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CONVERGENCE RATES OF SUMMATION PROCESSES OF CONVOLUTION TYPE OPERATORS

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ABSTRACT. We establish quantitative pointwise estimates of the rate of convergence of equi-uniform summation processes of convolution type operators in terms of the modulus of continuity of functions to be approximated and higher order absolute moments of approximate kernels. Furthermore, applications are presented for equi-uniform summation processes which are induced by various important summability methods, and several concrete important examples of approximating operators are also provided.

1. INTRODUCTION

Let $\mathbb{N}_0$ be the set of all nonnegative integers and let $\mathbb{R}$ denote the real line. Let $C_{2\pi}$ donote the Banach space of all 2π-periodic, continuous functions on $\mathbb{R}$ with the supremum norm. Let $\{\sigma_n\}_{n\in\mathbb{N}_0}$ be the sequence of Fejér operators defined by

$$\sigma_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x-t)f(t)\,dt \qquad (f \in C_{2\pi},\ x \in \mathbb{R}),$$

where

$$F_n(u) = \sum_{j=-n}^{n}\Big(1 - \frac{|j|}{n+1}\Big)e^{iju} \qquad (u \in \mathbb{R}).$$

Then for every $f \in C_{2\pi}$, $\{\sigma_n(f)(x)\}$ is almost convergent to $f(x)$ uniformly on $\mathbb{R}$, that is,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=m}^{n+m}\sigma_k(f)(x) = f(x) \qquad \text{uniformly in } m \in \mathbb{N}_0, x \in \mathbb{R}$$

(cf. [6], [8]).

In view of this result, we generally make the following situation:

Let $(E, \|\cdot\|)$ be a Banach space and let X be a compact convex subset of the r-dimensional metric linear space $\mathbb{R}^r$ of all r-tuples of real numbers, equipped with the usual metric

$$d_s(x,y) = \begin{cases} \Big(\sum_{i=1}^{r}|x_i - y_i|^s\Big)^{1/s} & (1 \le s < \infty) \\ \max\{|x_i - y_i| : 1 \le i \le r\} & (s = \infty) \end{cases}$$

$$(x = (x_1, x_2, \dots, x_r), y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r).$$

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Let $B(X,E)$ denote the Banach space of all E-valued bounded functions on X with the supremum norm $\|\cdot\|_X$. $C(X,E)$ stands for the closed linear subspace of $B(X,E)$ consisting of all E-valued continuous functions on X. Let Λ be an index set. Let $\mathcal{A}=\{a_{n,m}^{(\lambda)}: n,m\in\mathbb{N}_0, \lambda\in\Lambda\}$ be a family of nonnegative real numbers satisfying

$$\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)}=1 \quad \text{for all } n\in\mathbb{N}_0 \text{ and for all } \lambda\in\Lambda.$$

Let $\mathfrak{K}=\{K_n\}_{n\in\mathbb{N}_0}$ be a sequence of operators from $C(X,E)$ to $B(X,E)$. Then we define

$$K_{n,\lambda}(F)=\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} K_m(F) \qquad (F\in C(X,E)). \tag{1.1}$$

Here we assume that the series (1.1) absolutely converges in $B(X,E)$. The sequence $\mathfrak{K}$ is called an equi-uniform $\mathcal{A}$-summation process on $C(X,E)$ if for every $F\in C(X,E)$,

$$\lim_{n\to\infty}\|K_{n,\lambda}(F)-F\|_X=0 \qquad \text{uniformly in } \lambda\in\Lambda. \tag{1.2}$$

The purpose of this paper is to consider the rate of convergence behavior of (1.2) by giving quantitative pointwise estimates for the case where each K_m is a convolution type operator in $C(X,E)$ under certain suitable conditions. Consequently, we refine the estimates given in [13] for approximation by convolution type operators, which include the Korovkin type operators (cf. [2], [7]). Furthermore, applications are presented for various important summability methods which typically cover the almost convergent method due to Lorentz [8] (cf. [1], [16]), and several concrete important examples of approximating operators are also provided. Further extensive treatments of the rate of converegence for equi-uniform approximation processes of integral operators are considered in [15], of which results can be refinements of the estimates of the degree of approximation given in [13] (cf. [12], [14]).

2. Summation processes of integral operators

Let $F\in B(X,E)$ and let $\delta\geq 0$. Then we define

$$\omega_s(F,\delta)=\sup\{\|F(x)-F(y)\|: x,y\in X, d_s(x,y)\leq\delta\},$$

which is called the modulus of continuity of F. Obviously, $\omega_s(F,\cdot)$ is a monotone increasing function on $[0,\infty)$ and

$$\omega_s(F,0)=0, \quad \omega_s(F,\delta)\leq 2\|F\|_X \qquad (\delta\geq 0).$$

Note that

$$\omega_s(F,\delta)=\omega_s(F,\delta(X)) \qquad (\delta\geq\delta(X)),$$

where $\delta(X)$ denotes the diameter of X, and

$$\lim_{\delta\to+0}\omega_s(F,\delta)=0$$

for all $F \in C(X, E)$. For $i = 1, 2, \ldots, r$, p_i denotes the ith coordinate function on $\mathbb{R}^r$ defined by $p_i(x) = x_i$ for all $x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$. Then we have

$$d_s^p(x, y) \le c(p, r, s) \sum_{i=1}^{r} |p_i(x) - p_i(y)|^p \qquad (x, y \in \mathbb{R}^r,\ p > 0), \tag{2.1}$$

where

$$c(p, r, s) = \begin{cases} r^{p/s} & (1 \le s < \infty,\ s \ne p) \\ 1 & (1 \le s < \infty,\ s = p) \\ 1 & (s = \infty). \end{cases}$$

Let $L^1(X)$ denote the Banach space of all Lebesgue integrable functions g on X with the norm

$$\|g\|_1 = \int_X |g(x)|\, dx.$$

We make use of the key estimate in the following lemma for integral operators on $C(X, E)$.

Lemma 2.1. *Let $\{\chi(x; \cdot) : x \in X\}$ be a family of functions in $L^1(X), \tau$ a continuous mapping from X into itself and $p \ge 1$. Then for all $F \in C(X, E), x \in X$ and for all $\delta > 0$,*

$$\left\| \int_X \chi(x; y)(F(\tau(y)) - F(x))\, dy \right\| \le (\|\chi(x; \cdot)\|_1 + c(x; p, \delta))\omega_s(F, \delta),$$

where

$$c(x; p, \delta) = \min\{\delta^{-p}\|\chi(x; \cdot) d_s^p(x, \tau(\cdot))\|_1,\ \delta^{-1}\|\chi(x; \cdot)\|_1^{1-1/p} \|\chi(x; \cdot) d_s^p(x, \tau(\cdot))\|_1^{1/p}\}.$$

Proof. Since X is convex, by [13, Lemma 2.4 (b)] we have

$$\omega_s(G, \xi\delta) \le (1 + \xi)\omega_s(G, \delta)$$

for all $\xi, \delta \ge 0$ and for all $G \in B(X, E)$. Therefore, the desired result follows from [13, Lemma 2.7].

Let $\mathfrak{A} = \{\chi_n(x; \cdot) : n \in \mathbb{N}_0, x \in X\}$ be a family of nonnegative functions in $L^1(X)$ such that

$$\sup\{\|\chi_n(x; \cdot)\|_1 : n \in \mathbb{N}_0, x \in X\} < \infty.$$

We sometimes call $\mathfrak{A}$ a kernel. If

$$\|\chi_n(x; \cdot)\|_1 = \int_X \chi_n(x; y)\, dy = 1$$

for all $n \in \mathbb{N}_0$ and for all $x \in X$, then $\mathfrak{A}$ is said to be normal. If $\|\chi_n(x; \cdot)\|_1 \le 1$ for all $n \in \mathbb{N}_0$ and for all $x \in X$, then $\mathfrak{A}$ is said to be quasi-normal. We define

$$K_n(F)(x) = \int_X \chi_n(x; y) F(y)\, dy \qquad (F \in C(X, E),\ x \in X), \tag{2.2}$$

which exists as a Bochner integral and let $K_{n,\lambda}$ be defined by (1.1).

Let $p \ge 1$ be fixed and we define

$$\mu_{n,i}(x; p) = \|\chi_n(x; \cdot)|p_i(x) - p_i(\cdot)|^p\|_1 \qquad (n \in \mathbb{N}_0, x \in X, i = 1, 2, \ldots, r),$$

which is called the pth absolute moment of $\chi_n(x; \cdot)$.

For any $n \in \mathbb{N}_0, \lambda \in \Lambda$ and for any $x \in X$, we define

$$\tau_{n,\lambda}(x) = |b_{n,\lambda}(x) - 1|,$$

where

$$b_{n,\lambda}(x) := \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \int_X \chi_m(x;y)\,dy,$$

and

$$\theta_{n,\lambda,i}(x;p) = \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \mu_{m,i}(x;p) \qquad (i = 1, 2, \ldots, r).$$

From now on, let $\{\epsilon_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers.

Theorem 2.2. *For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in C(X,E)$ and for all $x \in X$,*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\tau_{n,\lambda}(x) + \zeta_{n,\lambda}(x)\omega_s(F,\epsilon_n), \tag{2.3}$$

where

$$\zeta_{n,\lambda}(x) = b_{n,\lambda}(x) + \eta_{n,\lambda}(x)$$

and

$$\eta_{n,\lambda}(x) = \min\Big\{c(p,r,s)\epsilon_n^{-p} \sum_{i=1}^{r} \theta_{n,\lambda,i}(x;p),$$
$$c(p,r,s)^{1/p}\epsilon_n^{-1}\Big(\sum_{i=1}^{r} \theta_{n,\lambda,i}(x;p)\Big)^{1/p} b_{n,\lambda}(x)^{1-1/p}\Big\}.$$

Proof. We have

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \Big\|\int_X \chi_m(x;y)(F(y) - F(x))\,dy\Big\| \tag{2.4}$$
$$+ \Big|\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \int_X \chi_m(x;y)\,dy - 1\Big|\|F(x)\| = I_{n,\lambda}^{(1)}(x) + I_{n,\lambda}^{(2)}(x).$$

Here $I_{n,\lambda}^{(1)}(x)$ and $I_{n,\lambda}^{(2)}(x)$ denote the first term and the second term in the above inequality, respectively. Then we have $I_{n,\lambda}^{(2)}(x) = \|F(x)\|\tau_{n,\lambda}(x)$. Taking $\chi(x;\cdot) = \chi_m(x;\cdot)$ and $\tau(y) = y$ in Lemma 2.1, we get

$$I_{n,\lambda}^{(1)}(x) \le \Big(b_{n,\lambda}(x) + \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} c_m(x;p,\delta)\Big)\omega_s(F,\delta), \tag{2.5}$$

where

$$c_m(x;p,\delta) = \min\{\delta^{-p}\|\chi_m(x;\cdot)d_s^p(x,\cdot)\|_1,\ \delta^{-1}\|\chi_m(x;\cdot)\|_1^{1-1/p}\|\chi_m(x;\cdot)d_s^p(x,\cdot)\|_1^{1/p}\}.$$

Now, if $p > 1$, then by Hölder's inequality we have

$$\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \|\chi_m(x;\cdot)\|_1^{1-1/p}\|\chi_m(x;\cdot)d_s^p(x;\cdot)\|_1^{1/p}$$
$$\le \Big(\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)}\|\chi_m(x;\cdot)\|_1\Big)^{1-1/p}\Big(\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)}\|\chi_m(x;\cdot)d_s^p(x;\cdot)\|_1\Big)^{1/p},$$

which clearly holds for $p = 1$. Also, by (2.1) we have

$$\|\chi_m(x;\cdot)d_s^p(x,\cdot)\|_1 \le c(p,r,s)\sum_{i=1}^{r}\mu_{m,i}(x;p) \qquad (m \in \mathbb{N}_0).$$

Therefore, we obtain

$$\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} c_m(x;p,\delta) \le \min\Big\{\delta^{-p}c(p,r,s)\sum_{i=1}^{r}\theta_{n,\lambda,i}(x;p),$$

$$\delta^{-1}b_{n,\lambda}(x)^{1-1/p}c(p,r,s)^{1/p}\Big(\sum_{i=1}^{r}\theta_{n,\lambda,i}(x;p)\Big)^{1/p}\Big\}\omega_s(F,\delta),$$

and so putting $\delta = \epsilon_n$ in the above inequality, (2.4) and (2.5) yield the desired estimate (2.3).

Corollary 2.3. *Suppose that $\mathfrak{A}$ is quasi-normal. Then for all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in C(X,E)$ and for all $x \in X$,*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\tau_{n,\lambda}(x) + (1+\gamma_{n,\lambda}(x))\omega_s(F,\epsilon_n), \tag{2.6}$$

where

$$\gamma_{n,\lambda}(x) = \min\Big\{c(p,r,s)\epsilon_n^{-p}\sum_{i=1}^{r}\theta_{n,\lambda,i}(x;p),\ c(p,r,s)^{1/p}\epsilon_n^{-1}\Big(\sum_{i=1}^{r}\theta_{n,\lambda,i}(x;p)\Big)^{1/p}\Big\}.$$

In particular, if $\mathfrak{A}$ is normal, then (2.6) reduces to

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le (1+\gamma_{n,\lambda}(x))\omega_s(F,\epsilon_n).$$

In the rest of this section, we restrict the integral operators K_n defined by (2.2) to the subclass of $C(X,E)$ defined as follows:

Let $\mathfrak{T} = \{T(x) : x \in X\}$ be a family of mappings from E to itself such that for each $f \in E$, the mapping from X to E defined by $x \mapsto T(x)(f) := (T(x))(f)$ is strongly continuous on X. Let L_n denote the restriction of K_n to the set $\{T(\cdot)(f) : f \in E\}$, that is,

$$L_n(x)(f) = \int_X \chi_n(x;y)T(y)(f)\,dy \qquad (f \in E,\ x \in X), \tag{2.7}$$

which exists as a Bochner integral. We define

$$L_{n,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} L_m(x)(f) \qquad (\lambda \in \Lambda), \tag{2.8}$$

which converges in E.

The family $\mathfrak{L} = \{L_n(x) : n \in \mathbb{N}_0, x \in X\}$ is called an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E if for every $f \in E$,

$$\lim_{n\to\infty}\|L_{n,\lambda}(\cdot)(f) - T(\cdot)(f)\|_X = 0 \qquad \text{uniformly in } \lambda \in \Lambda. \tag{2.9}$$

Concerning the rate of convergence behavior of (2.9), for each $f \in E$ and for each $\delta \ge 0$ we define

$$\omega_{s,\mathfrak{T}}(f,\delta) = \sup\{\|T(x)(f) - T(y)(f)\| : x, y \in X, d_s(x,y) \le \delta\},$$

which is called the modulus of continuity of f associated with $\mathfrak{T}$. Then we have the following result on the estimate for the rate of convergence of the equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process $\mathfrak{L}$ given by (2.8) with (2.7):

Theorem 2.4. *For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in E$ and for all $x \in X$,*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) + \zeta_{n,\lambda}(x)\omega_{s,\mathfrak{T}}(f, \epsilon_n).$$

Proof. Since

$$\omega_{s,\mathfrak{T}}(f, \delta) = \omega_s(T(\cdot)(f), \delta) \qquad (f \in X,\ \delta \geq 0),$$

this follows from Theorem 2.2.

Corollary 2.5. *Suppose that $\mathfrak{A}$ is quasi-normal. Then for all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in E$ and for all $x \in X$,*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) \ + \ (1 + \gamma_{n,\lambda}(x))\omega_{s,\mathfrak{T}}(f, \epsilon_n). \tag{2.10}$$

In particular, if $\mathfrak{A}$ is normal, then (2.10) reduces to

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq (1 + \gamma_{n,\lambda}(x))\omega_{s,\mathfrak{T}}(f, \epsilon_n).$$

Let ω be a monotone increasing continuous function on $[0, \infty)$ with $\omega(0) = 0$ and $M > 0$. Let $H_s(\omega, M)$ denote the class of all E-valued bounded functions F for which

$$\omega_s(F, \delta) \leq M\omega(\delta) \quad \text{for all } \delta \in [0, \infty).$$

In the special case $\omega(\delta) = \delta^\beta$ $(\beta > 0)$, we write $Lip_s(\beta, M)$ instead of $H_s(\omega, M)$. That is, $F \in Lip_s(\beta, M)$ implies

$$\omega_s(F, \delta) \leq M\delta^\beta \qquad (\delta \geq 0),$$

which is equivalent to

$$\|F(x) - F(y)\| \leq Md_s(x, y)^\beta \qquad (x,\ y \in X).$$

A function $F \in Lip_s(\beta, M)$ is sometimes said to satisfy a Lipschitz condition of order β with constant M with respect to d_s.

Let $H_{s,\mathfrak{T}}(\omega, M)$ denote the class of all elements $f \in E$ for which

$$\omega_{s,\mathfrak{T}}(f, \delta) \leq M\omega(\delta) \qquad \text{for all } \delta \in [0, \infty).$$

In the special case $\omega(\delta) = \delta^\beta$ $(\beta > 0)$, we write $Lip_{s,\mathfrak{T}}(\beta, M)$ instead of $H_{s,\mathfrak{T}}(\omega, M)$. That is, $f \in Lip_{s,\mathfrak{T}}(\beta, M)$ implies

$$\omega_{s,\mathfrak{T}}(f, \delta) \leq M\delta^\beta \qquad (\delta \geq 0),$$

which is equivalent to

$$\|T(x)(f) - T(y)(f)\| \leq Md_s(x, y)^\beta \qquad (x, y \in X).$$

An element $f \in Lip_{s,\mathfrak{T}}(\beta, M)$ is sometimes said to satisfy a Lipschitz condition of order β with constant M with respect to d_s.

Theorem 2.6. *For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in H_s(\omega, M)$ and for all $x \in X$,*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq \|F(x)\|\tau_{n,\lambda}(x) + M\zeta_{n,\lambda}(x)\omega(\epsilon_n). \tag{2.11}$$

In particular, if $F \in Lip_s(\beta, M)$, then (2.11) reduces to

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq \|F(x)\|\tau_{n,\lambda}(x) + M\zeta_{n,\lambda}\epsilon_n^\beta.$$

Proof. This follows from Theorem 2.2.

Corollary 2.7. *Suppose that* $\mathfrak{A}$ *is quasi-normal. Then for all* $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in H_s(\omega, M)$ *and for all* $x \in X$,

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq \|F(x)\|\tau_{n,\lambda}(x) + M(1 + \gamma_{n,\lambda}(x))\omega(\epsilon_n) \tag{2.12}$$

In particular, if $\mathfrak{A}$ *is normal, (2.12) reduces to*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq M(1 + \gamma_{n,\lambda}(x))\omega(\epsilon_n).$$

Also, if $F \in Lip_s(\beta, M)$, *then*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq \|F(x)\|\tau_{n,\lambda}(x) + M(1 + \gamma_{n,\lambda}(x))\epsilon_n^{\beta}. \tag{2.13}$$

In particular, if $\mathfrak{A}$ *is normal, then (2.13) reduces to*

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq M(1 + \gamma_{n,\lambda}(x))\epsilon_n^{\beta}.$$

Theorem 2.8. *For all* $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in H_{s,\mathfrak{T}}(\omega, M)$ *and for all* $x \in X$,

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) + M\zeta_{n,\lambda}(x)\omega(\epsilon_n). \tag{2.14}$$

In particular, if $f \in Lip_{s,\mathfrak{T}}(\beta, M)$, *then (2.14) reduces to*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) + M\zeta_{n,\lambda}(x)\epsilon_n^{\beta}.$$

Proof. This follows from Theorem 2.4.

Corollary 2.9. *Suppose that* $\mathfrak{A}$ *is quasi-normal. Then for all* $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in H_{s,\mathfrak{T}}(\omega, M)$ *and for all* $x \in X$,

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) + M(1 + \gamma_{n,\lambda}(x))\omega(\epsilon_n). \tag{2.15}$$

In particular, if $\mathfrak{A}$ *is normal, then (2.15) reduces to*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq M(1 + \gamma_{n,\lambda}(x))\omega(\epsilon_n).$$

Also, if $f \in Lip_{s,\mathfrak{T}}(\beta, M)$, *then*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq \|T(x)(f)\|\tau_{n,\lambda}(x) + M(1 + \gamma_{n,\lambda}(x))\epsilon_n^{\beta}. \tag{2.16}$$

In particular, if $\mathfrak{A}$ *is normal, then (2.16) reduces to*

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq M(1 + \gamma_{n,\lambda}(x))\epsilon_n^{\beta}.$$

3. Summation processes of convolution type operators

Let $c > 0$ and let $\{g_n\}_{n \in \mathbb{N}_0}$ be a sequence of nonnegative even continuous functions on $[-c, c]$ such that

$$\int_{-c}^{c} g_n(t)\,dt = 1 \qquad \text{for all } n \in \mathbb{N}_0.$$

Let

$$X = \prod_{i=1}^{r} [a_i, b_i], \quad 0 < b_i - a_i \leq c \qquad (i = 1, 2, \ldots, r)$$

and

$$X_0 = \prod_{i=1}^{r} [a_i + \delta_i, b_i - \delta_i], \quad 0 < \delta_i < \frac{1}{2}(b_i - a_i), \qquad (i = 1, 2, \ldots, r).$$

For each $x = (x_1, x_2, \dots, x_r) \in X$, we define

$$\eta(x_i) = \min\{x_i - a_i,\ b_i - x_i\} \qquad (i = 1, 2, \dots, r),$$

$$\xi(x_i) = \max\{x_i - a_i,\ b_i - x_i\} \qquad (i = 1, 2, \dots, r)$$

and

$$G_{n,\lambda}(x_i) = \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \int_{-\xi(x_i)}^{\xi(x_i)} |t|^p g_m(t)\, dt \qquad (i = 1, 2, \dots, r).$$

Now, we define

$$\chi_n(x; y) = \prod_{i=1}^{r} (g_n \circ p_i)(x - y) \qquad (x, y \in X,\ n \in \mathbb{N}_0)$$

and again, let $p \geq 1$ and let $\{\epsilon_n\}_{n \in \mathbb{N}_0}$ be a sequence of positive real numbers.

Theorem 3.1. *The following statements hold:*

(a) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in C(X, E)$ and for all $x = (x_1, x_2, \dots, x_r) \in X_0$,

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq 2\|F(x)\| \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} \nu_{n,\lambda,i}(x; p)$$
$$+ (A_{n,\lambda}(x) + B_{n,\lambda}(x)) \omega_s(F, \epsilon_n),$$

where

$$\nu_{n,\lambda,i}(x; p) = \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \int_{\eta(x_i)}^{c} t^p g_m(t)\, dt \qquad (i = 1, 2, \dots, r),$$

$$A_{n,\lambda}(x) = \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \Big(\prod_{i=1}^{r} \int_{-\xi(x_i)}^{\xi(x_i)} g_m(t)\, dt \Big)$$

and

$$B_{n,\lambda}(x) = \min\Big\{ c(p, r, s) \epsilon_n^{-p} \sum_{i=1}^{r} G_{n,\lambda}(x_i),$$
$$c(p, r, s)^{1/p} \epsilon_n^{-1} \Big(\sum_{i=1}^{r} G_{n,\lambda}(x_i) \Big)^{1/p} A_{n,\lambda}(x)^{1-1/p} \Big\}.$$

(b) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in E$ and for all $x = (x_1, x_2, \dots, x_r) \in X_0$,

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \leq 2\|T(x)(f)\| \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} \nu_{n,\lambda,i}(x; p)$$
$$+ (A_{n,\lambda}(x) + B_{n,\lambda}(x)) \omega_{s,\mathfrak{T}}(f, \epsilon_n).$$

Proof. For all $m \in \mathbb{N}_0$, we have

$$\int_X \chi_m(x; y) dy \leq \prod_{i=1}^{r} \int_{-\xi(x_i)}^{\xi(x_i)} g_m(t)\, dt \qquad (x \in X),$$

$$0 \leq 1 - \int_X \chi_m(x; y)\, dy \leq 2 \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} \int_{\eta(x_i)}^{c} t^p(t) g_m(t)\, dt \qquad (x \in X_0)$$

and

$$\sum_{i=1}^{r}\int_X \chi_m(x;y)|p_i(x)-p_i(y)|^p\,dy \le \sum_{i=1}^{r}\int_{-\xi(x_i)}^{\xi(x_i)}|t|^p g_m(t)\,dt \qquad (x\in X)$$

(cf. [11, Lemma 4]). Therefore, we obtain $b_{n,\lambda}(x)\le A_{n,\lambda}(x)$ and $\eta_{n,\lambda}(x)\le B_{n,\lambda}(x)$. Consequently, the desired results (a) and (b) follow from Theorems 2.2 and 2.4, respectively.

Corollary 3.2. *The following statements hold:*

(a) For all $n\in\mathbb{N}_0, \lambda\in\Lambda, F\in H_s(\omega,M)$ and all $x=(x_1,x_2,\dots,x_r)\in X_0$,

$$\|K_{n,\lambda}(F)(x)-F(x)\| \le 2\|F(x)\|\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)}\nu_{n,\lambda,i}(x;p) + M(A_{n,\lambda}(x)+B_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.1}$$

In particular, if $F\in Lip_s(\beta,M)$, then (3.1) reduces to

$$\|K_{n,\lambda}(F)(x)-F(x)\| \le 2\|F(x)\|\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)}\nu_{n,\lambda,i}(x;p) + M(A_{n,\lambda}(x)+B_{n,\lambda}(x))\epsilon_n^{\beta}.$$

(b) For all $n\in\mathbb{N}_0, \lambda\in\Lambda, f\in H_{s,\mathfrak{T}}(\omega,M)$ and all $x=(x_1,x_2,\dots,x_r)\in X_0$,

$$\|L_{n,\lambda}(x)(f)-T(x)(f)\| \le 2\|T(x)(f)\|\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)}\nu_{n,\lambda,i}(x;p) + M(A_{n,\lambda}(x)+B_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.2}$$

In particular, if $f\in Lip_{s,\mathfrak{T}}(\beta,M)$, then (3.2) reduces to

$$\|L_{n,\lambda}(x)(f)-T(x)(f)\| \le 2\|T(x)(f)\|\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)}\nu_{n,\lambda,i}(x;p) + M(A_{n,\lambda}(x)+B_{n,\lambda}(x))\epsilon_n^{\beta}.$$

For $\alpha>0$, we define

$$\mu_n(\alpha)=\int_{-c}^{c}|t|^\alpha g_n(t)\,dt \qquad (n\in\mathbb{N}_0),$$

which is called the αth absolute moment of g_n.

Theorem 3.3. *The following statements hold:*

(a) For all $n\in\mathbb{N}_0, \lambda\in\Lambda, F\in C(X,E)$ and for all $x=(x_1,x_2,\dots,x_r)\in X_0$,

$$\|K_{n,\lambda}(F)(x)-F(x)\| \le \|F(x)\|\zeta_{n,\lambda}(p)^p\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)} + (1+C_{n,\lambda}(x))\omega_s(F,\epsilon_n),$$

where

$$\zeta_{n,\lambda}(p)=\Big(\sum_{m=0}^{\infty}a_{n,m}^{(\lambda)}\mu_m(p)\Big)^{1/p}$$

and

$$C_{n,\lambda}(x) = \min\Big\{c(p,r,s)\epsilon_n^{-p}\sum_{i=1}^{r} G_{n,\lambda}(x_i),\ (c(p,r,s))^{1/p}\epsilon_n^{-1}\Big(\sum_{i=1}^{r} G_{n,\lambda}(x_i)\Big)^{1/p}\Big\}.$$

(b) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in E$ and for all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \le \|T(x)(f)\|\zeta_{n,\lambda}(p)^p \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + (1 + C_{n,\lambda}(x))\omega_{s,\mathfrak{T}}(f, \epsilon_n).$$

Proof. We have

$$2\nu_{n,\lambda,i}(x;p) \le 2\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} \int_0^c t^p g_m(t)\,dt = \zeta_{n,\lambda}(p)^p \qquad (i = 1, 2, \ldots, r),$$

$$\prod_{i=1}^{r}\int_{-\xi(x_i)}^{\xi(x_i)} g_m(t)\,dt \le \prod_{i=1}^{r}\int_{-c}^{c} g_m(t)\,dt = 1.$$

Thus, we obtain $A_{n,\lambda}(x) \le 1$, and so the desired result follows from Theorem 3.1.

Corollary 3.4. *The following statements hold:*

(a) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in H_s(\omega, M)$ and all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\zeta_{n,\lambda}(p)^p \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.3}$$

In particular, if $F \in Lip_s(\beta, M)$, then (3.3) reduces to

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\zeta_{n,\lambda}(p)^p \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\epsilon_n^{\beta}.$$

(b) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in H_{s,\mathfrak{T}}(\omega, M)$ and all $x = x_1, x_2, \ldots, x_r) \in X_0$,

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \le \|T(x)(f)\|\zeta_{n,\lambda}(p)^p \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.4}$$

In particular, if $f \in Lip_{s,\mathfrak{T}}(\beta, M)$, then (3.4) reduces to

$$\|L_{n,\lambda}(x)(f) - T(x)(f)\| \le \|T(x)(f)\|\zeta_{n,\lambda}(p)^p \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\epsilon_n^{\beta}.$$

Let φ be a nonnegative, continuous even function on $[-c, c]$ such that φ is decreasing on $(0, c]$ and

$$\varphi(0) = 1, \quad 0 \le \varphi(t) < 1 \qquad (0 < t \le c).$$

We define

$$g_n(t) = \rho_n \varphi^n(t) \qquad (|t| \le c,\ n \in \mathbb{N}_0),$$

where

$$\rho_n = \Big(\int_{-c}^{c} \varphi^n(t)\,dt\Big)^{-1} \qquad (n \in \mathbb{N}_0).$$

Then we have

$$\chi_n(x;y) = \rho_n^r \prod_{i=1}^{r} (\varphi^n \circ p_i)(x-y) \qquad (x, y \in X),$$

which reduces to the Korovkin kernel in case $r = 1$.

Suppose now that

$$\lim_{t\to+0} \frac{1-\varphi(t)}{t^q} = \kappa \tag{3.5}$$

for some $\kappa > 0$ and $q > 0$.

Lemma 3.5. *Let $0 < b - a \le c$ and $0 < \epsilon < \kappa$. Let $\alpha \ge 0$ and $\beta \ge 1$. Then the following inequality holds for all $n \in \mathbb{N}_0 \setminus \{0\}$ and there exists $\tau \in (0, (b-a)/\beta)$:*

$$A(\alpha, q, \kappa, \epsilon)\Big(\frac{1}{(n+1)q + \alpha + 1}\Big)^{(\alpha+1)/q} + B(\alpha, q, \kappa, \epsilon)e^{-n(k+\epsilon)\tau^q} \tag{3.6}$$

$$\le \int_0^c t^\alpha \varphi^n(t)\,dt \le C(\alpha, q, \kappa, \epsilon)\Big(\frac{1}{n}\Big)^{(\alpha+1)/q} + \frac{c^{\alpha+1} - \tau^{\alpha+1}}{\alpha+1} e^{-n(\kappa-\epsilon)\tau^q},$$

where

$$A(\alpha, q, \kappa, \epsilon) = \frac{1}{q}\Gamma\Big(\frac{\alpha+1}{q}\Big)\Big(\frac{q}{\kappa+\epsilon}\Big)^{(\alpha+1)/q},$$

$$B(\alpha, q, \kappa, \epsilon) = \frac{1}{\alpha+1}\Big(\tau^{\alpha+1} - \Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q}\Big),$$

$$C(\alpha, q, \kappa, \epsilon) = \frac{1}{q}\Gamma\Big(\frac{\alpha+1}{q}\Big)\Big(\frac{1}{\kappa-\epsilon}\Big)^{(\alpha+1)/q}$$

and

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt \qquad (x > 0)$$

is the gamma function.

Proof. By (3.5), there exists $\tau_0 \in (0, (b-a)/\beta)$ such that

$$(\kappa-\epsilon)t^q < 1 - \varphi(t) < (\kappa+\epsilon)t^q \qquad (0 < t \le \tau_0).$$

Take τ so that

$$0 < \tau < \min\Big\{\tau_0,\ \Big(\frac{1}{\kappa+\epsilon}\Big)^{1/q}\Big\}. \tag{3.7}$$

Then we have $0 < \tau < (b-a)/\beta \le c$ and

$$0 < 1 - (\kappa+\epsilon)t^q < \varphi(t) < 1 - (\kappa-\epsilon)t^q \qquad (0 < t \le \tau). \tag{3.8}$$

Since φ is decreasing on $[\tau, c]$, by (3.8) we have

$$\int_0^c t^\alpha \varphi^n(t)\,dt \le \int_0^\tau t^\alpha \varphi^n(t)\,dt + \varphi^n(\tau)\int_\tau^c t^\alpha\,dt$$

$$\le \int_0^\tau t^\alpha (1 - (\kappa-\epsilon)t^q)^n\,dt + \frac{c^{\alpha+1} - \tau^{\alpha+1}}{\alpha+1}\Big(1 - (\kappa-\epsilon)\tau^q\Big)^n = I_1 + I_2,$$

say. Since $1-x \le e^{-x}$ $(x \ge 0)$, we have

$$I_1 \le \int_0^\tau t^\alpha e^{-n(\kappa-\epsilon)t^q}\,dt = \Big(\frac{1}{n(\kappa-\epsilon)}\Big)^{(\alpha+1)/q} \int_0^{(n(\kappa-\epsilon))^{1/q}\tau} x^\alpha e^{-x^q}\,dx$$
$$= \frac{1}{q}\Big(\frac{1}{n(\kappa-\epsilon)}\Big)^{(\alpha+1)/q} \int_0^{n(\kappa-\epsilon)\tau^q} t^{(\alpha+1)/q-1} e^{-t}\,dt$$
$$\le \frac{1}{q}\Gamma\Big(\frac{\alpha+1}{q}\Big)\Big(\frac{1}{n(\kappa-\epsilon)}\Big)^{(\alpha+1)/q}.$$

Also, we have

$$I_2 \le \frac{c^{\alpha+1}-\tau^{\alpha+1}}{\alpha+1} e^{-n(\kappa-\epsilon)\tau^q}.$$

Therefore, the right-hand side of (3.6) holds.

Next, by (3.7) and (3.8) we have

$$\int_0^c t^\alpha \varphi^n(t)\,dt \ge \int_0^\tau t^\alpha (1-(\kappa+\epsilon)t^q)^n\,dt$$
$$= \frac{1}{q}\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} \int_0^{(\kappa+\epsilon)\tau^q} x^{(\alpha+1)/q-1}(1-x)^n\,dx$$
$$= \frac{1}{q}\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} \Big(\int_0^1 x^{(\alpha+1)/q-1}(1-x)^n\,dx$$
$$- \int_{(\kappa+\epsilon)\tau^q}^1 x^{(\alpha+1)/q-1}(1-x)^n\,dx\Big) = I_3 - I_4,$$

say. Since

$$\frac{\Gamma(x)}{\Gamma(y)} \le \frac{x^{x-1/2}e^y}{y^{y-1/2}e^x} \qquad (1 < x \le y)$$

(cf. [2]), we have

$$I_3 = \frac{1}{q}\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} \frac{\Gamma((\alpha+1)/q)\Gamma(n+1)}{\Gamma((\alpha+1)/q+n+1)} \quad \ge \frac{1}{q}\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q}\Gamma\Big(\frac{\alpha+1}{q}\Big)$$
$$\times\ e^{(\alpha+1)/q}\Big(1+\frac{\alpha+1}{(n+1)q}\Big)^{-(n+1/2)}\Big(\frac{\alpha+1}{q}+n+1\Big)^{-(\alpha+1)/q}$$
$$\ge\ \Big(\frac{1}{q}\Big)\Big(\frac{q}{\kappa+\epsilon}\Big)^{(\alpha+1)/q}\Gamma\Big(\frac{\alpha+1}{q}\Big)\Big(\frac{1}{\alpha+1+(n+1)q}\Big)^{(\alpha+1)/q}$$

Also, we have

$$I_4 \le \frac{1}{q}\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} \int_{(\kappa+\epsilon)\tau^q}^1 x^{(\alpha+1)/q-1}\big(1-(\kappa+\epsilon)\tau^q\big)^n\,dx$$
$$= \frac{1}{\alpha+1}\Big(\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} - \tau^{\alpha+1}\Big)\big(1-(\kappa+\epsilon)\tau^q\big)^n$$
$$\le \frac{1}{\alpha+1}\Big(\Big(\frac{1}{\kappa+\epsilon}\Big)^{(\alpha+1)/q} - \tau^{\alpha+1}\Big)e^{-n(\kappa+\epsilon)\tau^q}.$$

Consequently, the left-hand side of (3.6) holds.

Theorem 3.6. *The following statements hold:*

(a) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in C(X,E)$ and for all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq K(p,q)\|F(x)\| e^p_{n,\lambda}(p,q) \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + (1 + C_{n,\lambda}(x))\omega_s(F, \epsilon_n),$$

where

$$K(p,q) = \sup\Big\{(m+1)^{p/q}\mu_m(p) : m \in \mathbb{N}_0\Big\} < \infty$$

and

$$e_{n,\lambda}(p,q) = \Big(\sum_{m=0}^{\infty} \frac{a^{(\lambda)}_{n,m}}{(m+1)^{p/q}}\Big)^{1/p}.$$

(b) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in E$ and all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|L_{n,\lambda}(f)(x) - T(x)(f)\| \leq C_\varphi(p,q)\|T(x)(f)\| e^p_{\alpha,\lambda}(p,q) \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + (1 + C_{n,\lambda}(x))\omega_{s,\mathfrak{T}}(f, \epsilon_n).$$

Proof. Let

$$a = \max\{a_1, a_2, \ldots, a_r\},\ b = \min\{b_1, b_2, \ldots, b_r\},\ \alpha = p.$$

Then, by Lemma 3.5 we have

$$\rho_n \int_0^c \varphi^n(t)\, dt \leq \frac{C(p,q,\kappa,\epsilon)n^{-(\alpha+1)/q} + c^\alpha e^{-n(\kappa-\epsilon)\tau^q}}{A(0,q,\kappa,\epsilon)((n+1)q+1)^{-1/q} + B(0,q,\kappa,\epsilon)e^{-n(\kappa+\epsilon)\tau^q}},$$

and so $K(p,q)$ is finite. Therefore, we have

$$\zeta^p_{n,\lambda}(p) \leq K(p,q)e^p_{n,\lambda}(p,q)$$

for all $n \in \mathbb{N}_0$ and for all $\lambda \in \Lambda$. Thus, the desired result follows from Theorem 3.3.

Remark 3.7. Let $\alpha > 0$. Then we have

$$\mu_n(\alpha) \leq K(\alpha,p)\Big(\frac{1}{n+1}\Big)^{\alpha/q}$$

for all $n \in \mathbb{N}_0$, and so $\lim_{n\to\infty} \mu_n(\alpha) = 0$. In particular, we have

$$\mu_n(2) \leq K(2,q)\Big(\frac{1}{n+1}\Big)^{2/q}$$

for all $n \in \mathbb{N}_0$ (cf. [2, Proof of Theorem 1]), and so $\lim_{n\to\infty} \mu_n(2) = 0$ (cf. [7, Proof of Theorem 5]).

Corollary 3.8. *The following statements hold:*

(a) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, F \in H_s(\omega, M)$ and for all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|K_{n,\lambda}(F)(x) - F(x)\| \leq K(p,q)\|F(x)\| e^p_{n,\lambda}(p,q) \sum_{i=1}^{r} \frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.9}$$

In particular, if $F \in Lip_s(\beta, M)$, then (3.9) reduces to

$$\|K_{n,\lambda}(F)(x) - F(x)\| \le K(p,q)\|F(x)\|e^p_{n,\lambda}(p,q)\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\epsilon_n^{\beta}.$$

(b) For all $n \in \mathbb{N}_0, \lambda \in \Lambda, f \in H_{s,\mathfrak{X}}(\omega, M)$ and for all $x = (x_1, x_2, \ldots, x_r) \in X_0$,

$$\|L_{n,\lambda}(f)(x) - T(x)(f)\| \le K(p,q)\|T(x)(f)\|e^p_{\alpha,\lambda}(p,q)\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\omega(\epsilon_n). \tag{3.10}$$

In particular, if $f \in Lip_{s,\mathfrak{X}}(\beta, M)$, then (3.10) reduces to

$$\|L_{n,\lambda}(f)(x) - T(x)(f)\| \le K(p,q)\|T(x)(f)\|e^p_{\alpha,\lambda}(p,q)\sum_{i=1}^{r}\frac{1}{\eta^p(x_i)} + M(1 + C_{n,\lambda}(x))\epsilon_n^{\beta}.$$

We further assume that φ is continuously differentiable on $(0, c)$ and

$$\lim_{t\to+0}\frac{\varphi'(t)}{t^{q-1}} = A \tag{3.11}$$

for some $q > 0$ and $A < 0$. Then, by (3.5), we have $\kappa = -A/q$.

Now, let Φ be a nonnegative, continuous even function on $[-c, c]$, which satisfies the following conditions:

(Φ-1) $\Phi(0) = 0$;
(Φ-2) Φ is increasing and positive on $(0, c]$;
(Φ-3) Φ is continuously differentiable on $(0, c)$ and

$$\lim_{t\to+0}\frac{\Phi'(t)}{t^{q-1}} = B \tag{3.12}$$

for some $q > 0$ and $B > 0$.

We define

$$\varphi(t) = e^{-\Phi(t)} \qquad (-c \le t \le c).$$

Then (3.11) is satisfied with $A = -B$, and so $\kappa = B/q$. Let $\beta > 0$ and we specially define

$$\Phi(t) = \Phi_\beta(t) = |t|^\beta \qquad (-c \le t \le c).$$

Then (3.12) holds with $q = B = \beta$. Several examples of β induces the following important kernels:

(1°) *Picard*:

$$\beta = 1; \quad \varphi(t) = e^{-|t|}, \quad q = B = 1, \ \kappa = 1.$$

(2°) *Weierstrass*:

$$\beta = 2; \quad \varphi(t) = e^{-t^2}, \quad q = B = 2, \ \kappa = 1.$$

(3°) *Bui-Fedorov-Cervakov*:

$$\beta = 1/\nu,\ \nu > 0;\ \ \varphi(t) = e^{-|t|^{1/\nu}},\ \ q = B = 1/\nu,\ \kappa = 1.$$

The other important examples of φ satisfying (3.11) are the following:

(4°) *de la Vallée-Poussin*:

$$\varphi(t) = \cos^2 \frac{1}{2}t;\quad c = \pi,\ q = 2,\ A = -1/2,\ \kappa = 1/4.$$

(5°) Let $\nu > 0$ and

$$\varphi(t) = \Big(\cos \frac{1}{2}t\Big)^{\nu};\quad c = \pi,\ q = 2,\ A = -\nu/4,\ \kappa = \nu/8.$$

(6°) *Landau*:

$$\varphi(t) = 1 - t^2;\ \ c = 1,\ q = 2,\ A = -2,\ \kappa = 1.$$

(7°) *Mamedov*:

$$\varphi(t) = 1 - t^{2m};\ \ c = 1,\ m \in \mathbb{N},\ q = 2m,\ A = -2m,\ \kappa = 1.$$

(8°) Let $\nu > 0$ and

$$\varphi(t) = 1 - |t|^{\nu};\ \ c = 1,\ q = \nu,\ A = -\nu,\ \kappa = 1.$$

Note that the above examples (6°), (7°) and (8°) can be also particular cases of φ which is defined as follows:

Let $\mu > 0$ and let $\varPsi$ be a nonnegative, continuous even function on $[-1, 1]$, which satisfies the following conditions:

($\varPsi$-1) $\varPsi(0) = 0,\ \ 0 < \varPsi(t) \le 1\ \ \ (0 < t \le 1);$

($\varPsi$-2) $\varPsi$ is increasing on $(0, 1]$;

($\varPsi$-3) $\lim_{t\to+0} \varPsi(t)/t^{\mu} = K$ for some $K > 0$.

Then we define

$$\varphi(t) = 1 - \varPsi(t) \qquad (|t| \le 1),$$

and so (3.5) holds with $q = \mu$ and $\kappa = K$.

4. Various summability methods

Let $\mathcal{A} = \{a_{n,m}^{(\lambda)} : n, m \in \mathbb{N}_0, \lambda \in \varLambda\}$ be a family of scalars. $\mathcal{A}$ is said to be regular if it satisfies the following conditions:

(A-1) For each $m \in \mathbb{N}_0$, $\lim_{n\to\infty} a_{n,m}^{(\lambda)} = 0$ uniformly in $\lambda \in \varLambda$.

(A-2) $\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} = 1$ uniformly in $\lambda \in \varLambda$.

(A-3) For each $n \in \mathbb{N}_0$ and for each $\lambda \in \varLambda$,

$$a_n^{(\lambda)} := \sum_{m=0}^{\infty} |a_{n,m}^{(\lambda)}| < \infty,$$

and there exists $n_0 \in \mathbb{N}_0$ such that

$$\sup\{a_n^{(\lambda)} : n \ge n_0, n \in \mathbb{N}_0, \lambda \in \varLambda\} < \infty.$$

$\mathcal{A}$ is said to be stochastic if

$$a_{n,m}^{(\lambda)} \ge 0 \qquad (n, m \in \mathbb{N}_0,\ \lambda \in \varLambda)$$

and

$$\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} = 1 \qquad (n \in \mathbb{N}_0,\ \lambda \in \Lambda).$$

Obviously, if $\mathcal{A}$ is stochastic, then Conditions (A-2) and (A-3) are automatically satisfied.

A sequence $\{f_m\}_{m\in\mathbb{N}_0}$ of elements in E is said to be $\mathcal{A}$-summable to f if

$$\lim_{n\to\infty}\Big\|\sum_{m=0}^{\infty} a_{n,m}^{(\lambda)} f_m - f\Big\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, \tag{4.1}$$

where it is assumed that the series in (4.1) converges for each $n \in \mathbb{N}_0$ and for each $\lambda \in \Lambda$.

Concerning the relation between the regularity of $\mathcal{A}$ and $\mathcal{A}$-summability, $\mathcal{A}$ is regular if and only if every convergent sequence of elements in E is $\mathcal{A}$-summable to its limit (cf. [1], [10]).

As the following examples show, there is a wide variety of families $\mathcal{A}$ and their particular cases cover many important summability methods:

(1°) Given an infinite matrix $A = (a_{nm})_{n,m\in\mathbb{N}_0}$, if $a_{n,m}^{(\lambda)} = a_{nm}$ for all $n, m \in \mathbb{N}_0$ and for all $\lambda \in \Lambda$, then we obtain the usual matrix summability method by A.

(2°) If $\Lambda = \mathbb{N}_0$, then we obtain the summation method introduced by Petersen [16] (cf. [1]). In particular, if

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{n+1} & \text{if } \lambda \le m \le \lambda + n, \\ 0 & \text{otherwise,} \end{cases}$$

then we obtain the notion of almost convergent method (F-summability) introduced by Lorentz [8].

(3°) Let $Q = \{q^{(\lambda)} : \lambda \in \Lambda\}$ be a familiy of sequences $q^{(\lambda)} = \{q_n^{(\lambda)}\}_{n\in\mathbb{N}_0}$ of nonnegative real numbers such that

$$Q_n^{(\lambda)} := q_0^{(\lambda)} + q_1^{(\lambda)} + \cdots + q_n^{(\lambda)} > 0 \qquad (n \in \mathbb{N}_0,\ \lambda \in \Lambda).$$

We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{q_{n-m}^{(\lambda)}}{Q_n^{(\lambda)}} & \text{if } \quad m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Then $\mathcal{A}$-summability is called a (N, Q)-summability method, and this kind of summability is called the Nörlund summability method in the case where $q^{(\lambda)} = \{q_n\}_{n\in\mathbb{N}_0}$ is a fixed sequence of nonnegative real numbers satisfying $q_0 > 0$. The special case of interest is the following: We set $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$. Let $\Lambda \subseteq [0, \infty)$, $\beta > 0$ and

$$q_n^{(\lambda)} = C_n^{(\lambda+\beta-1)} \qquad (\lambda \in \Lambda,\ n \in \mathbb{N}_0),$$

where

$$C_0^{(\nu)} = 1, \quad C_n^{(\nu)} = \binom{n+\nu}{n} = \frac{(\nu+1)(\nu+2)\cdots(\nu+n)}{n!}$$
$$(\nu > -1,\ n \in \mathbb{N}).$$

In particular, if $\Lambda = \{0\}$, then we have the Cesàro summability method of order β.

(4°) *Cesàro type*:

Let $\Lambda \subseteq (0,\infty), \beta > -1$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} C_{n-m}^{(\lambda-1)} C_m^{(\beta)} / C_n^{(\beta+\lambda)} & \text{if } m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

(5°) *Euler-Knopp-Bernstein type*:
Let $\Lambda \subseteq [0,1]$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \binom{n}{m} \lambda^m (1-\lambda)^{n-m} & \text{if } m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Note that this can be a particular case of the generalized Lototsky matrix defined as follows (cf. [4], [5], [6], [17]): Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of continuous functions from $[0,1]$ to itself and let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative continuous functions on $[0,1]$. Then we define

$$a_{0,0}^{(\lambda)} = 1, \quad a_{n,m}^{(\lambda)} = 0 \qquad (m > n),$$

and

$$\psi_n(\lambda) \prod_{i=1}^{n} (x h_i(\lambda) + 1 - h_i(\lambda)) = \sum_{m=0}^{n} a_{n,m}^{(\lambda)} x^m. \tag{4.2}$$

In particular, if $\psi_n(\lambda) = 1$ for all $n \in \mathbb{N}, \lambda \in [0,1]$, and if h is a continuous function from $[0,1]$ to itself and $h_i = h$ for all $i \in \mathbb{N}$, then (4.2) implies

$$a_{n,m}^{(\lambda)} = \binom{n}{m} h(\lambda)^m (1 - h(\lambda))^{n-m}. \tag{4.3}$$

Therefore, if $h(\lambda) = \lambda^\beta$ $(\beta > 0)$, then (4.3) reduces to

$$a_{n,m}^{(\lambda)} = \binom{n}{m} \lambda^{\beta m} (1 - \lambda^\beta)^{n-m}.$$

(6°) *Meyer-König-Vermes-Zeller type*:
Let $\Lambda \subseteq [0,1)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m}{m} \lambda^m (1-\lambda)^{n+1}.$$

(7°) *Borel-Szász type*:
Let $\Lambda \subseteq [0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \exp(-n\lambda) \frac{(n\lambda)^m}{m!}.$$

(8°) *Baskakov type*:
Let $\Lambda \subseteq [0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m-1}{m} \lambda^m (1+\lambda)^{-n-m}.$$

This can be generalized as follows (cf. [3], [9]): Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions on $[0,\infty)$ which possess the following properties:

(φ-1) Each function φ_n is expanded in Taylor's series on $[0,\infty)$;

(φ-2) $\quad \varphi_n(0) = 1 \qquad (n \in \mathbb{N})$;

(φ-3) Each function φ_n is completely monotone, i.e.,

$$(-1)^m \varphi_n^{(m)}(t) \geq 0 \qquad (t \in [0,\infty), n \in \mathbb{N}, m \in \mathbb{N}_0);$$

(φ-4) There exists a strictly monotone increasing sequence $\{\ell_n\}_{n\in\mathbb{N}}$ of positive integers and a sequence $\{\alpha_{n,m}\}_{n,m\in\mathbb{N}}$ of real-valued functions on $[0,\infty)$ such that

$$\varphi_n^{(m)}(t) = -n\varphi_{\ell_n}^{(m-1)}(1+\alpha_{n,m}(t)) \qquad (t \in [0,\infty)).$$

Now we define

$$a_{0,0}^{(\lambda)} = 1, \quad a_{0,m}^{(\lambda)} = 0 \qquad (m \in \mathbb{N}),$$

$$a_{n,m}^{(\lambda)} = (-1)^m \frac{\varphi_n^{(m)}(\lambda)}{m!}\lambda^m \qquad (n \in \mathbb{N}, m \in \mathbb{N}_0).$$

(9°) *Abel type*:

Let $\{r_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers converging to one such that $0 \leq r_n < 1$ for all $n \in \mathbb{N}_0$ and Let $\Lambda \subseteq (-1,\infty)$. We define

$$a_{n,m}^{(\lambda)} = (1-r_n)^{\lambda+1}\binom{m+\lambda}{m} r_n^m.$$

(10°) *logarithmic type*:

Let $\{r_n\}_{n\in\mathbb{N}_0}$ be as in (9°) and $\Lambda \subseteq (0,1]$. We define

$$a_{n,m}^{(\lambda)} = -\frac{(\lambda r_n)^{m+1}}{(m+1)\log(1-\lambda r_n)}.$$

(11°) Let $\{h_n\}_{n\in\mathbb{N}_0}$ and $\{v_n\}_{n\in\mathbb{N}_0}$ be sequences of positive continuous functions on $(0,\infty)$. Let $\Lambda \subseteq (0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \left(\frac{h_n(\lambda)}{h_n(\lambda)+v_n(\lambda)}\right)^n \binom{n}{m} \left(\frac{v_n(\lambda)}{h_n(\lambda)}\right)^m & \text{if } 0 \leq m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

Special selections of h_n and v_n yield the following summability methods:

(12°) Let $\{b_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers and let g be a sequence of positive continuous function on $(0,\infty)$. Let $\Lambda \subseteq (0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{(1+b_n g(\lambda))^n}\binom{n}{m}(b_n g(\lambda))^m & \text{if } 0 \leq m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

(13°) Let $\{b_n\}_{n\in\mathbb{N}_0}$ be as in (12°) and $\beta > 0$. We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{(1+b_n \lambda^\beta)^n}\binom{n}{m} b_n^m \lambda^{m\beta} & \text{if } 0 \leq m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

(14°) *Balázs type*:

Let $\{b_n\}_{n\in\mathbb{N}_0}$ be as in (12°) and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{(1+b_n \lambda)^n}\binom{n}{m} b_n^m \lambda^{m} & \text{if } 0 \leq m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

(15°) *Bleimann-Butzer-Hahn type*:

We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{(1+\lambda)^n}\binom{n}{m}\lambda^m & \text{if } 0 \le m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Remark 4.1. All the families $\mathcal{A}$ of the generic entries $a_{n,m}^{(\lambda)}$ given in (2°)-(9°) and (11°)-(15°) are stochastic and all the families $\mathcal{A}$ of the generic entries $a_{n,m}^{(\lambda)}$ given in (4°)-(10°) are regular for any bounded closed interval Λ. Let Λ be a bounded closed interval of $(0, \infty)$. If

$$\sup\Big\{\Big(\frac{h_n(\lambda)}{h_n(\lambda) + v_n(\lambda)}\Big)^n : \lambda \in \Lambda,\ n \in \mathbb{N}_0\Big\} < 1$$

and

$$\sup\Big\{\frac{v_n(\lambda)}{h_n(\lambda)} : \lambda \in \Lambda,\ n \in \mathbb{N}_0\Big\} < \infty,$$

then $\mathcal{A}$ is regular. If $0 < a \le b_n \le b < \infty$ for all $n \in \mathbb{N}_0$, then $\mathcal{A}$ given in (12°) is regular, and so the summability method given in (13°) is regular. In particular, the summability method of Balázs type given (14°) is regular, and so the summability method of Bleimann-Butzer-Hahn type given in (15°) is regular.

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