why measuring a distance several times and averaging out the results yields a better approximation for the distance.

Invariants on the stochastic plane. The *stochastic plane* is a plane with all points stochastic, i.e. points for which the coordinates are random variables. The stochastic plane will be denoted by $\hat{\mathcal{P}}$ to make the distinction from the real plane denoted by $\mathcal{P}$. If $\hat{M}$ is a stochastic point in the plane, then its coordinates are

$$\hat{X}_M = X_M + \epsilon_M, \qquad \hat{Y}_M = Y_M + \epsilon'_M,$$

where (X_M, Y_M) are the coordinates of the expected point $M = E(\hat{M})$. The random errors are considered independent and normally distributed, with $\epsilon \sim N(0, k)$, $\epsilon' \sim N(0, k')$. Next we shall provide a few examples of stochastic invariants of the plane.

The midpoint of a line segment. Let $\hat{A}$ and $\hat{B}$ be two stochastic points in the stochastic plane. We can easily see that the coordinates of the midpoint of the line segment AB

$$\begin{aligned} f_X(\hat{A}, \hat{B}) &= \frac{1}{2}(\hat{X}_A + \hat{X}_B), \\ f_Y(\hat{A}, \hat{B}) &= \frac{1}{2}(\hat{Y}_A + \hat{Y}_B) \end{aligned}$$

are stochastic invariants. The variance of the first invariant is $k^2/2$ and the variance of the second one is $k'^2/2$.

The center of mass. Given a polygon with the vertices at the stochastic points $\hat{A}_i$, $i \in \{1, \dots, n\}$, with the stochastic coordinates

$$\hat{X}_{A_i} = X_{A_i} + \epsilon_i, \quad \hat{Y}_{A_i} = Y_{A_i} + \epsilon'_i,$$

then the coordinates of the center of mass

$$f_X(\hat{A}_1, \dots, \hat{A}_i) = \frac{1}{n}\sum \hat{X}_{A_i}, \qquad f_Y(\hat{A}_1, \dots, \hat{A}_i) = \frac{1}{n}\sum \hat{Y}_{A_i}$$

are stochastic invariants since

$$\begin{aligned} E\big(f_X(\hat{A}_1, \dots, \hat{A}_i)\big) &= \frac{1}{n}\sum E(\hat{X}_{A_i}) = \frac{1}{n}\sum X_{A_i} \\ &= f_X(A_1, \dots, A_i) = f_X\big(E(\hat{A}_1), \dots, E(\hat{A}_n)\big). \end{aligned}$$

Since $\epsilon_1, \dots, \epsilon_n$ are independent and normally distributed with variance k^2, then their average $\frac{1}{n}\sum \epsilon_i$ has variance $\frac{k^2}{n}$. It follows that $f_X(\hat{A}_1, \dots, \hat{A}_i)$ has variance $\frac{k^2}{n}$. Similarly, $f_Y(\hat{A}_1, \dots, \hat{A}_i)$ has variance $\frac{k'^2}{n}$. We notice that when $n \to \infty$, the variance tends to zero, which means that the coordinates of the center of mass tend to become deterministic in this limit case.

The angular argument. If $\hat{M}$ is a stochastic point on the fixed unit circle, its angular argument is given by $\tau = t + \epsilon$, where $\epsilon \sim N(0, k)$. Then $f(\hat{M}) = \tau$ is a stochastic invariant with variance k^2, since

$$E\big(f(\hat{M})\big) = E(\tau) = E(t + \epsilon) = t = f\big(E(\hat{M})\big).$$

The inner product function. For any two stochastic points $\hat{M}$ and $\hat{N}$ in the plane $\hat{\mathcal{P}}$ define

$$f(\hat{M}, \hat{N}) = \hat{X}_M \hat{X}_N + \hat{Y}_M \hat{Y}_N.$$

A computation shows

$$\begin{aligned} f(\hat{M}, \hat{N}) &= (X_M + \epsilon_M)(X_N + \epsilon_N) + (Y_M + \epsilon'_M)(Y_N + \epsilon'_N) \\ &= X_M X_N + Y_M Y_N + \epsilon_N X_M + \epsilon_M X_N + \epsilon_M \epsilon_N \\ &\quad + Y_M \epsilon'_M + \epsilon'_M Y_N + \epsilon'_M \epsilon'_N, \end{aligned}$$

and taking the expectation yields

$$E\big(f(\hat{M}, \hat{N})\big) = X_M X_N + Y_M Y_N = f(M, N) = f\big(E(\hat{M}), E(\hat{N})\big),$$

i.e. the function f is a stochastic invariant.

The area of a triangle. In the following we shall show that the area of a triangle is a stochastic invariant. Consider three stochastic points $\hat{A}$, $\hat{B}$, $\hat{C}$ with $E(\hat{A}) = A$, $E(\hat{B}) = B$, $E(\hat{C}) = C$. Consider the oriented area function $f : \hat{\mathcal{P}} \times \hat{\mathcal{P}} \times \hat{\mathcal{P}} \to \mathbb{R}$ given by

$$f(\hat{A}, \hat{B}, \hat{C}) = \sigma(\hat{A}, \hat{B}, \hat{C}) = \frac{1}{2} \begin{vmatrix} 1 & \hat{X}_A & \hat{Y}_A \\ 1 & \hat{X}_B & \hat{Y}_B \\ 1 & \hat{X}_C & \hat{Y}_C \end{vmatrix},$$

where

$$\begin{aligned} \hat{X}_A &= X_A + \epsilon_A, \quad \hat{X}_B = X_B + \epsilon_B, \quad \hat{X}_C = X_C + \epsilon_C \\ \hat{Y}_A &= Y_A + \epsilon_A, \quad \hat{Y}_B = Y_B + \epsilon_B, \quad \hat{Y}_C = Y_C + \epsilon_C. \end{aligned}$$

Using the linearity property of the determinant we have

$$\begin{aligned} f(\hat{A}, \hat{B}, \hat{C}) &= \frac{1}{2} \begin{vmatrix} 1 & \hat{X}_A & \hat{Y}_A \\ 1 & \hat{X}_B & \hat{Y}_B \\ 1 & \hat{X}_C & \hat{Y}_C \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} 1 & X_A & Y_A \\ 1 & X_B & Y_B \\ 1 & X_C & Y_C \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & X_A & \epsilon'_A \\ 1 & X_B & \epsilon'_B \\ 1 & X_C & \epsilon'_C \end{vmatrix} \qquad (4.1) \\ &\quad + \frac{1}{2} \begin{vmatrix} 1 & \epsilon_A & Y_A \\ 1 & \epsilon_B & Y_B \\ 1 & \epsilon_C & Y_C \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & \epsilon_A & \epsilon'_A \\ 1 & \epsilon_B & \epsilon'_B \\ 1 & \epsilon_C & \epsilon'_C \end{vmatrix} \\ &= \sigma(A, B, C) + \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 + \frac{1}{2}\Delta_3. \qquad (4.2) \end{aligned}$$

The expected value of the last three determinants vanish. For instance,

$$E(\Delta_1) = E\left[\begin{vmatrix} 1 & X_A & \epsilon'_A \\ 1 & X_B & \epsilon'_B \\ 1 & X_C & \epsilon'_C \end{vmatrix}\right] = \begin{vmatrix} 1 & X_A & E(\epsilon'_A) \\ 1 & X_B & E(\epsilon'_B) \\ 1 & X_C & E(\epsilon'_C) \end{vmatrix} = \begin{vmatrix} 1 & X_A & 0 \\ 1 & X_B & 0 \\ 1 & X_C & 0 \end{vmatrix} = 0.$$

Then taking the expected value in relation (4.2) yields

$$E\big(f(\hat{A},\hat{B},\hat{C})\big) = \sigma(A,B,C) = f(A,B,C),$$

i.e. the oriented area of a triangle is a stochastic invariant. Next we shall compute its variance. Using (4.2) we get

$$\begin{aligned} Varf &= E\big(f^2(\hat{A},\hat{B},\hat{C})\big) - \sigma^2(A,B,C) \\ &= \frac{1}{4}E(\Delta_1^2+\Delta_2^2+\Delta_3^2) + \sigma(A,B,C)E(\Delta_1+\Delta_2+\Delta_3) \\ &\quad + \frac{1}{2}E(\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_3\Delta_1). \end{aligned} \tag{4.3}$$

Expanding the determinant

$$\Delta_1 = X_B(\epsilon'_C-\epsilon'_A) + X_A(\epsilon'_B-\epsilon'_C) + X_C(\epsilon'_A-\epsilon'_B),$$

and then

$$\begin{aligned} \Delta_1^2 &= X_B^2(\epsilon'_C-\epsilon'_A)^2 + X_A^2(\epsilon'_B-\epsilon'_C)^2 + X_C^2(\epsilon'_A-\epsilon'_B)^2 \\ &\quad +2X_AX_B(\epsilon'_C-\epsilon'_A)(\epsilon'_B-\epsilon'_C) + 2X_BX_C(\epsilon'_C-\epsilon'_A)(\epsilon'_A-\epsilon'_B) \\ &\quad +2X_AX_C(\epsilon'_B-\epsilon'_C)(\epsilon'_A-\epsilon'_B). \end{aligned}$$

Using

$$\begin{aligned} E\big((\epsilon'_B-\epsilon'_C)^2\big) &= E\big((\epsilon'_A-\epsilon'_B)^2\big) = E\big((\epsilon'_C-\epsilon'_A)^2\big) = 2k'^2, \\ E\big((\epsilon'_C-\epsilon'_A)(\epsilon'_B-\epsilon'_C)\big) &= -E(\epsilon'^2_C) = -k'^2, \\ E\big((\epsilon'_C-\epsilon'_A)(\epsilon'_A-\epsilon'_B)\big) &= -E(\epsilon'^2_A) = -k'^2, \end{aligned}$$

taking the expectation yields

$$E(\Delta_1^2) = 2k'^2(X_A^2+X_B^2+X_C^2-X_AX_B-X_BX_C-X_CX_A). \tag{4.4}$$

Similarly we can show that

$$E(\Delta_2^2) = 2k^2(Y_A^2+Y_B^2+Y_C^2-Y_AY_B-Y_BY_C-Y_CY_A). \tag{4.5}$$

Expanding the last determinant

$$\Delta_3 = \epsilon_B(\epsilon'_C-\epsilon'_A) + \epsilon_A(\epsilon'_B-\epsilon'_C) + \epsilon_C(\epsilon'_A-\epsilon'_B)$$

and taking the square yields

$$\begin{aligned} \Delta_3^2 &= \epsilon_B^2(\epsilon'_C-\epsilon'_A)^2 + \epsilon_A^2(\epsilon'_B-\epsilon'_C)^2 + \epsilon_C^2(\epsilon'_A-\epsilon'_B)^2 \\ &= 2\epsilon_B\epsilon_A(\epsilon'_C-\epsilon'_A)(\epsilon'_B-\epsilon'_C) + 2\epsilon_B\epsilon_C(\epsilon'_C-\epsilon'_A)(\epsilon'_A-\epsilon'_B) \\ &\quad +2\epsilon_A\epsilon_C(\epsilon'_B-\epsilon'_C)(\epsilon'_A-\epsilon'_B). \end{aligned}$$

Using the independence of the variables ϵ and ϵ' we have

$$E\big(\epsilon_B^2(\epsilon'_C-\epsilon'_A)^2\big) = E(\epsilon_B^2)E(\epsilon'_C-\epsilon'_A)^2 = k^2\cdot 2k'^2,$$

$$E\big(\epsilon_B\epsilon_A(\epsilon'_C-\epsilon'_A)(\epsilon'_B-\epsilon'_C)\big) = E(\epsilon_B)E(\epsilon_A)E\big((\epsilon'_C-\epsilon'_A)(\epsilon'_B-\epsilon'_C)\big) = 0,$$

and the similar relations. Then the previous relation yields

$$E(\Delta_3^2) = 6k^2k'^2.$$

Using the properties of independent random variables we also have

$$E(\Delta_1\Delta_2) = E(\Delta_2\Delta_3) = E(\Delta_3\Delta_1) = 0,$$
$$E(\Delta_1) = E(\Delta_2) = E(\Delta_3) = 0.$$

Substituting in formula (4.3) we obtain

$$\begin{aligned} Varf &= \frac{1}{2}k'^2(X_A^2 + X_B^2 + X_C^2 - X_AX_B - X_BX_C - X_CX_A) \\ &+\frac{1}{2}k^2(Y_A^2 + Y_B^2 + Y_C^2 - Y_AY_B - Y_BY_C - Y_CY_A) + \frac{3}{2}k^2k'^2. \end{aligned}$$

To conclude, a good estimation of the area of a triangle with stochastic vertices is the area of the triangle formed by the centers of the distributions of the aforementioned stochastic vertices.

Since any convex polygon can be partitioned into triangles, using that the area function is additive, we obtain that the signed area function associated with any convex polygon is a stochastic invariant.

Operations with stochastic invariants.
It follows easily from the definition that the linear combination of two stochastic invariants is also a stochastic invariant. However, the usual product is not necessarily a stochastic invariant, but the tensorial product is.

Tensorial product of stochastic invariants. Let $f(\hat{M}_1, \dots, \hat{M}_r)$ and $g(\hat{N}_1, \dots, \hat{N}_s)$ be two stochastic invariants. Define the new stochastic invariant

$$(f \otimes g)(\hat{M}_1, \dots, \hat{M}_r, \hat{N}_1, \dots, \hat{N}_s) = f(\hat{M}_1, \dots, \hat{M}_r)g(\hat{N}_1, \dots, \hat{N}_s).$$

Using the independence we can check that

$$\begin{aligned} E[(f \otimes g)(\hat{M}_1, \dots, \hat{M}_r, \hat{N}_1, \dots, \hat{N}_s)] &= E[f(\hat{M}_1, \dots, \hat{M}_r)]E[g(\hat{N}_1, \dots, \hat{N}_s)] \\ &= f(M_1, \dots, M_r)g(N_1, \dots, N_s) \\ &= (f \otimes g)(M_1, \dots, M_r, N_1, \dots, N_s). \end{aligned}$$

5. Stochastic transforms

A mapping $F = (F_1, F_2) : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ is called a *stochastic transform* if its components $F_1, F_2 : \hat{\mathcal{P}} \to \hat{\mathbb{R}}$ are stochastic invariants. This means

$$E\big(F(\hat{M})\big) = F\big(E(\hat{M})\big), \qquad \forall \hat{M} \in \hat{\mathcal{P}}.$$

Next we shall encounter a few examples. Since E is a linear operator, then any transform with linear components is a stochastic transform. In particular, a translation, a rotation or an affine transform is a stochastic transform. For instance, if $\mathcal{T}$ is the translation by a fixed vector (u, v) given by

$$\begin{aligned} x' &= x + u \\ y' &= y + v, \end{aligned}$$

then

$$\begin{aligned} E\big(\mathcal{T}(\hat{X}_M,\hat{Y}_M)\big) &= \big(E(\hat{X}_M+u), E(\hat{Y}_M+v)\big) = \big(E(\hat{X}_M)+u, E(\hat{Y}_M)+v\big) \\ &= \mathcal{T}\big(E(\hat{X}_M), E(\hat{Y}_M)\big). \end{aligned}$$

Proposition 5.1. *Let $\mathcal{S}$ denote the set of all stochastic transforms of the plane $\hat{\mathcal{P}}$. Then*

(i) *If $f, g \in \mathcal{S}$, then $\alpha f + \beta g \in \mathcal{S}$, for all $\alpha, \beta \in \mathbb{R}$.*

(ii) *If $f, g \in \mathcal{S}$, then $f \circ g \in \mathcal{S}$.*

(iii) *If $f \in \mathcal{S}$ and f is invertible, then $f^{-1} \in \mathcal{S}$.*

Proof. (i) It comes from the linearity of the expectation operator E.

(ii) It follows from the commutativity between E and the functions f and g.

(iii) Applying the expectation to $f\big(f^{-1}(\hat{M})\big) = \hat{M}$ yields $E\big(f(f^{-1}(\hat{M})\big) = M$, which after using the commutativity between E and f yields

$$f\big(E(f^{-1}(\hat{M}))\big) = M.$$

Taking the inverse of f yields

$$E(f^{-1}(\hat{M})) = f^{-1}(M),$$

which is $E(f^{-1}(\hat{M})) = f^{-1}\big(E(\hat{M})\big)$, so $f^{-1} \in \mathcal{S}$. □

The invertible elements of $\mathcal{S}$ will be denoted by $U(\mathcal{S})$. This forms a group of transforms which is noncommutative. For instance, as it follows from the linearity, any transformation of the plane of type

$$\begin{aligned} x' &= ax + by + c_1 \\ y' &= cx + dy + c_2, \end{aligned}$$

with $ad \neq bc$, is an element of $U(\mathcal{S})$. We shall show in the following that there are elements of $U(\mathcal{S})$ which are neither isometries nor affine transforms. In order to show this, it suffices to provide an example of a stochastic transform of the plane which is not linear.

A nonlinear stochastic transform. Consider $F = (F_1, F_2) : \hat{\mathcal{P}} = \hat{\mathbb{R}} \times \hat{\mathbb{R}} \to \hat{\mathcal{P}}$ given by

$$\begin{aligned} F_1(\hat{x},\hat{y}) &= e^{\hat{x}} \sin \hat{y} \\ F_2(\hat{x},\hat{y}) &= e^{\hat{x}} \cos \hat{y}, \end{aligned}$$

which is invertible since $\frac{\partial(F_1,F_2)}{\partial(\hat{x},\hat{y})} = -e^{2\hat{x}} \neq 0$. Consider the stochastic coordinates

$$\hat{x} = x + \epsilon, \qquad \hat{y} = y + \epsilon',$$

with $\epsilon, \epsilon' \sim N(0,k)$ independent variables.

We recall from section 2 that $E(\cos \epsilon') = e^{-k^2/2}$ and $E(\sin \epsilon') = 0$. Using a similar method we shall show that $E(e^{\epsilon}) = e^{k^2/2}$. Since $E(e^{\epsilon}) = \int e^x \varphi(x)\, dx$, with $\varphi(x) = \frac{1}{\sqrt{2\pi}k} e^{-x^2/2k^2}$, the expected value $E(e^{\epsilon})$ is uniquely determined by the density function $\varphi(x)$, so if the random variable ϵ is replaced by another one with the same density function, then the expected values are the same. We choose to replace ϵ by $\frac{k}{\sqrt{t}} W_t \sim N(0,k)$, where W_t stands for the 1-dimensional Brownian motion. Denote $\frac{k}{\sqrt{t}} = \sigma_t$. Using (2.3) we have

$$E(e^\epsilon) = E\big(\sum \frac{\sigma_t^n W_t^n}{n!}\big) = \sum \frac{1}{n!}\Big(\frac{\sigma_t^2 t}{2}\Big)^n = e^{t\sigma_t^2/2} = e^{k^2/2}.$$

Next we shall check that F_1 is a stochastic invariant.

$$\begin{aligned} E\big(F_1(\hat{x},\hat{y})\big) &= E(e^{\hat{x}} \sin \hat{y}) = E\big(e^x e^\epsilon \sin(y+\epsilon')\big) \\ &= e^x \sin y\, E(e^\epsilon \cos \epsilon') + e^x \cos y\, E(e^\epsilon \sin \epsilon') \\ &= e^x \sin y\, E(e^\epsilon) E(\cos \epsilon') + e^x \cos y\, E(e^\epsilon) E(\sin \epsilon') \\ &= e^x \sin y\, e^{k^2/2} e^{-k^2/2} + e^x \cos y\, e^{k^2/2} \cdot 0 \\ &= e^x \sin y = F_1(x,y) = F_1\big(E(\hat{x}), E(\hat{y})\big). \end{aligned}$$

Similarly, one can show that $E\big(F_2(\hat{x},\hat{y})\big) = F_2\big(E(\hat{x}), E(\hat{y})\big)$. It follows that $F = (F_1, F_2)$ is a stochastic transform, with $F \in U(\mathcal{S})$.

Another example of nonlinear stochastic transform is $F = (F_1, F_2)$ with

$$\begin{aligned} F_1(\hat{x},\hat{y}) &= \hat{x} - \hat{x}\hat{y} + \hat{y} \\ F_2(\hat{x},\hat{y}) &= \hat{x} + \hat{x}\hat{y} + \hat{y}, \end{aligned}$$

which is invertible on $\hat{\mathcal{P}} \backslash \{(\hat{x},\hat{y})\}$. This follows from the linearity of E and the properties of ϵ and ϵ'

$$E(\hat{x} \pm \hat{x}\hat{y} + \hat{y}) = x \pm E(xy + x\epsilon' + y\epsilon + \epsilon\epsilon') + y = x \pm xy + y.$$

It is worth to notice that in the last two examples the components F_1 and F_2 are harmonic functions, i.e., $(\partial_x^2 + \partial_y^2) F_i(x,y) = 0$, $i = 1,2$. In the next section we shall deal with this property of the stochastic transforms.

6. The main result on stochastic transforms

The first result deals with a characterization of the stochastic invariants on the stochastic line $\hat{\mathbb{R}}$.

Theorem 6.1. *$f : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ is a stochastic invariant if and only if $f(x)$ is a linear function.*

Proof. Let $\hat{x} = x + \epsilon$, with $\epsilon \sim N(0, \sqrt{t})$. We have

$$E\big(f(x+\epsilon)\big) = \int f(u)\varphi_t(u-x)\, du = \int f(x+v)\varphi_t(v)\, dv.$$

Since f is a stochastic invariant

$$E\big(f(x+\epsilon)\big) = f(x),$$

and hence

$$f(x) = \int f(x+v)\varphi_t(v)\, dv.$$

Differentiating twice with respect to x and integrating by parts yields

$$f''(x) = \int f''(x+v)\varphi_t(v)\, dv = \int f(x+v)\partial_v^2 \varphi_t(v)\, dv = 2\int f(x+v)\partial_t \varphi_t(v)\, dv$$

$$= \quad 2\partial_t \int f(x+v)\varphi_t(v)\,dv = 2\partial_t f(x) = 0,$$

and hence $f(x)$ is linear in x.
The converse statement follows from the linearity of the expectation operator E. $\square$

The next result extends the previous one to the 2-dimensional case.

Theorem 6.2. *Let $\hat{\mathcal{P}} = \hat{\mathbb{R}} \times \hat{\mathbb{R}}$ be the stochastic plane. Then $F = (F_1, F_2) : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ is a stochastic transform if and only if the components F_1 and F_2 are harmonic functions, i.e. $(\partial_x^2 + \partial_y^2)F_i = 0$, $i = 1, 2$.*

Proof. Assume $F = (F_1, F_2)$ is a stochastic transform. Let $\hat{x} = x + \epsilon$, $\hat{y} = y + \epsilon'$, with $\epsilon, \epsilon \sim N(0, \sqrt{t})$, independent stochastic variables. Let $\varphi_t(x)$ be the density function of a normally distributed random variable with zero mean and variance t. It is known that this is the heat kernel for the operator $\partial_t - \frac{1}{2}\partial_x^2$, so

$$2\partial_t \varphi_t(x) = \partial_x^2 \varphi_t(x), \qquad t > 0. \tag{6.1}$$

Since $\hat{x}$ and $\hat{y}$ are independent random variables, their joint distribution function is the product of their density functions. Using this fact we have

$$\begin{aligned} E\big(F_1(\hat{x}, \hat{y})\big) &= \iint F_1(u, v)\varphi_t(u - x)\varphi_t(v - y)\,dudv \\ &= \iint F_1(w + x, z + y)\varphi_t(w)\varphi_t(z)\,dwdz. \end{aligned}$$

Since F is a stochastic transform, $E\big(F_1(\hat{x}, \hat{y})\big) = F_1(x, y)$. Then the previous relation yields

$$F_1(x, y) = \iint F_1(w + x, z + y)\varphi_t(w)\varphi_t(z)\,dwdz.$$

Differentiating twice with respect to x and using integration by parts yields

$$\begin{aligned} \partial_x^2 F_1(x, y) &= \partial_x^2 \iint F_1(w + x, z + y)\varphi_t(w)\varphi_t(z)\,dwdz \\ &= \iint F_1(w + x, z + y)\partial_w^2\varphi_t(w)\varphi_t(z)\,dwdz, \end{aligned}$$

and using relation (6.1) we have

$$\partial_x^2 F_1(x, y) = 2\iint F_1(w + x, z + y)\partial_t\varphi_t(w)\varphi_t(z)\,dwdz. \tag{6.2}$$

Similarly, we obtain

$$\partial_y^2 F_1(x, y) = 2\iint F_1(w + x, z + y)\varphi_t(w)\partial_t\varphi_t(z)\,dwdz. \tag{6.3}$$

Adding (6.2) and (6.3) we get

$$\begin{aligned} \frac{1}{2}\big(\partial_x^2 + \partial_y^2\big)F_1(x, y) &= \iint F_1(w + x, z + y)\partial_t\big(\varphi_t(w)\varphi_t(z)\big)\,dwdz \\ &= \partial_t \iint F_1(w + x, z + y)\big(\varphi_t(w)\varphi_t(z)\big)\,dwdz \end{aligned}$$

$$= \partial_t E\big(F_1(\hat{x},\hat{y})\big) = \partial_t F_1(x,y) = 0,$$

and hence F_1 is a harmonic function. Similar considerations apply to the component F_2.

In order to prove the converse, we assume the functions F_i harmonic and show that F is a stochastic transform. It suffices to show that the components F_i are stochastic invariants.

We make the remark that the definition property of the stochastic invariant $f : \hat{\mathbb{R}} \times \hat{\mathbb{R}} \to \hat{\mathbb{R}}$

$$E(f(\hat{x})) = f(E(\hat{x}))$$

corresponds to the *mean property* of harmonic functions. If $\hat{x} \in \hat{\mathbb{R}} \times \hat{\mathbb{R}}$ is a Gaussian random variable centered at x, then $E(f(\hat{x}))$ is the average of $f(\hat{x})$ over the entire stochastic plane. For f harmonic we have

$$E(f(\hat{x})) = average(f(\hat{x})) = \lim_{R\to\infty} \frac{1}{\pi R^2} \int_{|u-x|\le R} f(u)\,du = f(x) = f(E(\hat{x})).$$

Applying this argument for $f = F_i$, $i = 1, 2$, shows that F_i are stochastic invariants. □

Corollary 6.3. *Any stochastic transform $F = (F_1, F_2)$ has analytic components.*

Proof. It follows from the fact that harmonic functions on $\mathbb{R}^2$ are real parts of holomorphic functions. □

Corollary 6.4. *Let U be an open set in the plane. If F and H are two stochastic transforms with $F_{|U} = H_{|U}$, then $F = H$.*

Proof. It follows from Corollary 6.3 and the identity theorem of two analytic functions. □

7. The geometry induced by $U(\mathcal{S})$

We are interested in the study of those properties of the plane that remain unchanged when the points of the plane are subject to the transformations of the group $U(\mathcal{S})$. If *Isom* is the isometries group of the plane, then *Isom* is a proper subgroup of $U(\mathcal{S})$, and hence all the properties invariant by $U(\mathcal{S})$ are also invariant by *Isom*.

Next we shall provide an example of a class of transformations of $U(\mathcal{S})$.

Proposition 7.1. *Let $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonconstant function which satisfies the Cauchy-Riemann system of equations*

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}$$
$$\frac{\partial F_1}{\partial y} = -\frac{\partial F_2}{\partial x}.$$

Then $F \in U(\mathcal{S})$.

Proof. Using the Cauchy-Riemann equations, we have

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial}{\partial y}\frac{\partial F_2}{\partial x} = 0,$$

so F_1 is harmonic. In a similar way F_2 is harmonic. In order to show that F is invertible we compute its Jacobian using the Cauchy-Riemann equations

$$|J_F| = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_2}{\partial x} \\ \frac{\partial F_1}{\partial y} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \Big(\frac{\partial F_1}{\partial x}\Big)^2 + \Big(\frac{\partial F_1}{\partial y}\Big)^2 = \Big(\frac{\partial F_2}{\partial x}\Big)^2 + \Big(\frac{\partial F_2}{\partial y}\Big)^2 \neq 0.$$

□

If consider $\mathbb{R}^2 = \mathbb{C}$ and write $F = F_1 + iF_2$, then F becomes a biholomorphic function. It is known that the biholomorphic transforms are conformal, i.e. preserve angles between lines and curves. Hence a transformation given by Proposition 7.1 preserves angles. For instance, if choose the holomorphic function $F(z) = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$, then $F = (F_1, F_2) = (e^x \cos y, e^x \sin y)$ is a stochastic transform, which preserves angles.

8. Stochastic invariants and Brownian motions

Assume that the coordinates of the stochastic point $\hat{M}$ change with respect to time t according to the laws

$$\hat{x}_t = x + W_1(t), \qquad \hat{y}_t = y + W_2(t), \qquad t \geq 0,$$

where $W_1(t)$, $W_2(t)$ are independent 1-dimensional Brownian motions starting at 0. Consider the stochastic transform $F = (F_1, F_2) : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$. The stochastic point $F(\hat{M})$ has the coordinates

$$(\hat{u}_t, \hat{v}_t) = \big(F_1(\hat{x}_t, \hat{y}_t), F_2(\hat{x}_t, \hat{y}_t)\big).$$

Applying Ito's formula yields

$$\begin{aligned} d\hat{u}_t &= \frac{\partial F_1}{\partial x} dW_1(t) + \frac{\partial F_1}{\partial y} dW_2(t) + \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2}\right) dt \\ d\hat{v}_t &= \frac{\partial F_2}{\partial x} dW_1(t) + \frac{\partial F_2}{\partial y} dW_2(t) + \frac{1}{2}\left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2}\right) dt. \end{aligned}$$

Since the functions F_1, F_2 are harmonic, see Theorem 6.2, we have

$$\begin{aligned} d\hat{u}_t &= \frac{\partial F_1}{\partial x} dW_1(t) + \frac{\partial F_1}{\partial y} dW_2(t) \\ d\hat{v}_t &= \frac{\partial F_2}{\partial x} dW_1(t) + \frac{\partial F_2}{\partial y} dW_2(t), \end{aligned}$$

which after integration yields

$$\hat{u}_t = \hat{u}_0 + \int_0^t \frac{\partial F_1}{\partial x} dW_1(s) + \int_0^t \frac{\partial F_1}{\partial y} dW_2(s) \tag{8.1}$$

$$\hat{v}_t = \hat{v}_0 + \int_0^t \frac{\partial F_2}{\partial x} dW_1(s) + \int_0^t \frac{\partial F_2}{\partial y} dW_2(s), \tag{8.2}$$

which are the coordinates of the point $F(\hat{M})$. Since the Ito integral is a martingale, these coordinates satisfy

$$E(\hat{u}_t, \hat{v}_t) = (\hat{u}_0, \hat{v}_0) = (x, y) = F\big(E(\hat{M})\big).$$

Moreover, the coordinates (8.1–8.2) represent versions of the 1-dimensional Brownian motions starting at (x, y) with a certain time scale. Applying Corollary 8.5.3 of [2], p. 154 yields

$$\hat{u}_t = x + W_1\big(\alpha_1(t)\big), \qquad \hat{v}_t = y + W_2\big(\alpha_2(t)\big),$$

with $\alpha_i(\beta_i(t)) = \beta_i(\alpha_i(t)) = t$, $t \geq 0$, where

$$\beta_i(t) = \int_0^t \|\nabla F_i\|^2 (W_1(s), W_2(s))\, ds.$$

In the particular case, when $F = F_1 + iF_2$ is holomorphic, then after a certain change of the time scale, the process $(\hat{u}_t, \hat{v}_t)$ becomes a 2-dimensional Brownian motion, see [2], p. 158.

For more properties of Brownian motion the reader may consult [3] and [1].

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