Journal of Nonlinear and Convex Analysis
Volume 11, Number 1, 2010, 13–33

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

UNIFORM NON-ℓ_1^n-NESS OF ψ-DIRECT SUMS OF BANACH SPACES

MIKIO KATO, KICHI-SUKE SAITO, AND TAKAYUKI TAMURA

Abstract. We shall characterize the uniform non-ℓ_1^n-ness of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y, where ψ is a convex function on the unit interval satisfying certain conditions. A previous result for the uniform non-squareness will be obtained as a corollary. As extreme cases we shall treat the ℓ_1-sum and the ℓ_∞-sum of finitely many Banach spaces.

1. Introduction

The ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y is the direct sum $X \oplus Y$ equipped with the norm $\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$, where the $\|(\cdot,\cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to a convex (continuous) function ψ with some conditions on the unit interval. This extends the notion of the ℓ_p-sum $X \oplus_p Y$. Since it was introduced in Takahashi, Kato and Saito [33], the ψ-direct sum of Banach spaces have been attracting a good deal of attention and been treated by several authors ([5, 6, 7, 8, 18, 19, 21, 22, 23, 25, 27, 28, 29, 30, etc.]; cf. [31, 32, 27]). In particular the present authors [19] showed that $X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and neither $\psi = \psi_1$ nor $\psi = \psi_\infty$, where $\psi_1(t) = 1$ and $\psi_\infty(t) = \max\{1-t, t\}$ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms respectively.

The purpose of this paper is to characterize the uniform non-ℓ_1^n-ness of $X \oplus_\psi Y$. In comparison with uniform non-squareness the situation will be much more complicated than expected. Section 2 is devoted to some definitions and preliminary results.

In Section 3 we shall show that under the assumption $\psi \neq \psi_1, \psi_\infty$, $X \oplus_\psi Y$ is uniformly non-ℓ_1^n if and only if X and Y are uniformly non-ℓ_1^n. Keeping in mind the result on uniform non-squareness mentioned above, the following question arises: Let X and Y be uniformly non-ℓ_1^n. Then is it possible that $X \oplus_\psi Y$ is uniformly non-ℓ_1^n with $\psi = \psi_1$ or $\psi = \psi_\infty$? Our next result (Theorem 3.5) will answer this question as follows: Under the assumption that both X and Y are not uniformly non-ℓ_1^{n-1}, $X \oplus_\psi Y$ is uniformly non-ℓ_1^n if and only if X and Y are uniformly non-ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$. This assumption on X and Y cannot be removed; we shall present some counterexamples in the final section. Theorem 3.5 covers the above-mentioned result concerning uniform non-squareness as the case $n = 2$. Another corollary states that the $\ell_{p,q}$-sum $X \oplus_{p,q} Y$, $1 \le q \le p \le \infty$, $q < \infty$, is uniformly

---

2000 *Mathematics Subject Classification.* 46B20, 46B99.

Key words and phrases. Absolute norm, convex function, direct sum of Banach spaces, uniformly non-ℓ_1^n space, uniformly non-square space.

The authors are supported in part by Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science (the first and third authors (20540179), the second author (20540158)).

non-ℓ_1^n if and only if X and Y are uniformly non-ℓ_1^n. The same is true for the ℓ_p-sum $X \oplus_p Y$, $1 < p < \infty$, as the case $p = q$.

In the next two sections we shall treat the extreme cases. Some results obtained there will be applied to construct the examples stated above. According to Theorem 3.5 *the ℓ_1-sum $X \oplus_1 Y$ and the ℓ_∞-sum $X \oplus_\infty Y$ can be uniformly non-ℓ_1^n, $n \geq 3$.* In Section 4 we shall first show that the ℓ_1-sum $X \oplus_1 Y$ is uniformly non-ℓ_1^n if and only if there exist positive integers n_1, n_2 with $n_1 + n_2 = n - 1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$ (Theorem 4.2). This was recently extended for finitely many Banach spaces in [22] with a different proof. We shall present another proof of this result by induction based on Theorem 4.2 (Theorem 4.3). A corollary states that if the ℓ_1-sum $(X_1 \oplus \cdots \oplus X_m)_1$ of Banach spaces $X_1, \ldots, X_m$ is uniformly non-ℓ_1^n, then each X_i is uniformly non-ℓ_1^{n-1}. Theorem 4.2 says the converse of this statement holds true for $m = 2$ and $n = 3$, that is, $X \oplus_1 Y$ is uniformly non-ℓ_1^3 if and only if X and Y are uniformly non-square (recall that $X \oplus_1 Y$ cannot be uniformly non-square for all X and Y).

In Section 5 we shall show that for m uniformly non-square spaces $X_1, \ldots, X_m$, the ℓ_∞-sum $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n if and only if $m < 2^{n-1}$ (Theorem 5.2). In particular $X \oplus_\infty Y$ with uniformly non-square spaces X, Y is uniformly non-ℓ_1^n if and only if $n \geq 3$, or equivalently, if X and Y are uniformly non-square, then $X \oplus_\infty Y$ is uniformly non-ℓ_1^3. In contrast with the ℓ_1-sum case the converse of this result is not valid. Instead we shall obtain that for three Banach spaces X, Y and Z, $(X \oplus Y \oplus Z)_\infty$ is uniformly non-ℓ_1^3 if and only if X, Y and Z are uniformly non-square. Theorem 5.2 also yields that ℓ_∞^m is uniformly non-ℓ_1^n if and only if $m < 2^{n-1}$, which will be useful to construct various examples. (Refer to the recent paper [23] for some further results on ℓ_∞-sums.) In the final Section 6 we shall present some counterexamples for Theorem 3.5.

2. Definitions and preliminary results

Let Ψ be the family of all convex (continuous) functions ψ on $[0,1]$ satisfying

$$\psi(0) = \psi(1) = 1 \text{ and } \max\{1-t, t\} \leq \psi(t) \leq 1 \ (0 \leq t \leq 1). \tag{2.1}$$

For any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$, that is, $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z, w \in \mathbb{C}$ and $\|(1,0)\| = \|(0,1)\| = 1$, let

$$\psi(t) = \|(1-t, t)\| \ (0 \leq t \leq 1). \tag{2.2}$$

Then $\psi \in \Psi$. Conversely for any $\psi \in \Psi$ define

$$\|(z,w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{2.3}$$

Then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ and satisfies (2.2) (Bonsall and Duncan [2], see also [31, 32]). The ℓ_p-norms $\|\cdot\|_p$ are such examples and for all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$ we have

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1 \tag{2.4}$$

([2]). By (2.2) the convex functions corresponding to the ℓ_p-norms are given by

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \tag{2.5}$$

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ-direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,\, y)\|_\psi = \|(\|x\|,\, \|y\|)\|_\psi, \tag{2.6}$$

where the $\|(\cdot,\, \cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to the convex function ψ ([33, 18]; see [30] for several examples). This extends the notion of the ℓ_p-sum $X \oplus_p Y$ and provides a plenty of concrete non ℓ_p-type norms on $X \oplus Y$.

A Banach space X is said to be *uniformly non-ℓ_1^n* (cf. [1, 24]) provided there exists ϵ $(0 < \epsilon < 1)$ such that for any $x_1, \ldots, x_n \in S_X$, the unit sphere of X, there exists an n-tuple of signs $\theta = (\theta_j)$ for which

$$\left\| \sum_{j=1}^n \theta_j x_j \right\| \le n(1-\epsilon). \tag{2.7}$$

As is well known, we may take $x_1, \ldots, x_n$ from the unit ball B_X of X in the definition (This is immediately seen from Lemma 3.1 below; see [20, Corollary 4]). In case of $n = 2$ X is called *uniformly non-square* ([15]; cf. [1, 24]). Though we can consider the case $n = 1$ formally, no Banach space is uniformly non-ℓ_1^1. The following fundamental fact was proved in Brown [3] (see also Hudzik [13]).

Proposition A ([3, 13]). *Let X be a Banach space. If X is uniformly non-ℓ_1^n, then X is uniformly non-ℓ_1^{n+1} for every $n \in \mathbb{N}$.*

For convenience of the reader we shall present a proof. Assume that X is uniformly non-ℓ_1^n. Then there exists $\epsilon_0 > 0$ such that for any $x_1, \ldots, x_n \in S_X$ there exists an n-tuple of signs (θ_j) such that $\| \sum_{j=1}^n \theta_j x_j \| \le n(1-\epsilon_0)$. Then for $\theta_{n+1} = \pm 1$ we have

$$\left\| \sum_{j=1}^{n+1} \theta_j x_j \right\| \le \left\| \sum_{j=1}^{n} \theta_j x_j \right\| + \|\theta_{n+1} x_{n+1}\| \le n(1-\epsilon_0) + 1 = (n+1)(1 - \frac{n\epsilon_0}{n+1}).$$

Now we recall a sequence of monotonicity properties of absolute norms on $\mathbb{C}^2$, which is essential in our later discussion.

Lemma 2.1 (2, p.36, Lemma 2)**.** *Let $\psi \in \Psi$.*
(i) *If $|p| \le |r|$ and $|q| \le |s|$, then $\|(p,q)\|_\psi \le \|(r,s)\|_\psi$.*
(ii) *If $|p| < |r|$ and $|q| < |s|$, then $\|(p,q)\|_\psi < \|(r,s)\|_\psi$.*

The following assertion is not true in general:

(2.8) *Let $|p| \le |r|$ and $|q| \le |s|$. If $|p| < |r|$ or $|q| < |s|$, then $\|(p,q)\|_\psi < \|(r,s)\|_\psi$.*

Indeed the ℓ_∞-norm does not satisfy (2.8). Those norms satisfying (2.8) are characterized as follows.

Proposition 2.2 (Takahashi, Kato and Saito [33]). *Let $\psi \in \Psi$. Then the following assertions are equivalent*:

(i) *If* $|z| \le |u|$ *and* $|w| < |v|$, *or* $|z| < |u|$ *and* $|w| \le |v|$, *then* $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.

(ii) $\psi(t) > \psi_\infty(t)$ *for all* $t \in (0,1)$.

In particular, if ψ is strictly convex, that is, if, for any $s,\ t \in [0,1]$ $(s \neq t)$ and for any c $(0 < c < 1)$, one has $\psi((1-c)s + ct) < (1-c)\psi(s) + c\psi(t)$, then the assertion (i) holds true. A more precise (component-wise) result is given in [33]. The next proposition presents a condition for specified (z,w) and (u,v) to satisfy the above assertion (i) for a general $\psi \in \Psi$.

Proposition 2.3 (Kato-Saito-Tamura [20]). *Let $\psi \in \Psi$. Let $(z,w),\ (u,v) \in \mathbb{C}^2$.*

(i) *Let* $|z| < |u|$ *and* $|w| = |v|$. *Then* $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |w|$.

(ii) *Let* $|z| = |u|$ *and* $|w| < |v|$. *Then* $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |z|$.

3. Uniform non-ℓ_1^n-ness of $X \oplus_\psi Y,\ \psi \neq \psi_1, \psi_\infty$

We need a sequence of lemmas. The first lemma, a recent result of the present authors [20], is of independent interest as it provides a sharper inequality than the triangle inequality and its reverse (see also [9, 26]).

Lemma 3.1 (Kato-Saito-Tamura [20]). *For all nonzero elements $x_1, x_2, \dots, x_n$ in a Banach space X*

$$\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le n} \|x_j\| \tag{3.1}$$

$$\le \sum_{j=1}^n \|x_j\| \le \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le n} \|x_j\|.$$

Lemma 3.2. *Let $\{x_1^{(k)}\}_k, \dots,\ \{x_n^{(k)}\}_k$ be n sequences with nonzero terms in a Banach space X for which $\{\|x_1^{(k)}\|\}_k, \dots,\ \{\|x_n^{(k)}\|\}_k$ converge to nonzero limits, respectively. Then the following are equivalent.*

(i) $\displaystyle\lim_{k\to\infty} \left\|\sum_{j=1}^n x_j^{(k)}\right\| = \lim_{k\to\infty} \sum_{j=1}^n \|x_j^{(k)}\|.$

(ii) $\displaystyle\lim_{k\to\infty} \left\|\sum_{j=1}^n \frac{x_j^{(k)}}{\|x_j^{(k)}\|}\right\| = n.$

Proof. Let $\lim_{k\to\infty} \|x_j^{(k)}\| = a_j > 0$. Suppose (i) to be true. Then by (3.1) we have

$$0 \le n - \left\|\sum_{j=1}^n \frac{x_j^{(k)}}{\|x_j^{(k)}\|}\right\| \le \frac{1}{\min_{1\le j\le n} \|x_j^{(k)}\|} \left(\sum_{j=1}^n \|x_j^{(k)}\| - \left\|\sum_{j=1}^n x_j^{(k)}\right\|\right) \to 0$$

as $k \to \infty$, where it should be noted that $\lim_{k\to\infty} \min_{1\le j\le n} \|x_j^{(k)}\| = \min\{a_1, \ldots, a_n\}$. Hence we obtain (ii). The converse implication is immediate from Lemma 3.1. $\square$

Lemma 3.3. *Let $\{x_1^{(k)}\}_k, \ldots, \{x_n^{(k)}\}_k$ be n sequences in a Banach space X for which the sequences of their norms are convergent. Then the following are equivalent.*

(i) $\displaystyle\lim_{k\to\infty} \left\| \sum_{j=1}^n x_j^{(k)} \right\| = \lim_{k\to\infty} \sum_{j=1}^n \|x_j^{(k)}\|.$

(ii) $\displaystyle\lim_{k\to\infty} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| = \lim_{k\to\infty} \left[\alpha \|x_1^{(k)}\| + \sum_{j=2}^n \|x_j^{(k)}\| \right]$ *for all* $\alpha > 0$.

(iii) $\displaystyle\lim_{k\to\infty} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| = \lim_{k\to\infty} \left[\alpha \|x_1^{(k)}\| + \sum_{j=2}^n \|x_j^{(k)}\| \right]$ *for some* $\alpha > 0$.

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds. Then, since for any $\alpha \ge 1$

$$\begin{aligned} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| &= \left\| \alpha \sum_{j=1}^n x_j^{(k)} - (\alpha - 1) \sum_{j=2}^n x_j^{(k)} \right\| \\ &\ge \alpha \left\| \sum_{j=1}^n x_j^{(k)} \right\| - (\alpha - 1) \sum_{j=2}^n \|x_j^{(k)}\|, \end{aligned}$$

we have

$$\begin{aligned} \liminf_{k\to\infty} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| &\ge \alpha \lim_{k\to\infty} \sum_{j=1}^n \|x_j^{(k)}\| - (\alpha - 1) \lim_{k\to\infty} \sum_{j=2}^n \|x_j^{(k)}\| \\ &= \alpha \lim_{k\to\infty} \|x_1^{(k)}\| + \lim_{k\to\infty} \sum_{j=2}^n \|x_j^{(k)}\|, \end{aligned}$$

from which it follows that

$$\lim_{k\to\infty} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| = \lim_{k\to\infty} \left[\alpha \|x_1^{(k)}\| + \sum_{j=2}^n \|x_j^{(k)}\| \right]. \tag{3.2}$$

If $0 < \alpha < 1$ we have

$$\begin{aligned} \left\| \alpha x_1^{(k)} + \sum_{j=2}^n x_j^{(k)} \right\| &= \left\| \sum_{j=1}^n x_j^{(k)} - (1 - \alpha) x_1^{(k)} \right\| \\ &\ge \left\| \sum_{j=1}^n x_j^{(k)} \right\| - (1 - \alpha) \|x_1^{(k)}\|. \end{aligned}$$

Hence

$$\begin{aligned}\liminf_{k\to\infty}\left\|\alpha x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right\| &\geq \lim_{k\to\infty}\sum_{j=1}^n\|x_j^{(k)}\|-(1-\alpha)\lim_{k\to\infty}\|x_1^{(k)}\| \\ &= \alpha\lim_{k\to\infty}\|x_1^{(k)}\|+\lim_{k\to\infty}\sum_{j=2}^n\|x_j^{(k)}\|,\end{aligned}$$

which implies (3.2). The implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Assume that (3.2) is true for some $\alpha_0>0$. If $\alpha_0>1$, since

$$\begin{aligned}\left\|x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right\| &= \left\|\alpha_0 x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}-(\alpha_0-1)x_1^{(k)}\right\| \\ &\geq \left\|\alpha_0 x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right\|-(\alpha_0-1)\|x_1^{(k)}\|,\end{aligned}$$

we have

$$\begin{aligned}\liminf_{k\to\infty}\left\|\sum_{j=1}^n x_j^{(k)}\right\| &\geq \alpha_0\lim_{k\to\infty}\|x_1^{(k)}\|+\lim_{k\to\infty}\sum_{j=2}^n\|x_j^{(k)}\|-(\alpha_0-1)\lim_{k\to\infty}\|x_1^{(k)}\| \\ &= \lim_{k\to\infty}\sum_{j=1}^n\|x_j^{(k)}\|,\end{aligned}$$

from which we have (i). If $0<\alpha_0<1$, we have

$$\begin{aligned}\left\|\sum_{j=1}^n x_j^{(k)}\right\| &= \left\|\frac{1}{\alpha_0}\alpha_0 x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right\| \\ &= \left\|\frac{1}{\alpha_0}\left(\alpha_0 x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right)-\left(\frac{1}{\alpha_0}-1\right)\sum_{j=2}^n x_j^{(k)}\right\| \\ &\geq \frac{1}{\alpha_0}\left\|\alpha_0 x_1^{(k)}+\sum_{j=2}^n x_j^{(k)}\right\|-\left(\frac{1}{\alpha_0}-1\right)\sum_{j=2}^n\|x_j^{(k)}\|.\end{aligned}$$

Hence

$$\begin{aligned}&\liminf_{k\to\infty}\left\|\sum_{j=1}^n x_j^{(k)}\right\| \\ &\geq \frac{1}{\alpha_0}\left(\alpha_0\lim_{k\to\infty}\|x_1^{(k)}\|+\lim_{k\to\infty}\sum_{j=2}^n\|x_j^{(k)}\|\right)-\left(\frac{1}{\alpha_0}-1\right)\lim_{k\to\infty}\sum_{j=2}^n\|x_j^{(k)}\| \\ &= \lim_{k\to\infty}\|x_1^{(k)}\|+\lim_{k\to\infty}\sum_{j=2}^n\|x_j^{(k)}\|,\end{aligned}$$

and therefore we obtain (i). This completes the proof. $\square$

Now we are in a position to present our first main result.

Theorem 3.4. *Let X and Y be Banach spaces and let $\psi \in \Psi, \psi \neq \psi_1, \psi_\infty$. Then the following are equivalent.*
(i) *$X \oplus_\psi Y$ is uniformly non-ℓ_1^n.*
(ii) *X and Y are uniformly non-ℓ_1^n.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial because X and Y are identified with subspaces of $X \oplus_\psi Y$. We show (ii) $\Rightarrow$ (i). Assume that X and Y are uniformly non-ℓ_1^n and $X \oplus_\psi Y$ is not uniformly non-ℓ_1^n. Then we have n sequences $\{(x_j^{(k)}, y_j^{(k)})\}_k$ in $X \oplus_\psi Y$ $(j = 1, \ldots, n)$ such that

$$\|(x_j^{(k)}, y_j^{(k)})\|_\psi = 1 \text{ for all } 1 \leq j \leq n \text{ and } k \in \mathbb{N} \tag{3.3}$$

and

$$\left\| \sum_{j=1}^{n} \theta_j (x_j^{(k)},\ y_j^{(k)}) \right\|_\psi = \left\| \left(\sum_{j=1}^{n} \theta_j x_j^{(k)},\ \sum_{j=1}^{n} \theta_j y_j^{(k)} \right) \right\|_\psi \to n \text{ as } k \to \infty \tag{3.4}$$

for all n-tuples of signs $\theta = (\theta_j)$. Since $\|x_j^{(k)}\| \leq \|(x_j^{(k)}, y_j^{(k)})\|_\psi = 1$, the sequence $\{\|x_j^{(k)}\|\}_k$ is bounded for all j. So $\{\|x_j^{(k)}\|\}_k$ has a convergent subsequence. For simplicity we assume that $\{\|x_j^{(k)}\|\}_k$ itself converges. As the same argument works for the sequences $\{\|y_j^{(k)}\|\}_k$, $\{\| \sum_{j=1}^n \theta_j x_j^{(k)} \|\}_k$, and $\{\| \sum_{j=1}^n \theta_j y_j^{(k)} \|\}_k$, we may assume that

$$\|x_j^{(k)}\| \to a_j,\ \|y_j^{(k)}\| \to b_j \text{ as } k \to \infty \tag{3.5}$$

and

$$\left\| \sum_{j=1}^{n} \theta_j x_j^{(k)} \right\| \quad \to \quad A_\theta, \tag{3.6}$$

$$\left\| \sum_{j=1}^{n} \theta_j y_j^{(k)} \right\| \quad \to \quad B_\theta \tag{3.7}$$

as $k \to \infty$. Then letting $k \to \infty$ in (3.3), we have

$$\|(a_j, b_j)\|_\psi = 1 \text{ for all } 1 \leq j \leq n. \tag{3.8}$$

By (3.4) we have

$$\|(A_\theta, B_\theta)\|_\psi = n \text{ for all } \theta = (\theta_j). \tag{3.9}$$

Hence

$$
\begin{aligned}
n = \|\,(A_\theta, B_\theta)\,\|_\psi &= \lim_{k\to\infty} \left\| \left(\left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\|, \left\| \sum_{j=1}^n \theta_j y_j^{(k)} \right\| \right) \right\|_\psi \\
&\le \lim_{k\to\infty} \left\| \left(\sum_{j=1}^n \|x_j^{(k)}\|, \sum_{j=1}^n \|y_j^{(k)}\| \right) \right\|_\psi \\
&= \left\| \left(\sum_{j=1}^n a_j, \sum_{j=1}^n b_j \right) \right\|_\psi = \left\| \sum_{j=1}^n (a_j, b_j) \right\|_\psi \\
&\le \sum_{j=1}^n \|(a_j, b_j)\|_\psi = n
\end{aligned}
$$

and thus we have

$$
\|\,(A_\theta, B_\theta)\,\|_\psi = \left\| \left(\sum_{j=1}^n a_j, \sum_{j=1}^n b_j \right) \right\|_\psi = n \text{ for all } \theta = (\theta_j). \tag{3.10}
$$

We note here that from the condition $\psi \neq \psi_1$ it follows that $a_j > 0$ for all j, or $b_j > 0$ for all j. Indeed suppose that $a_{j_1} = b_{j_2} = 0$ with some j_1 and j_2. Then by (3.8), $j_1 \neq j_2$ and $b_{j_1} = a_{j_2} = 1$. Since

$$
\begin{aligned}
\left\| \sum_{j=1}^n (a_j, b_j) \right\|_\psi &= \left\| (a_{j_1}, b_{j_1}) + (a_{j_2}, b_{j_2}) + \sum_{j\neq j_1, j_2} (a_j, b_j) \right\|_\psi \\
&\le \|(0,1) + (1,0)\|_\psi + \left\| \sum_{j\neq j_1, j_2} (a_j, b_j) \right\|_\psi \\
&= \|(1,1)\|_\psi + \left\| \sum_{j\neq j_1, j_2} (a_j, b_j) \right\|_\psi,
\end{aligned}
$$

we have

$$
2 \ge \|(1,1)\|_\psi \ge \left\| \sum_{j=1}^n (a_j, b_j) \right\|_\psi - \left\| \sum_{j\neq j_1, j_2} (a_j, b_j) \right\|_\psi \ge n - (n-2) = 2
$$

by (3.10), whence $\|(1,1)\|_\psi = 2$. Consequently we have

$$
\psi\left(\frac{1}{2}\right) = \left\| \left(\frac{1}{2}, \frac{1}{2} \right) \right\|_\psi = 1, \tag{3.11}
$$

which implies that $\psi = \psi_1$, a contradiction.

Now we obviously have $A_\theta \le \sum_{j=1}^n a_j$ and $B_\theta \le \sum_{j=1}^n b_j$. Consequently in view of Lemma 2.1, for all $\theta = (\theta_j)$ there is no case that

$$
A_\theta < \sum_{j=1}^n a_j \text{ and } B_\theta < \sum_{j=1}^n b_j.
$$

Case 1. Let $A_\theta = \sum_{j=1}^n a_j$ and $B_\theta = \sum_{j=1}^n b_j$ for all $\theta = (\theta_j)$.

(a) Assume first that $a_j > 0$ for all j. Then

$$\lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\| = A_\theta = \sum_{j=1}^n a_j = \lim_{k\to\infty} \sum_{j=1}^n \|\theta_j x_j^{(k)}\|.$$

Therefore by Lemma 3.2 we have

$$\lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j \frac{x_j^{(k)}}{\|x_j^{(k)}\|} \right\| = n \text{ for all } \theta = (\theta_j),$$

which implies that X is not uniformly non-ℓ_1^n, a contradiction.

(b) If $b_j > 0$ for all j, the parallel argument works for Y.

Case 2. Let (a) $A_\theta = \sum_{j=1}^n a_j$ for all $\theta = (\theta_j)$ and $B_{\theta'} < \sum_{j=1}^n b_j$ with some $\theta' = (\theta'_j)$, or (b) $A_{\theta'} < \sum_{j=1}^n a_j$ with some $\theta' = (\theta'_j)$ and $B_\theta = \sum_{j=1}^n b_j$ for all $\theta = (\theta_j)$. It is enough to see the case (a). Since

$$\|(A_{\theta'}, B_{\theta'})\|_\psi = \left\| \left(\sum_{j=1}^n a_j, \sum_{j=1}^n b_j \right) \right\|_\psi = n,$$

we have by Proposition 2.3 $\sum_{j=1}^n a_j = \|(A_{\theta'}, B_{\theta'})\|_\psi = n$ and hence $a_j = 1$ for all j (recall $0 \le a_j \le 1$). Consequently X is not uniformly non-ℓ_1^n as in Case 1(a).

Case 3. Let $A_\theta < \sum_{j=1}^n a_j$ and $B_{\theta'} < \sum_{j=1}^n b_j$ with different $\theta = (\theta_j)$ and $\theta' = (\theta'_j)$. Then we have

$$A_\theta < \sum_{j=1}^n a_j \text{ and } B_\theta = \sum_{j=1}^n b_j \tag{3.12}$$

and

$$A_{\theta'} = \sum_{j=1}^n a_j \text{ and } B_{\theta'} < \sum_{j=1}^n b_j. \tag{3.13}$$

By Proposition 2.3, (3.10), (3.12) and (3.13) we have

$$\sum_{j=1}^n a_j = A_{\theta'} = \|(A_{\theta'}, B_{\theta'})\|_\psi = n \quad \text{and} \quad \sum_{j=1}^n b_j = B_\theta = \|(A_\theta, B_\theta)\|_\psi = n.$$

Therefore by (3.10)

$$n = \left\| \left(\sum_{j=1}^n a_j, \sum_{j=1}^n b_j \right) \right\|_\psi = \|(n, n)\|_\psi = n\|(1,1)\|_\psi,$$

and thus $\|(1,1)\|_\psi = 1$. Consequently we have

$$\psi\left(\frac{1}{2}\right) = \left\| \left(\frac{1}{2}, \frac{1}{2}\right) \right\|_\psi = \frac{1}{2},$$

which implies that $\psi = \psi_\infty$, a contradiction. This completes the proof. □

The foregoing Theorem 3.4 does not answer the following: Let X and Y be uniformly non-ℓ_1^n. Is it possible for $X \oplus_\psi Y$ to be uniformly non-ℓ_1^n with $\psi = \psi_1$ or $\psi = \psi_\infty$? The next theorem will answer this question.

Theorem 3.5. *Let X and Y be Banach spaces and let $\psi \in \Psi$. Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*
(i) *$X \oplus_\psi Y$ is uniformly non-ℓ_1^n.*
(ii) *X and Y are uniformly non-ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$.*

Proof. We merely see the assertion (i) $\Rightarrow$ (ii). Assume that $X \oplus_\psi Y$ is uniformly non-ℓ_1^n. Then X and Y are uniformly non-ℓ_1^n as mentioned before. Since X is not uniformly non-ℓ_1^{n-1}, there exist $n-1$ sequences $\{x_1^{(k)}\}_k, \ldots, \{x_{n-1}^{(k)}\}_k \subset S_X$ such that

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{n-1} \theta_j x_j^{(k)} \right\| = n-1 \tag{3.14}$$

for all $\theta_j = \pm 1$. We first assume that $\psi = \psi_1$. Take $y \in S_Y$. Then the n sequences $\{(x_1^{(k)}, 0)\}, \ldots, \{(x_{n-1}^{(k)}, 0)\}, \{(0, y)\}$ are in the unit sphere of $X \oplus_\psi Y$ and

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{n-1} \theta_j (x_j^{(k)}, 0) + \theta_n (0, y) \right\|_1 = \lim_{k\to\infty} \left[\left\| \sum_{j=1}^{n-1} \theta_j x_j^{(k)} \right\| + \|\theta_n y\| \right] = n,$$

from which it follows that $X \oplus_1 Y$ is not uniformly non-ℓ_1^n, a contradiction. Thus we have $\psi \neq \psi_1$. Next assume that $\psi = \psi_\infty$. Since Y is not uniformly non-ℓ_1^{n-1}, there exist $n-1$ sequences $\{y_1^{(k)}\}_k, \ldots, \{y_{n-1}^{(k)}\}_k \subset S_Y$ such that

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{n-1} \theta_j y_j^{(k)} \right\| = n-1 \tag{3.15}$$

for all $\theta_j = \pm 1$. Then the sequences $\{(x_1^{(k)}, y_1^{(k)})\}_k, \ldots, \{(x_{n-1}^{(k)}, y_{n-1}^{(k)})\}_k$, $\{(-x_1^{(k)}, y_1^{(k)})\}_k$ are in the unit sphere of $X \oplus_\infty Y$. Owing to Lemma 3.3 it follows from (3.15) that

$$\lim_{k\to\infty} \left\| 2\theta_1 y_1^{(k)} + \sum_{j=2}^{n-1} \theta_j y_j^{(k)} \right\| = \lim_{k\to\infty} \left[2\|y_1^{(k)}\| + \sum_{j=2}^{n-1} \|y_j^{(k)}\| \right] = n.$$

Hence, if $\theta_1 = \theta_n$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \left\| \sum_{j=1}^{n-1} \theta_j (x_j^{(k)}, y_j^{(k)}) + \theta_n(-x_1^{(k)}, y_1^{(k)}) \right\|_\infty \\
= &\lim_{k\to\infty} \left\| \left(\sum_{j=2}^{n-1} \theta_j x_j^{(k)}, \sum_{j=2}^{n-1} \theta_j y_j^{(k)} + 2\theta_1 y_1^{(k)} \right) \right\|_\infty \\
= &\lim_{k\to\infty} \max \left\{ \left\| \sum_{j=2}^{n-1} \theta_j x_j^{(k)} \right\|, \left\| \sum_{j=2}^{n-1} \theta_j y_j^{(k)} + 2\theta_1 y_1^{(k)} \right\| \right\} \\
= &\max \left\{ \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j x_j^{(k)} \right\|, \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j y_j^{(k)} + 2\theta_1 y_1^{(k)} \right\| \right\} \\
= &\max \left\{ \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j x_j^{(k)} \right\|, n \right\} = n.
\end{aligned}$$

If $\theta_1 = -\theta_n$, by (3.14) we have

$$\begin{aligned}
&\lim_{k\to\infty} \left\| \sum_{j=1}^{n-1} \theta_j (x_j^{(k)}, y_j^{(k)}) + \theta_n(-x_1^{(k)}, y_1^{(k)}) \right\|_\infty \\
= &\lim_{k\to\infty} \max \left\{ \left\| \sum_{j=2}^{n-1} \theta_j x_j^{(k)} + 2\theta_1 x_1^{(k)} \right\|, \left\| \sum_{j=2}^{n-1} \theta_j y_j^{(k)} \right\| \right\} \\
= &\max \left\{ \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j x_j^{(k)} + 2\theta_1 x_1^{(k)} \right\|, \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j y_j^{(k)} \right\| \right\} \\
= &\max \left\{ n, \lim_{k\to\infty} \left\| \sum_{j=2}^{n-1} \theta_j y_j^{(k)} \right\| \right\} = n.
\end{aligned}$$

Consequently $X \oplus_\infty Y$ is not uniformly non-ℓ_1^n, a contradiction. Thus we have $\psi \neq \psi_\infty$, which completes the proof. □

Remark 3.6. In Theorem 3.5 we cannot remove the condition that neither X nor Y is uniformly non-ℓ_1^{n-1}. We shall see this in the final section.

Theorem 3.5 yields the following recent result of the authors.

Corollary 3.7 (Kato-Saito-Tamura [19]). *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-square.*

(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

Now we consider the (Lorentz) $\ell_{p,q}$-norm $\|\cdot\|_{p,q}$, $1 \le q \le p \le \infty$:

$$\|(z_1, z_2)\|_{p,q} = \left\{ z_1^{*q} + 2^{(q/p)-1} z_2^{*q} \right\}^{1/q},$$

where $\{z_1^*, z_2^*\}$ is the non-increasing rearrangement of $\{|z_1|, |z_2|\}$. (Note that in case of $1 \le p < q \le \infty$, $\|\cdot\|_{p,q}$ is not a norm but a quasi-norm (cf. [16], [34, p.126]). Clearly $\|\cdot\|_{p,q}$ is an absolute normalized norm and the corresponding convex function $\psi_{p,q}$ is given by

$$\psi_{p,q}(t) = \begin{cases} \{(1-t)^q + 2^{q/p-1}t^q\}^{1/q} & \text{if } 0 \le t \le 1/2, \\ \{t^q + 2^{q/p-1}(1-t)^q\}^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases} \tag{3.16}$$

Thus $\psi_{p,q}$ yields the $\ell_{p,q}$-sum $X \oplus_{p,q} Y$:

$$\|(x, y)\|_{p,q} = \left\{ \max(\|x\|^q, \|y\|^q) + 2^{(q/p)-1} \min(\|x\|^q, \|y\|^q) \right\}^{1/q}. \tag{3.17}$$

Corollary 3.8. *Let $1 \le q \le p \le \infty$, $q < \infty$. Then the $\ell_{p,q}$-sum $X_1 \oplus_{p,q} X_2$ is uniformly non-ℓ_1^n if and only if X_1 and X_2 are uniformly non-ℓ_1^n. In particular the ℓ_p-sum $X_1 \oplus_p X_2$, $1 < p < \infty$, is uniformly non-ℓ_1^n if and only if X_1 and X_2 are uniformly non-ℓ_1^n. The same is true for the uniform non-squareness.*

4. ℓ_1-SUMS

The ℓ_1-sum $X \oplus_1 Y$ cannot be uniformly non-square for all X and Y, whereas Theorem 3.5 indicates that it can be uniformly non-ℓ_1^n, $n \ge 3$. In this section we shall treat the uniform non-ℓ_1^n-ness of the ℓ_1-sum of finitely many Banach spaces. We shall denote by $(X_1 \oplus \cdots \oplus X_m)_1$ the ℓ_1-sum of m Banach spaces $X_1, \ldots, X_m$ though it is not consistent with the notation $X \oplus_1 Y$. First we shall prove the following.

Proposition 4.1. *Let X and Y be Banach spaces and let $n \ge 3$ and $n = n_1 + n_2$ with positive integers n_1, n_2. Let $X \oplus_1 Y$ be uniformly non-ℓ_1^n. Then X is uniformly non-$\ell_1^{n_1}$ or Y is uniformly non-$\ell_1^{n_2}$.*

Proof. Assume that X is not uniformly non-$\ell_1^{n_1}$ and Y is not uniformly non-$\ell_1^{n_2}$. Then there exist $\{x_1^{(k)}\}_k, \ldots, \{x_{n_1}^{(k)}\}_k$ in S_X and $\{y_1^{(k)}\}_k, \ldots, \{y_{n_2}^{(k)}\}_k$ in S_Y such that $\lim_{k\to\infty} \|\sum_{j=1}^{n_1} \theta'_j x_j^{(k)}\| = n_1$ for all (θ'_j) of n_1 signs and $\lim_{k\to\infty} \|\sum_{j=1}^{n_2} \theta''_j y_j^{(k)}\| = n_2$ for all (θ''_j) of n_2 signs, respectively. Define $(z_1^{(k)}, w_1^{(k)}), \ldots, (z_{n_1+n_2}^{(k)}, w_{n_1+n_2}^{(k)})$ in $X \oplus_1 Y$ by

$$(z_1^{(k)}, w_1^{(k)}) = (x_1^{(k)}, 0), \ldots\ldots, (z_{n_1}^{(k)}, w_{n_1}^{(k)}) = (x_{n_1}^{(k)}, 0),$$

$$(z_{n_1+1}^{(k)}, w_{n_1+1}^{(k)}) = (0, y_1^{(k)}), \ldots\ldots, (z_{n_1+n_2}^{(k)}, w_{n_1+n_2}^{(k)}) = (0, y_{n_2}^{(k)}).$$

Let $\theta = (\theta_j)$ be an arbitrary (n_1+n_2)-tuple of signs. Then $\|(z_j^{(k)}, w_j^{(k)})\|_1 = 1$ for all $1 \le j \le n$ and $k \in \mathbb{N}$ and

$$\begin{aligned}\left\| \sum_{j=1}^{n_1+n_2} \theta_j (z_j^{(k)}, w_j^{(k)}) \right\|_1 &= \left\| \left(\sum_{j=1}^{n_1} \theta_j x_j^{(k)}, \sum_{j=n_1+1}^{n_1+n_2} \theta_j y_{j-n_1}^{(k)} \right) \right\|_1 \\ &= \left\| \sum_{j=1}^{n_1} \theta_j x_j^{(k)} \right\| + \left\| \sum_{j=n_1+1}^{n_1+n_2} \theta_j y_{j-n_1}^{(k)} \right\| \to n_1 + n_2\end{aligned}$$

as $k \to \infty$. This implies that $X \oplus_1 Y$ is not uniformly non-$\ell_1^{n_1+n_2}$, which completes the proof. $\square$

Theorem 4.2. *Let X and Y be Banach spaces. The following are equivalent.*

(i) *$X \oplus_1 Y$ is uniformly non-ℓ_1^n.*

(ii) *There exist positive integers n_1 and n_2 with $n_1 + n_2 = n-1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $X \oplus_1 Y$ is uniformly non-ℓ_1^n. Let $n_1 = \min\{m \in \mathbb{N} : X \text{ is uniformly non-}\ell_1^{m+1}\}$ (note that X is uniformly non-ℓ_1^n). Then X is uniformly non-$\ell_1^{n_1+1}$, but not uniformly non-$\ell_1^{n_1}$. Therefore Y is uniformly non-$\ell_1^{n-n_1}$ by Proposition 4.1 and hence $n - n_1 \ge 2$. Letting $n_2 = n - n_1 - 1$, we have the conclusion.

(ii) $\Rightarrow$ (i). Assume that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$ with $n_1 + n_2 = n-1$. Suppose that $X \oplus_1 Y$ is not uniformly non-ℓ_1^n. Then we have n sequences $\{(x_j^{(k)}, y_j^{(k)})\}_k$ in $X \oplus_1 Y$ $(j = 1, \dots, n)$ such that

$$\|(x_j^{(k)}, y_j^{(k)})\|_1 = 1 \text{ for all } 1 \le j \le n \text{ and } k \in \mathbb{N} \tag{4.1}$$

and

$$\left\| \sum_{j=1}^{n} \theta_j (x_j^{(k)}, y_j^{(k)}) \right\|_1 = \left\| \left(\sum_{j=1}^{n} \theta_j x_j^{(k)}, \sum_{j=1}^{n} \theta_j y_j^{(k)} \right) \right\|_1 \to n \text{ as } k \to \infty \tag{4.2}$$

for all n-tuples of signs $\theta = (\theta_j)$. As in the proof of Theorem 3.4 we may assume that each of the sequences $\{\|x_j^{(k)}\|\}_k$, $\{\|y_j^{(k)}\|\}_k$, $\{\|\sum_{j=1}^n \theta_j x_j^{(k)}\|\}_k$, and $\{\|\sum_{j=1}^n \theta_j y_j^{(k)}\|\}_k$ has a limit and

$$\|x_j^{(k)}\| \to a_j, \ \|y_j^{(k)}\| \to b_j \text{ as } k \to \infty \tag{4.3}$$

and

$$\left\| \sum_{j=1}^{n} \theta_j x_j^{(k)} \right\| \to A_\theta, \ \left\| \sum_{j=1}^{n} \theta_j y_j^{(k)} \right\| \to B_\theta \text{ as } k \to \infty. \tag{4.4}$$

Letting $k \to \infty$ in (4.1), we have

$$\|(a_j, b_j)\|_1 = 1 \text{ for all } 1 \le j \le n, \tag{4.5}$$

and by (4.2)

$$\|(A_\theta, B_\theta)\|_1 = n \text{ for all } \theta = (\theta_j). \tag{4.6}$$

Then as (3.10)

$$\|(A_\theta, B_\theta)\|_1 = \left\| \left(\sum_{j=1}^n a_j, \sum_{j=1}^n b_j \right) \right\|_1 = n \text{ for all } \theta = (\theta_j).$$

Since $A_\theta \le \sum_{j=1}^n a_j$ and $B_\theta \le \sum_{j=1}^n b_j$, we have

$$A_\theta = \sum_{j=1}^n a_j, \; B_\theta = \sum_{j=1}^n b_j \tag{4.7}$$

for all $\theta = (\theta_j)$. Let $L = \{j : a_j = 0\}$ and $M = \{j : b_j = 0\}$. Since X is uniformly non-$\ell_1^{n_1+1}$, we have

$$\mathrm{card}(L^c) = \lim_{k\to\infty} \left\| \sum_{j\in L^c} \theta_j \frac{x_j^{(k)}}{\|x_j^{(k)}\|} \right\| < n_1 + 1$$

by (4.7) and Lemma 3.2. In the same way $\mathrm{card}(M^c) < n_2 + 1$. Therefore

$$\mathrm{card}(L) > n - n_1 - 1 = n_2.$$

On the other hand, as $L \subset M^c$ (recall that $\|(a_j, b_j)\| = 1$ for all j), we obtain that $\mathrm{card}(L) \le \mathrm{card}(M^c) < n_2 + 1$, a contradiction. This completes the proof. □

Recently Theorem 4.2 was extended for finitely many Banach spaces in [22] as follows (with the different proof). We shall present another proof by induction based on Theorem 4.2.

Theorem 4.3 (Kato and Tamura [22, Theorem 1])**.** *Let $X_1, \dots, X_m$ be Banach spaces. Let n be an arbitrary positive integer with $n \ge 2$. Then the following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_m)_1$ *is uniformly non-*ℓ_1^n.

(ii) *There exist positive integers $n_1, \dots, n_m$ with $n_1 + n_2 + \cdots + n_m = n - 1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$.*

Proof. According to Theorem 4.2 our assertion is valid for $m = 2$. Assume that the assertion holds for m. Then, since the space $(X_1 \oplus \cdots \oplus X_{m+1})_1$ is expressed as $(X_1 \oplus \cdots \oplus X_{m+1})_1 = (X_1 \oplus \cdots \oplus X_m)_1 \oplus_1 X_{m+1}$, we have that $(X_1 \oplus \cdots \oplus X_{m+1})_1$ is uniformly non-ℓ_1^n if and only if there exist positive integers n_0 and n_{m+1} with $n_0 + n_{m+1} = n - 1$ such that $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-$\ell_1^{n_0+1}$ and X_{m+1} is uniformly non-$\ell_1^{n_{m+1}+1}$. By the induction assumption, $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-$\ell_1^{n_0+1}$ if and only if there exist positive integers $n_1, \dots, n_m$ with $n_1 + n_2 + \cdots + n_m = n_0 - 1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$. This implies that our assertion holds true for $m + 1$, which completes the proof. □

From Theorem 4.3 it follows that if even one of $X_1, \dots, X_m$ is not uniformly non-ℓ_1^{n-1}, then $(X_1 \oplus \cdots \oplus X_m)_1$ cannot be uniformly non-ℓ_1^n, that is:

Corollary 4.4. *Let $X_1, \dots, X_m$ be Banach spaces. If $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^n, then each of X_i is uniformly non-ℓ_1^{n-1}.*

Indeed, assume that $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^n. Then by Theorem 4.3 there exist positive integers $n_1, \dots, n_m$ with $n_1 + \cdots + n_m = n-1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$. As $n_i + 1 \le n_1 + \cdots + n_m = n-1$, X_i is uniformly non-ℓ_1^{n-1} for each i by Proposition A.

As the case $m = 2$ and $n = 3$ Theorem 4.3 yields the following interesting result.

Theorem 4.5. *Let X and Y be Banach spaces. Then the following are equivalent.*
(i) *$X \oplus_1 Y$ is uniformly non-ℓ_1^3.*
(ii) *X and Y are uniformly non-square.*

5. ℓ_∞-sums

In this section we shall discuss the uniform non-ℓ_1^n-ness of the ℓ_∞-sum of a finite number of uniformly non-square Banach spaces. The ℓ_∞-sum of Banach spaces $X_1, \dots, X_m$, which we denote by $(X_1 \oplus \cdots \oplus X_m)_\infty$, is their direct sum equipped with the norm $\|\cdot\|_\infty = \max\{\|\cdot\|_{X_1}, \dots, \|\cdot\|_{X_m}\}$.

Proposition 5.1. *Let X be a uniformly non-square Banach space and let $\{x_1^{(k)}\}_k$, ..., $\{x_n^{(k)}\}_k$ be n sequences with nonzero terms in the closed unit ball of X. Let*

$$(5.1)\qquad \begin{aligned} &B(\{x_1^{(k)}\}, \dots, \{x_n^{(k)}\}) \\ := &\left\{(\theta_j) : \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\| = n,\ \theta_1 = 1,\ \theta_j = \pm 1\ (2 \le j \le n)\right\}. \end{aligned}$$

Then $\operatorname{card}(B(\{x_1^{(k)}\}, \dots, \{x_n^{(k)}\})) \le 1$.

Proof. We shall prove this by induction. In case of $n = 2$ our assertion is valid as X is uniformly non-square. Assume that our assertion holds true for any n sequences in B_X, $n \ge 2$. Let $\{x_1^{(k)}\}_k, \dots, \{x_{n+1}^{(k)}\}_k$ be $n+1$ sequences with nonzero terms in B_X. Suppose that $(\theta_j), (\theta'_j) \in B(\{x_1^{(k)}\}, \dots, \{x_{n+1}^{(k)}\})$. Then

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{n+1} \theta_j x_j^{(k)} \right\| = \lim_{k\to\infty} \left\| \sum_{j=1}^{n+1} \theta'_j x_j^{(k)} \right\| = n+1.$$

Denote by $B(n)$ the set $B(\{x_1^{(k)}\}, \dots, \{x_n^{(k)}\})$ for the first n sequences $\{x_1^{(k)}\}_k, \dots,$ $\{x_n^{(k)}\}_k$. Then

$$\begin{aligned} n \ge \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\| &\ge \lim_{k\to\infty} \left[\left\| \sum_{j=1}^{n+1} \theta_j x_j^{(k)} \right\| - \|\theta_{n+1} x_{n+1}^{(k)}\| \right] \\ &\ge (n+1) - 1 = n. \end{aligned}$$

Thus we have $\lim_{k\to\infty} \|\sum_{j=1}^n \theta_j x_j^{(k)}\| = n$. The same is true for $(\theta'_j)_{j=1}^n$. Therefore $(\theta_j)_{j=1}^n, (\theta'_j)_{j=1}^n \in B(n)$, which implies that $\theta_j = \theta'_j$ for all $1 \le j \le n$ by the induction hypothesis. If $\theta_{n+1} \neq \theta'_{n+1}$, we have $\lim_{k\to\infty} \| \sum_{j=1}^{n} \theta_j x_j^{(k)} \pm x_{n+1}^{(k)} \| = n+1$. Consequently

$$\begin{aligned}\lim_{k\to\infty} \left\| n\left(\frac{1}{n}\sum_{j=1}^{n}\theta_j x_j^{(k)}\right) \pm x_{n+1}^{(k)} \right\| &= \lim_{k\to\infty} \left\| \sum_{j=1}^{n}\theta_j x_j^{(k)} \pm x_{n+1}^{(k)} \right\| = n+1 \\ &= \lim_{k\to\infty} \left\| \sum_{j=1}^{n}\theta_j x_j^{(k)} \right\| + 1 \\ &= \lim_{k\to\infty} \left[n\left\| \frac{1}{n}\sum_{j=1}^{n}\theta_j x_j^{(k)} \right\| + \| \pm x_{n+1}^{(k)} \| \right]\end{aligned}$$

(note that $\lim_{k\to\infty} \|x_{n+1}^{(k)}\| = 1$), from which it follows by Lemma 3.3 that

$$\begin{aligned}\lim_{k\to\infty} \left\| \left(\frac{1}{n}\sum_{j=1}^{n}\theta_j x_j^{(k)}\right) \pm x_{n+1}^{(k)} \right\| &= \lim_{k\to\infty} \left[\left\| \frac{1}{n}\sum_{j=1}^{n}\theta_j x_j^{(k)} \right\| + \|x_{n+1}^{(k)}\| \right] \\ &= 2.\end{aligned}$$

This contradicts the uniform non-squareness of X. Therefore we obtain $(\theta_j) = (\theta'_j)$, i.e. $\mathrm{card}(B(\{x_1^{(k)}\}, \dots, \{x_{n+1}^{(k)}\})) \leq 1$, which completes the proof. □

Theorem 5.2. *Let $X_1, \dots, X_m$ be uniformly non-square Banach spaces. Then $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n if and only if $m < 2^{n-1}$.*

Proof. Assume first that $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n. Suppose that $m \geq 2^{n-1}$. Let $t = 2^{n-1}$. Then ℓ_∞^t is uniformly non-ℓ_1^n as ℓ_∞^t is imbedded into $(X_1 \oplus \cdots \oplus X_m)_\infty$. We recall *Rademacher matrices* $R_n = (r_{ij}^{(n)})$ ($2^n \times n$ matrices; see [17]):

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \quad (n = 1, 2, \dots) \tag{5.2}$$

Take $x_1 = (r_{11}^{(n)}, \dots, r_{t1}^{(n)}), \dots, x_n = (r_{1n}^{(n)}, \dots, r_{tn}^{(n)})$ from the unit sphere of ℓ_∞^t (we write n columns of the upper half submatrix of R_n in row). Let $\theta = (\theta_j)$ be arbitrary n signs with $\theta_1 = 1$. By the definition of R_n there exists an i_0, $1 \leq i_0 \leq m$, such that $\theta_j = r_{i_0 j}^{(n)}$ for all $1 \le j \le n$. Then we have

$$\begin{aligned}
\left\| \sum_{j=1}^n \theta_j x_j \right\|_\infty &= \left\| \sum_{j=1}^n \theta_j (r_{1j}^{(n)}, \dots, r_{i_0 j}^{(n)}, \dots, r_{tj}^{(n)}) \right\|_\infty \\
&= \left\| \left(\sum_{j=1}^n \theta_j r_{1j}^{(n)}, \dots, \sum_{j=1}^n \theta_j r_{i_0 j}^{(n)}, \dots, \sum_{j=1}^n \theta_j r_{tj}^{(n)} \right) \right\|_\infty \\
&= \max \left\{ \left| \sum_{j=1}^n \theta_j r_{1j}^{(n)} \right|, \dots, \left| \sum_{j=1}^n \theta_j r_{i_0 j}^{(n)} \right|, \dots, \left| \sum_{j=1}^n \theta_j r_{tj}^{(n)} \right| \right\} \\
&= \max \left\{ \left| \sum_{j=1}^n \theta_j r_{1j}^{(n)} \right|, \dots, n, \dots, \left| \sum_{j=1}^n \theta_j r_{tj}^{(n)} \right| \right\} = n
\end{aligned}$$

and also $\| \sum_{j=1}^n (-\theta_j) x_j \|_\infty = n$. Since θ is arbitary, ℓ_∞^t is not uniformly non-ℓ_1^n, a contradiction. Consequently, if $(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n, we have $m < 2^{n-1}$.

Conversely assume that $m < 2^{n-1}$. Let

$$K = \sup \left\{ \min_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|_\infty : x_1, \dots, x_n \in S_{(X_1 \oplus \cdots \oplus X_m)_\infty} \right\}. \tag{5.3}$$

Then there exist n sequences $\{x_1^{(k)}\}_k, \dots, \{x_n^{(k)}\}_k$ in the unit sphere of $(X_1 \oplus \cdots \oplus X_m)_\infty$ such that $K = \lim_{k\to\infty} \min_{\theta_j = \pm 1} \| \sum_{j=1}^n \theta_j x_j^{(k)} \|_\infty$. Put $x_1^{(k)} = (x_{11}^{(k)}, \dots, x_{m1}^{(k)})$, $\dots, x_n^{(k)} = (x_{1n}^{(k)}, \dots, x_{mn}^{(k)})$. By choosing subsequences if necessary, we may assume that $\lim_{k\to\infty} \| \sum_{j=1}^n \theta_j x_{ij}^{(k)} \|$ exists for each $1 \le i \le m$. Let $(\theta_j) \in B(\{x_1^{(k)}\}, \dots, \{x_n^{(k)}\})$. Then as

$$\begin{aligned}
n &= \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\|_\infty = \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j (x_{1j}^{(k)}, \dots, x_{mj}^{(k)}) \right\|_\infty \\
&= \lim_{k\to\infty} \left\| \left(\sum_{j=1}^n \theta_j x_{1j}^{(k)}, \dots, \sum_{j=1}^n \theta_j x_{mj}^{(k)} \right) \right\|_\infty \\
&= \max \left\{ \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_{1j}^{(k)} \right\|, \dots, \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_{mj}^{(k)} \right\| \right\},
\end{aligned}$$

there exists $1 \le i_0 \le m$ such that $\lim_{k\to\infty} \| \sum_{j=1}^n \theta_j x_{i_0 j}^{(k)} \| = n$. Let

$$B_i(n) := B(\{x_{i1}^{(k)}\}, \dots, \{x_{in}^{(k)}\}) = \left\{ (\theta_j) : \ \theta_1 = 1, \ \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta_j x_{ij}^{(k)} \right\| = n \right\} \tag{5.4}$$

for the space X_i and let $B = \bigcup_{i=1}^n B_i(n)$. Then by Proposition 5.1 $\mathrm{card}(B_i(n)) \le 1$ and hence $\mathrm{card}(B) \le m$. Therefore denoting by A the set of all n-tuples (θ_j) of signs with $\theta_1 = 1$, we have $\mathrm{card}(A) - \mathrm{card}(B) \geq 2^{n-1} - m > 0$. Consequently there exists $(\theta'_j) \in A$ such that $\lim_{k\to\infty} \| \sum_{j=1}^n \theta'_j x_{ij}^{(k)} \| < n$ for all $1 \leq i \leq m$, whence we have $\lim_{k\to\infty} \| \sum_{j=1}^n \theta'_j x_j^{(k)} \|_\infty < n$. Since

$$K = \lim_{k\to\infty} \min_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j^{(k)} \right\|_\infty \leq \lim_{k\to\infty} \left\| \sum_{j=1}^n \theta'_j x_j^{(k)} \right\|_\infty < n,$$

$(X_1 \oplus \cdots \oplus X_m)_\infty$ is uniformly non-ℓ_1^n. This completes the proof. □

As the case $m = 2$ in Theorem 5.2 we have the next result.

Corollary 5.3. *Let X and Y be uniformly non-square Banach spaces. Then $X \oplus_\infty Y$ is uniformly non-ℓ_1^n if and only if $n \geq 3$.*

This is equivalent to:

Corollary 5.3 bis. *Let X and Y be uniformly non-square Banach spaces. Then $X \oplus_\infty Y$ is uniformly non-ℓ_1^3.*

According to Theorem 4.5 the ℓ_1-sum $X \oplus_1 Y$ is uniformly non-ℓ_1^3 if and only if X and Y are uniformly non-square, while the converse assertion of Corollary 5.3 bis for the ℓ_∞-sum $X \oplus_\infty Y$ is not true as we shall see in Remark 5.5 below. Instead we shall obtain the following result which is interesting in contrast with Theorem 4.5.

Theorem 5.4. *Let X, Y and Z be Banach spaces. Then the following are equivalent.*

(i) *$(X \oplus Y \oplus Z)_\infty$ is uniformly non-ℓ_1^3.*

(ii) *X, Y and Z are uniformly non-square.*

Proof. The implication (ii) ⇒ (i) is a consequence of Theorem 5.2. We shall prove that (i) implies (ii). Assume that $(X \oplus Y \oplus Z)_\infty$ is uniformly non-ℓ_1^3 and the assertion (ii) does not hold. We may assume that X is not uniformly non-square without loss of generality. Let $W = Y \oplus_\infty Z$. Then W is not uniformly non-square by Corollary 3.7. Therefore there exist $\{x_1^{(k)}\}_k, \{x_2^{(k)}\}_k \subset S_X$ and $\{w_1^{(k)}\}_k, \{w_2^{(k)}\}_k \subset S_W$ such that

$$\lim_{k\to\infty} \|x_1^{(k)} \pm x_2^{(k)}\| = 2 \tag{5.5}$$

and

$$\lim_{k\to\infty} \|w_1^{(k)} \pm w_2^{(k)}\| = 2, \tag{5.6}$$

respectively. Then $(x_1^{(k)}, w_1^{(k)}),\ (x_2^{(k)}, w_2^{(k)}),\ (x_2^{(k)}, -w_2^{(k)}) \in S_{X\oplus_\infty W}$. Since

$$\begin{aligned} & \|(x_1^{(k)}, w_1^{(k)}) \pm (x_2^{(k)}, w_2^{(k)}) + (x_2^{(k)}, -w_2^{(k)})\|_\infty \\ = \ & \|(x_1^{(k)} \pm x_2^{(k)} + x_2^{(k)}, w_1^{(k)} \pm w_2^{(k)} - w_2^{(k)})\|_\infty, \end{aligned}$$

owing to Lemma 3.3 with (5.5) and (5.6) we have

$$\|(x_1^{(k)}, w_1^{(k)}) + (x_2^{(k)}, w_2^{(k)}) + (x_2^{(k)}, -w_2^{(k)})\|_\infty = \|(\|x_1^{(k)} + 2x_2^{(k)}\|, \|w_1^{(k)}\|)\|_\infty \to 3$$

and

$$\|(x_1^{(k)}, w_1^{(k)}) - (x_2^{(k)}, w_2^{(k)}) + (x_2^{(k)}, -w_2^{(k)})\|_\infty = \|(\|x_1^{(k)}\|, \|w_1^{(k)} - 2w_1^{(k)}\|)\|_\infty \to 3$$

as $k \to \infty$. In the same way

$$\|(x_1^{(k)}, w_1^{(k)}) \pm (x_2^{(k)}, w_2^{(k)}) - (x_2^{(k)}, -w_2^{(k)})\|_\infty \to 3 \text{ as } k \to \infty.$$

Consequently we obtain that $(X \oplus Y \oplus Z)_\infty = X \oplus_\infty W$ is not uniformly non-ℓ_1^3, a contradiction, which implies that X is uniformly non-square. This completes the proof. □

Remark 5.5. Let X, Y and Z be uniformly non-square Banach spaces and let $W = Y \oplus_\infty Z$. Then $X \oplus_\infty W$ is uniformly non-ℓ_1^3 by Theorem 5.4, whereas W is not uniformly non-square. Thus the converse assertion of Corollary 5.3 bis is not true.

We shall close this section with the following extremely useful result to construct various examples.

Corollary 5.6. *ℓ_∞^m is uniformly non-ℓ_1^n if and only if $m < 2^{n-1}$.*

6. Examples and problems

In Theorem 3.5 we have seen that *if $X \oplus_\psi Y$ is uniformly non-ℓ_1^n and if neither X nor Y is uniformly non-ℓ_1^{n-1}, then $\psi \neq \psi_1, \psi_\infty$.* We shall give some examples below which show that we cannot remove the assumption that X and Y are not uniformly non-ℓ_1^{n-1}.

Examples. (i) Let $X = \ell_\infty^3, Y = \ell_\infty^4$ and $\psi = \psi_\infty$. Then $X \oplus_\infty Y = \ell_\infty^7$. Owing to Corollary 5.6, $X \oplus_\infty Y$ is uniformly non-ℓ_1^4, whereas X is uniformly non-ℓ_1^3 and Y is not uniformly non-ℓ_1^3.

(ii) Let $X = \ell_\infty^2, Y = \ell_\infty^3$ and $\psi = \psi_1$. Then by Corollary 5.6 both of X and Y are uniformly non-ℓ_1^3. By Theorem 4.2 (let $n_1 = n_2 = 2$), $X \oplus_1 Y$ is uniformly non-ℓ_1^5. whereas both of X and Y are uniformly non-ℓ_1^4. (Recall that Corollary 4.4 says that for general Banach spaces X and Y, if $X \oplus_1 Y$ is uniformly non-ℓ_1^n, then X and Y are uniformly non-ℓ_1^{n-1}.)

Problem 6.1. Characterize the uniform non-ℓ_1^n-ness or the uniform non-squareness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_m)_\psi$ (cf. [18, 32, 27]).

Problem 6.2. Characterize the uniform non-ℓ_1^n-ness of $(X_1 \oplus \cdots \oplus X_m)_\infty$ without the assumption that $X, \ldots, X_m$ are uniformly non-square.

References

[1] B. Beauzamy, *Introduction to Banach Spaces and their Geometry*, 2nd ed., North-Holland, 1985.

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10**, 1973.

[3] D. R. Brown, *B-convexity in Banach spaces*, Doctoral dissertation, Ohaio State University, 1970.

[4] M. Denker and H. Hudzik, *Uniformly non-$l_n^{(1)}$ Musielak-Orlicz sequence spaces*, Proc. Indian Acad. Sci. Math. Sci. **101** (1991), 71–86
[5] S. Dhompongsa, A. Kaewkhao and S. Saejung, *Uniform smoothness and U-convexity of ψ-direct sums*, J. Nonlinear Convex Anal. **6** (2005), 327–338.
[6] S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao, *Fixed point property of direct sums*, Nonlinear Anal. **63** (2005), e2177-e2188.
[7] P. N. Dowling, *On convexity properties of ψ-direct sums of Banach spaces*, J. Math. Anal. Appl. **288** (2003), 540–543.
[8] P. N. Dowling and B. Turett, *Complex strict convexity of absolute norms on $\mathbb{C}^n$ and direct sums of Banach spaces*, J. Math. Anal. Appl. **323** (2006), 930–937.
[9] M. Fujii, M. Kato, K.-S. Saito and T. Tamura, *Sharp mean triangle inequality*, to appear in Math. Inequal. Appl.
[10] D. P. Giesy, *On a convexity condition in normed linear spaces*, Trans. Amer. Math. Soc. **125** (1966), 114–146.
[11] D. P. Giesy and R. C. James, *Uniformly non-$l_n^{(1)}$ and B-convex spaces*, Studia Math. **48** (1973), 61–69.
[12] R. Grząślewicz, H. Hudzik and W. Orlicz, *Uniformly non-$\ell_n^{(1)}$ property in some Orlicz spaces*, Bull. Acad. Polon. Sci. Math. **34** (1986), 161–171.
[13] H. Hudzik, *Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm*, Studia Math. **81** (1985), 271–284.
[14] H. Hudzik, A. Kamińska and W. Kurc, *Uniformly non-$\ell_n^{(1)}$ Musielak-Orlicz spaces*, Bull. Acad. Polon. Sci. Math. **35** (1987), 7–8.
[15] C. James, *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542–550.
[16] M. Kato, *On Lorentz spaces $\ell_{p,q}\{E\}$*, Hiroshima Math. J. **6** (1976), 73–93.
[17] M. Kato, L. E. Persson and Y. Takahashi, *Clarkson type inequalities and their relations to the concepts of type and cotype*, Collect. Math. **51** (2000), 327–346.
[18] M. Kato, K.-S. Saito and T. Tamura, *On ψ-direct sums of Banach spaces and convexity*, J. Aust. Math. Soc. **75** (2003), 413–422.
[19] M. Kato, K.-S. Saito and T. Tamura, *Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, Math. Inequal. Appl. **7** (2004), 429–437.
[20] M. Kato, K.-S. Saito and T. Tamura, *Sharp triangle inequality and its reverse in Banach spaces*, Math. Inequal. Appl. **10** (2007), 451–460.
[21] M. Kato and T. Tamura, *Weak nearly uniform smoothness and worth property of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, Comment. Math. Prace Mat. **46** (2006), 113–129.
[22] M. Kato and T. Tamura, *Uniform non-ℓ_1^n-ness of ℓ_1-sums of Banach spaces*, Comment. Math. Prace Mat. **47** (2007), 161–169.
[23] M. Kato and T. Tamura, *Uniform non-ℓ_1^n-ness of ℓ_∞-sums of Banach spaces*, Comment. Math. **49** (2009), 179–187.
[24] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, 1998.
[25] K.-I. Mitani and K.-S. Saito, *A note on geometrical properties of Banach spaces using ψ-direct sums*, J. Math. Anal. Appl. **327** (2007), 898–907.
[26] K.-I. Mitani, K.-S. Saito M. Kato and T. Tamura, *On sharp triangle inequalities in Banach spaces*, J. Math. Anal. Appl. **336** (2007), 1178–1186.
[27] K. Mitani, K.-S. Saito and T. Suzuki, *Smoothness of absolute norms on $\mathbb{C}^n$*, J. Convex Anal. **10** (2003), 89–107.
[28] K.-I. Mitani, S. Oshiro and K.-S. Saito, *Smoothness of ψ-direct sums of Banach spaces*, Math. Inequal. Appl. **8** (2005), 147–157.
[29] S. Saejung, *Extreme points, smooth points and noncreasiness of ψ-direct sum of Banach spaces*, Nonlinear Anal. **66** (2007), 2459-2469.
[30] K.-S. Saito and M. Kato, *Uniform convexity of ψ-direct sums of Banach spaces*, J. Math. Anal. Appl. **277** (2003), 1–11
[31] K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$*, J. Math. Anal. Appl. **244** (2000), 515–532.

[32] K.-S. Saito, M. Kato and Y. Takahashi, *On absolute norms on* $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879–905.

[33] Y. Takahashi, M. Kato and K.-S. Saito, *Strict convexity of absolute norms on* $\mathbb{C}^2$ *and direct sums of Banach spaces*, J. Inequal. Appl. **7** (2002), 179–186.

[34] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.

Manuscript received November 9, 2009
revised November 11, 2009

Mikio Kato
Department of Basic Sciences, Kyushu Institute of Technology, Kitakyushu 804-8550, Japan
E-mail address: katom@tobata.isc.kyutech.ac.jp

Kichi-Suke Saito
Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
E-mail address: saito@math.sc.niigata-u.ac.jp

Takayuki Tamura
Graduate School of Humanities and Social Sciences, Chiba University, Chiba 263-8522, Japan
E-mail address: tamura@le.chiba-u.ac.jp