Journal of Nonlinear and Convex Analysis
Volume 11, Number 1, 2010, 101–114

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A QUASI-FIXED POLYNOMIAL PROBLEM FOR A POLYNOMIAL FUNCTION

HANG-CHIN LAI* AND YI-CHOU CHEN

Abstract. Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a real-valued polynomial function of the form

$$F(\overline{x}, y) = \sum_{i=0}^{s} f_i(\overline{x}) y^i, \qquad \text{the degree of } y \text{ in } F(\overline{x}, y) = s \geq 1.$$

Given an irreducible real-valued polynomial function $p(\overline{x})$, $\overline{x} \in \mathbb{R}^n$ and a non-negative integer m, we will find a polynomial function $y(\overline{x}) \in \mathbb{R}[\overline{x}]$ to satisfy the following equation:

$$(*) \qquad F(\overline{x}, y(\overline{x})) = ap^m(\overline{x})$$

for some constant $a \in \mathbb{R}$. The constant a is dependent on the solution $y(\overline{x})$, namely a quasi-fixed (polynomial) solution of the polynomial equation $(*)$.

In this paper, we prove that (i) If the equation $(*)$ has infinitely many quasi-fixed solutions, then the leading coefficient of y in $F(\overline{x}, y)$ must be of the form: $f_s(\overline{x}) = cp^k(\overline{x})$ for some $c \in \mathbb{R}$, $k \in \mathbb{N}$, and the solutions of $(*)$ are $y_\lambda(\overline{x}) = -f_{s-1}(\overline{x})/sf_s(\overline{x}) + \lambda p^t(\overline{x}), \lambda \in \mathbb{R}$ and $t = (m-k)/s$. (ii) If the equation $(*)$ has finitely many quasi-fixed solutions, the number of all quasi-fixed solutions does not exceed the number $s+2$.

1. Introduction and Preliminaries

In 1987, Lenstra [1] proved that for a polynomial function $F(x, y) \in \mathbb{Q}(\alpha)[x, y]$, there exists a polynomial function $y(x) \in \mathbb{Q}(\alpha)[x]$ to satisfy the equation

$$F(x, y(x)) = 0.$$

This polynomial equation can be derived to a polynomial $y(x)$ to satisfy the equation

$$F(x, y(x)) = x.$$

This means that x is a fixed point of the polynomial function $F(x, y(x))$.
In other words, one will search a polynomial $y = y(x)$ such that

$$(1.1) \qquad F(x, y(x)) = x, \qquad x \in \mathbb{Q}(\alpha) \quad \text{(rational algebraic number)}.$$

Recently, Tung [2] studied this problem by considering:

$$(1.2) \qquad F(x, y(x)) = cx^m, \qquad x \in \mathbb{K} \quad \text{(field)}, \quad m \in \mathbb{N}$$

where $F(x, y) \in \mathbb{K}[x, y]$, c is a constant depending on $y(x)$ and $m \in \mathbb{N}$ is a given positive integer. From (1.1) and (1.2), we are motivated to consider an irreducible

2000 *Mathematics Subject Classification.* Primary 47H10.
Key words and phrases. Quasi-fixed solution, quasi-fixed value.
This research was partly supported by NSC, Taiwan.
*Corresponding author.

polynomial $p(x)$ to instead of x in (1.2). That is a question to ask whether we could solve a polynomial function $y = y(x)$ to satisfy the polynomial equation:

$$F(x, y(x)) = ap^m(x) \qquad x \in \mathbb{R} \quad \text{(real number)} \tag{1.3}$$

where a is a real constant depending on the solution $y(x)$.

Remark. This variable x may be considered in a unique factorization domain of algebra terminology.

Now we let $F(x, y)$ be a polynomial function, $p : \mathbb{R} \to \mathbb{R}$ an irreducible polynomial in $\mathbb{R}$ and $m \in \mathbb{N}$.

Definition 1.1. A polynomial function $y = y(x)$ satisfying equation (1.3) is called a quasi-fixed (polynomial) solution corresponding to some real number a. This a is called a quasi-fixed value corresponding to the polynomial solutions $y = y(x)$.

Since there may have many solutions corresponding to the number a, for convenience, we use the following notations to represent different situations:

(1) $\mathbf{Qs}_F\mathbf{(a)}$, the set of all **q**uasi-fixed **s**olutions $y(x)$ corresponding to a fixed quasi-fixed value a.
(2) $\mathbf{Qs}_F$, the set of all **q**uasi-fixed **s**olutions satisfying equation (1.3).
(3) $\mathbf{Qv}_F$, the set of all **q**uasi-fixed **v**alues satisfying equation (1.3).

Evidently,

$$\mathbf{Qs}_F = \bigcup_{a \in \mathrm{Qv}_F} \Big(\mathbf{Qs}_F\mathbf{(a)}\Big) \text{ and } \mathbf{Qs}_F\mathbf{(a)} \bigcap \mathbf{Qs}_F\mathbf{(b)} = \emptyset \text{ for any } a \neq b \text{ in } \mathbf{Qv}_F.$$

In this paper, we consider a more general quasi-fixed polynomial problem in which the $x \in \mathbb{R}$ is replaced by a vector $\overline{x} \in \mathbb{R}^n$ and the polynomial $p(x)$ is replaced by the irreducible polynomial function $p(\overline{x})$. Then we restate the equation (1.3) as the following equation :

$$F(\overline{x}, y) = ap^m(\overline{x}). \tag{1.4}$$

It is a new developed fixed point-like problem. We call the polynomial solution $y = y(\overline{x})$ for equation (1.4) as a quasi fixed (polynomial) solution. Precisely, we rewrite as Definition 1.1 as following.

Definition 1.2. If the equation (1.4) is solvable and $y(\overline{x})$ is a polynomial solution of equation (1.4), then $y(\overline{x})$ is called a **quasi-fixed solution** of the the polynomial function $F(\overline{x}, y)$ given in (1.4) and call the constant $a \in \mathbb{R}$ a **quasi-fixed value** corresponding to some solutions $y(\overline{x})$ (not only one solution).

The number of all solutions to equation (1.4) may exist infinitely many, or finitely many, or not solvable.

In Section 2, we derive some properties of quasi-fixed solutions of $F(\overline{x}, y)$. If the equation (1.4) exists infinitely many or finitely many quasi-fixed solutions, the relative properties will be described in Section 3.

Throughout the paper, we denote the polynomial function as the form:

$$\begin{aligned} F(\overline{x}, y) &= f_s(\overline{x})y^s + f_{s-1}(\overline{x})y^{s-1} + \cdots + f_1(\overline{x})y + f_0(\overline{x}) \\ &= \sum_{i=0}^{s} f_i(\overline{x})y^i. \end{aligned} \tag{1.5}$$

2. Some Lemmas

For convenience, we explain some interesting properties of quasi-fixed polynomial solutions as the following lemmas. Throughout this paper, $p(\overline{x})$ is an irreducible polynomial.

Lemma 2.1. *Let $y_1(\overline{x}) \in \mathbf{Qs}_F(a)$, $y_2(\overline{x}) \in \mathbf{Qs}_F(b)$, $a \neq b$ in $\mathbf{Qv}_F$. Then*

$$y_1(\overline{x}) - y_2(\overline{x}) = dp^t(\overline{x}) \quad \textit{for some } d \in \mathbb{R},\ t \in \mathbb{N}.$$

Proof. Since $y_1(\overline{x}) \neq y_2(\overline{x})$ in $\mathbf{Qs}_F$ corresponds to $a \neq b$ in $\mathbf{Qv}_F$ respectively, thus,

$$F(\overline{x}, y_1(\overline{x})) = ap^m(\overline{x}) \quad \text{and} \quad F(\overline{x}, y_2(\overline{x})) = bp^m(\overline{x}).$$

Subtracting the above two equations and using binomial formula, it yields

$$\begin{aligned}
&F(\overline{x}, y_1(\overline{x})) - F(\overline{x}, y_2(\overline{x}))\\
&= f_s(\overline{x})[y_1^s(\overline{x}) - y_2^s(\overline{x})] + f_{s-1}(\overline{x})[y_1^{s-1}(\overline{x}) - y_2^{s-1}(\overline{x})] + \cdots + f_1(\overline{x})[y_1(\overline{x}) - y_2(\overline{x})]\\
&= [y_1(\overline{x}) - y_2(\overline{x})][f_s(\overline{x})G_s(y_1(\overline{x}), y_2(\overline{x})) + f_{s-1}(\overline{x})G_{s-1}(y_1(\overline{x}), y_2(\overline{x})) + \cdots + f_1(\overline{x})]\\
&= [y_1(\overline{x}) - y_2(\overline{x})]Q(\overline{x}, y_1(\overline{x}), y_2(\overline{x}))\\
&= (a-b)p^m(\overline{x}),
\end{aligned}$$

where

$$G_j(y_1(\overline{x}), y_2(\overline{x})) = y_1^{j-1}(\overline{x}) + y_1^{j-2}(\overline{x})y_2(\overline{x}) + \cdots + y_2^{j-1}(\overline{x}), \qquad \text{for } j = s, s-1, \ldots, 2, 1.$$

Evidently, the factor $y_1(\overline{x}) - y_2(\overline{x})$ is divisible to the term $(a-b)p^m(\overline{x})$. Since $a \neq b$,

$$y_1(\overline{x}) - y_2(\overline{x}) = dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}, \text{ and } t \leq m \in \mathbb{N}.$$

This completes the proof. □

If two distinct quasi-fixed solutions corresponding the same value a in $\mathbf{Qv}_F$, then we have the following Lemma :

Lemma 2.2. *Let $y_1(\overline{x}) \neq y_2(\overline{x}) \in \mathbf{Qs}_F(a)$ and $y_1(\overline{x}) - y_2(\overline{x}) = dp^t(\overline{x})$ for some $d \in \mathbb{R}$, $t \in \mathbb{N}$. Then any quasi-fixed solution $y(\overline{x}) \notin \mathbf{Qs}_F(a)$ can be represented by*

$$y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x}) \qquad \textit{for some } d \in \mathbb{R}.$$

(This power t of $p^t(\overline{x})$ is independent of the choice $y(\overline{x}) \notin \mathbf{Qs}_F(a)$.)

Proof. If $y(\overline{x}) \notin \mathbf{Qs}_F(a)$, by Lemma 2.1, there exist some d_1, $d_2 \in \mathbb{R}$ such that

$$\begin{aligned}
y(\overline{x}) - y_1(\overline{x}) &= d_1 p^{t_1}(\overline{x})\\
\text{and} \quad y(\overline{x}) - y_2(\overline{x}) &= d_2 p^{t_2}(\overline{x}).
\end{aligned}$$

Then

$$\begin{aligned}
d_1 p^{t_1}(\overline{x}) - d_2 p^{t_2}(\overline{x}) &= (y(\overline{x}) - y_1(\overline{x})) - (y(\overline{x}) - y_2(\overline{x}))\\
&= (y_2(\overline{x}) - y_1(\overline{x})) = -d_{12}p^t(\overline{x}).
\end{aligned}$$

Since $p(\overline{x})$ is irreducible, $t_1 = t_2 = t$, . It follows that

$$y(\overline{x}) - y_1(\overline{x}) = dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

□

Lemma 2.3. *Let $a \neq b$ in $\mathbf{Qv}_F$, and assume that*

(i) *$y_1(\overline{x}) \neq y_2(\overline{x})$ in $\mathbf{Qs}_F(a)$ with $y_1(\overline{x}) - y_2(\overline{x}) = c_{12}p^t(\overline{x})$ for some $c_{12} \in \mathbb{R}$.*
(ii) *another two solutions $h_1(\overline{x}) \neq h_2(\overline{x})$ in $\mathbf{Qs}_F(b)$ with $h_1(\overline{x}) - h_2(\overline{x}) = d_{12}p^t(\overline{x})$ for some $d_{12} \in \mathbb{R}$.*

Then any $y(\overline{x}) \in \mathbf{Qs}_F$ can be represented by

$$y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x}) \quad \textit{for some } d \in \mathbb{R}.$$

Proof. 1° If $y(\overline{x}) \notin \mathbf{Qs}_F(a)$, by Lemma 2.2, for $y_1(\overline{x}) \in \mathbf{Qs}_F(a)$, we have

$$y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

2° If $y(\overline{x}) \in \mathbf{Qs}_F(a)$, then $y(\overline{x}) \notin \mathbf{Qs}_F(b)$. Since $h_1(\overline{x}) \in \mathbf{Qs}_F(b)$, the Lemma 2.2 yields

$$y(\overline{x}) = h_1(\overline{x}) + dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}. \tag{2.1}$$

Since $y_1(\overline{x}) \notin \mathbf{Qs}_F(b)$, we also have

$$y_1(\overline{x}) = h_1(\overline{x}) + d_1p^t(\overline{x}) \quad \text{for some } d_1 \in \mathbb{R}. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$\begin{aligned} y(\overline{x}) &= h_1(\overline{x}) + dp^t(\overline{x}) \\ &= (y_1(\overline{x}) - d_1p^t(\overline{x})) + dp^t(\overline{x}) \\ &= y_1(\overline{x}) + (d - d_1)p^t(\overline{x}). \end{aligned}$$

Hence any $y(\overline{x}) \in \mathbf{Qs}_F$ can be represented by

$$y(\overline{x}) = y_1(\overline{x}) + \widetilde{d}p^t(\overline{x}), \quad \text{for } d = d - d_1 \in \mathbb{R}.$$

This completes the proof. □

Lemma 2.4. *Suppose that the cardinal number $|\mathbf{Qv}_F| \geq 3$ and for any $y(\overline{x}) \neq h(\overline{x})$ in $\mathbf{Qs}_F$. Then there exists a fixed $t \in \mathbb{N}$ such that*

$$y(\overline{x}) - h(\overline{x}) = dp^t(\overline{x}) \quad \textit{for some} d \in \mathbb{R}.$$

Proof. Since $|\mathbf{Qv}_F| \geq 3$, we may choose any three distinct quasi-fixed values a_1, a_2 and a_3 in $\mathbf{Qv}_F$, and three quasi-fixed solutions $y_1(\overline{x})$, $y_2(\overline{x})$ and $y_3(\overline{x})$ in $\mathbf{Qs}_F$ such that

$$F(\overline{x}, y_i(\overline{x})) = a_ip^m(\overline{x}), \quad \imath = 1,\ 2,\ 3.$$

According to the assumption of Lemma 2.1, we have

$$\begin{aligned} y_1(\overline{x}) - y_2(\overline{x}) &= d_{12}(p(\overline{x}))^{t_{12}}, \\ y_1(\overline{x}) - y_3(\overline{x}) &= d_{13}(p(\overline{x}))^{t_{13}}, \\ y_2(\overline{x}) - y_3(\overline{x}) &= d_{23}(p(\overline{x}))^{t_{23}}, \end{aligned}$$

for some d_{12}, d_{13}, $d_{23} \in \mathbb{R}$ and t_{12}, t_{13}, $t_{23} \in \mathbb{N}$. Then

$$\begin{aligned} d_{13}(p(\overline{x}))^{t_{13}} &= y_1(\overline{x}) - y_3(\overline{x}) \\ &= \Big(y_1(\overline{x}) - y_2(\overline{x})\Big) + \Big(y_2(\overline{x}) - y_3(\overline{x})\Big) \\ &= d_{12}(p(\overline{x}))^{t_{12}} + d_{23}(p(\overline{x}))^{t_{23}}. \end{aligned}$$

This reduces $t_{12} = t_{13} = t_{23} = t$.
Now for any $y(\overline{x}) \in \mathbf{Qs}_F$, it corresponds a real number $a \in \mathbf{Qv}_F$ such that

$$F(\overline{x}, y(\overline{x})) = ap^m(\overline{x}).$$

Since $|\mathbf{Qv}_F| \geq 3$, we may suppose that a_1, a_2 and a_3 are distinct. Thus, at least two of the three values are distinct to a, say $a_1 \neq a$ and $a_2 \neq a$. From Lemma 2.1, we have

$$y(\overline{x}) - y_1(\overline{x}) = d_1 p^{t_1}(\overline{x}) \quad \text{and} \quad y(\overline{x}) - y_2(\overline{x}) = d_2 p^{t_2}(\overline{x}).$$

Thus

$$\begin{aligned} d_{12}(p(\overline{x}))^{t_{12}} = y_1(\overline{x}) - y_2(\overline{x}) &= \Big(y_1(\overline{x}) - y(\overline{x})\Big) + \Big(y(\overline{x}) - y_2(\overline{x})\Big) \\ &= -d_1(p(\overline{x}))^{t_1} + d_2(p(\overline{x}))^{t_2}. \end{aligned}$$

This reduces $t_1 = t_2 = t$. It follows that

$$y(\overline{x}) = y_1(\overline{x}) + d_1 p^t(\overline{x}).$$

Similarly, any $h(\overline{x}) \in \mathbf{Qs}_F$ can be represented by

$$h(\overline{x}) = y_1(\overline{x}) + \widetilde{d_1} p^t(\overline{x}) \quad \text{for some } \widetilde{d_1} \in \mathbb{R}.$$

Consequently,

$$y(\overline{x}) - h(\overline{x}) = (d_1 - \widetilde{d_1}) p^t(\overline{x}) = dp^t(\overline{x}) \quad \text{with } d = d_1 - \widetilde{d_1}.$$

Hence

$$y(\overline{x}) = h(\overline{x}) + dp^t(\overline{x}).$$

□

Lemma 2.5. *Suppose that $a \neq b$ in $\mathbf{Qv}_F$. If there exist two distinct solutions $y_1(\overline{x})$, $y_2(\overline{x}) \in \mathbf{Qs}_F(a)$ and another two distinct solutions $h_1(\overline{x})$, $h_2(\overline{x}) \in \mathbf{Qs}_F(b)$, then there exists $t \in \mathbb{N}$ such that for any $y(\overline{x}) \in \mathbf{Qs}_F$, either $y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x})$ for some $d \in \mathbb{R}$ or $y_1(\overline{x}) + y_2(\overline{x}) = h_1(\overline{x}) + h_2(\overline{x})$.*

Proof. Let $i, j \in \{1, 2\}$ and by Lemma 2.1, the difference $y_i(\overline{x}) - h_j(\overline{x})$ be of the following forms

$$\begin{aligned} y_1(\overline{x}) - h_1(\overline{x}) &= d_{11}(p(\overline{x}))^{t_{11}}, \\ y_1(\overline{x}) - h_2(\overline{x}) &= d_{12}(p(\overline{x}))^{t_{12}}, \\ y_2(\overline{x}) - h_1(\overline{x}) &= d_{21}(p(\overline{x}))^{t_{21}}, \\ y_2(\overline{x}) - h_2(\overline{x}) &= d_{22}(p(\overline{x}))^{t_{22}}, \end{aligned}$$

for some d_{11}, d_{12}, d_{21}, $d_{22} \in \mathbb{R}$ and t_{11}, t_{12}, t_{21}, $t_{22} \in \mathbb{N}$. Then

$$d_{11}(p(\overline{x}))^{t_{11}} - d_{21}(p(\overline{x}))^{t_{21}} = y_1(\overline{x}) - y_2(\overline{x}) \;=\; d_{12}(p(\overline{x}))^{t_{12}} - d_{22}(p(\overline{x}))^{t_{22}}.$$

1.° If $t_{11} \neq t_{12}$, then $t_{11} = t_{22}$ and $t_{12} = t_{21}$. Therefore, $d_{11} = -d_{22}$ and $d_{21} = -d_{12}$. Thus

$$\begin{aligned}&2\Big(y_1(\overline{x}) + y_2(\overline{x})\Big) - 2\Big(h_1(\overline{x}) + h_2(\overline{x})\Big)\\ =&\Big(y_1(\overline{x}) - h_1(\overline{x})\Big) + \Big(y_2(\overline{x}) - h_2(\overline{x})\Big) + \Big(y_2(\overline{x}) - h_1(\overline{x})\Big) + \Big(y_1(\overline{x}) - h_2(\overline{x})\Big)\\ =&\Big(d_{11} + d_{22}\Big)(p(\overline{x}))^{t_{11}} + \Big(d_{21} + d_{12}\Big)(p(\overline{x}))^{t_{12}} = 0.\end{aligned}$$

It follows that

$$y_1(\overline{x}) + y_2(\overline{x}) = h_1(\overline{x}) + h_2(\overline{x}).$$

2.° If $t_{11} = t_{12}$, then $t_{11} = t_{12} = t_{21} = t_{22} = t$. By Lemma 2.3, any $y(\overline{x}) \in \mathbf{Qs}_F$ can be presented by

$$y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

□

If there are three distinct solutions in $\mathbf{Qs}_F(a)$ and another two solutions in $\mathbf{Qs}_F(b)$, then how about the results in Lemma 2.5?

To answer this question, we state a corollary as follows.

Corollary 2.6. *Suppose that $a \neq b$ in $\mathbf{Qv}_F$. If there are three distinct solutions $y_1(\overline{x})$, $y_2(\overline{x})$, $y_3(\overline{x}) \in \mathbf{Qs}_F(a)$ and other two distinct solutions $h_1(\overline{x})$, $h_2(\overline{x}) \in \mathbf{Qs}_F(b)$. Then there exists an integer $t \in \mathbb{N}$ such that any $y(\overline{x}) \in \mathbf{Qs}_F$ can be represented by*

$$y(\overline{x}) = y_1(\overline{x}) + dp^t(\overline{x}) \quad \textit{for some } d \in \mathbb{R}.$$

(The power t of $p^t(\overline{x})$ is independent to the choice of $y(\overline{x}) \in \mathbf{Qs}_F$.)

Proof. By Lemma 2.5, if $y_1(\overline{x}) + y_2(\overline{x}) = h_1(\overline{x}) + h_2(\overline{x})$ and $y_1(\overline{x}) + y_3(\overline{x}) = h_1(\overline{x}) + h_2(\overline{x})$, then $y_2(\overline{x}) = y_3(\overline{x})$. This contradicts three distinct solutions in $\mathbf{Qs}_F(a)$. Hence for any $y(\overline{x})$ in $\mathbf{Qs}_F$, it is represented by:

$$y(\overline{x}) - h(\overline{x}) = dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

□

By the above preparations, the polynomial function y in $F(\overline{x}, y)$ with $\deg_y F = 1$ as the form

$$F(\overline{x}, y) = f_1(\overline{x})y + f_0(\overline{x}),$$

then the solutions in the problem:

$$F(\overline{x}, y) = ap^m(\overline{x})$$

have the following essential theorem.

Theorem 2.7. *Let $F(\overline{x}, y)$ be a polynomial function with* $\deg_y F = 1$ *as the form $F(\overline{x}, y) = f_1(\overline{x})y + f_0(\overline{x}) = ap^m(\overline{x})$ for some $a \in \mathbb{R}$.* (The irreducible polynomial function $p(\overline{x})$ and $m \in \mathbb{N}$ are given.) *Suppose that the cardinal number* $|\mathbf{Qs}_F| \geq 2(=\deg_y F + 1)$. *Then*

(i) $f_1(\overline{x}) = cp^k(\overline{x})$ *for some* $c \in \mathbb{R}$, $k \in \mathbb{N}$.

(ii) *any solution of equation (1.4) is of the form :*

$$y(\overline{x}) = -f_0(\overline{x})/f_1(\overline{x}) + \lambda p^{m-k}(\overline{x}) \quad \textit{for some } \lambda \in \mathbb{R} \quad \textit{and}$$

(iii) *the cardinal number* $|\mathbf{Qs}_F| = \infty$.

Proof. Since $|\mathbf{Qs}_F| \geq 2$, we see that there are two distinct solutions $y_1(\overline{x})$, $y_2(\overline{x})$ in $\mathbf{Qs}_F$ such that

$$F(\overline{x}, y_1(\overline{x})) = ap^m(\overline{x})$$
$$\text{and} \quad F(\overline{x}, y_2(\overline{x})) = bp^m(\overline{x}).$$

(i) According to the fundamental theorem of algebra, since $\deg_y F = 1$, it may have $a \neq b$. Thus,

$$F(x, y_1(\overline{x})) = f_1(\overline{x})y_1(\overline{x}) + f_0(\overline{x}) = ap^m(\overline{x}) \tag{2.3}$$
$$F(x, y_2(\overline{x})) = f_1(\overline{x})y_2(\overline{x}) + f_0(\overline{x}) = bp^m(\overline{x}). \tag{2.4}$$

By (2.3) – (2.4), we get

$$f_1(\overline{x})(y_1(\overline{x}) - y_2(\overline{x})) = (a-b)p^m(\overline{x}).$$

It follows that $f_1(\overline{x})$ must have form

$$f_1(\overline{x}) = cp^k(\overline{x}) \tag{2.5}$$
$$\text{and} \quad y_1 - y_2 = \frac{a-b}{c} p^{m-k}(\overline{x})$$

for some $c \neq 0$ in $\mathbb{R}$ and $k \leq m$ in $\mathbb{N}$.

(ii) Since

$$\begin{aligned} f_0(\overline{x}) &= ap^m(\overline{x}) - f_1(\overline{x})y_1(\overline{x}) &(2.6)\\ &= ap^m(\overline{x}) - cp^k(\overline{x})y_1(\overline{x}) \\ &= p^k(\overline{x})\big(ap^{m-k}(\overline{x}) - cy_1(\overline{x})\big) \\ \text{by (2.5)} \quad &= f_1(\overline{x})\big(a/c\, p^{m-k}(\overline{x}) - y_1(\overline{x})\big). &(2.7)\end{aligned}$$

Thus (2.7) implies $f_1(\overline{x}) \mid f_0(\overline{x})$, and (2.6) can be written as

$$y_1(\overline{x}) = \frac{ap^m(\overline{x}) - f_0(\overline{x})}{f_1(\overline{x})}. \tag{2.8}$$

Moreover, we derive

$$\begin{aligned} F(\overline{x}, y) &= f_1(\overline{x})y + f_0(\overline{x}) \\ &= f_1(\overline{x})(y - y_1(\overline{x})) + (f_1(\overline{x})y_1(\overline{x}) + f_0(\overline{x})) \\ \text{by (2.3)} \quad &= f_1(\overline{x})(y - y_1(\overline{x})) + ap^m(\overline{x}) \\ \text{by (2.5)} \quad &= cp^k(\overline{x})(y - y_1(\overline{x})) + ap^m(\overline{x}). \end{aligned}$$

Since $y = y(\overline{x}) \in \mathbf{Qs}_F$ and $F(x, y(\overline{x})) = \widetilde{a}p^m(\overline{x})$ for some $\widetilde{a} \in \mathbb{R}$, we have

$$cp^k(\overline{x})(y(\overline{x}) - y_1(\overline{x})) + ap^m(\overline{x}) = F(\overline{x}, y(\overline{x})) = \widetilde{a}p^m(\overline{x}).$$

Hence

$$y(\overline{x}) - y_1(\overline{x}) = \frac{(\widetilde{a}-a)p^m(\overline{x})}{cp^k(\overline{x})} = \frac{\widetilde{a}-a}{c}p^{m-k}(\overline{x}), \quad \text{and}$$

$$\begin{aligned} y(\overline{x}) &= y_1(\overline{x}) + \frac{\widetilde{a}-a}{c}p^{m-k}(\overline{x}) \\ \text{by (2.8)} \quad &= \frac{ap^m(\overline{x}) - f_0(\overline{x})}{f_1(\overline{x})} + \frac{\widetilde{a}-a}{c}p^{m-k}(\overline{x}) \\ &= \frac{-f_0(\overline{x})}{f_1(\overline{x})} + \frac{ap^m(\overline{x})}{f_1(\overline{x})} + \frac{\widetilde{a}-a}{c}p^{m-k}(\overline{x}) \\ \text{by (2.5)} \quad &= \frac{-f_0(\overline{x})}{f_1(\overline{x})} + \frac{ap^m(\overline{x})}{cp^k(\overline{x})} + \frac{\widetilde{a}-a}{c}p^{m-k}(\overline{x}) \\ &= \frac{-f_0(\overline{x})}{f_1(\overline{x})} + \frac{\widetilde{a}}{c}p^{m-k}(\overline{x}). \end{aligned}$$

Therefore

$$y(\overline{x}) = \frac{-f_0(\overline{x})}{f_1(\overline{x})} + \lambda p^{m-k}(\overline{x}) \quad \text{for some } \lambda = \tfrac{\widetilde{a}}{c} \in \mathbb{R}.$$

(iii) Actually in (ii), for any $\lambda \in \mathbb{R}$, $y(\overline{x}) = \frac{-f_0(\overline{x})}{f_1(\overline{x})} + \lambda p^{m-k}(\overline{x})$ is also a quasi-fixed solution for $F(\overline{x}, y)$. The reason is

$$\begin{aligned} F(x, y(\overline{x})) &= f_1(\overline{x})y(\overline{x}) + f_0(\overline{x}) \\ &= f_1(\overline{x})[\frac{-f_0(\overline{x})}{f_1(\overline{x})} + \lambda p^{m-k}(\overline{x})] + f_0(\overline{x}) \\ &= \lambda f_1(\overline{x})p^{m-k}(\overline{x}) \\ \text{by (2.5)} \quad &= cp^m(\overline{x}). \end{aligned}$$

This shows that (1.4) has infinitely many solutions(i.e. $|\mathbf{Qs}_F| = \infty$).

□

Remark. Notice that in this case of $\deg_y F = 1$, and $|\mathbf{Qs}_F| < \infty$, the solution number can not be larger than 1, otherwise, like in Theorem 2.7, the case (iii) means that "$|\mathbf{Qs}_F| \geq s+2$, then $|\mathbf{Qs}_F| = \infty$". We have to prove that if $\deg_y F \geq 2$, any quasi-fixed value a may correspond to at most s quasi-fixed solutions $y(\overline{x})$, that is, if $\deg_y F = s$ and $|\mathbf{Qv}_F| = u < \infty$, the cardinal $|\mathbf{Qs}_F| \leq su$.

Question. Does the number su be best bound for $|\mathbf{Qs}_F|$?

The answer is " no", we see the following theorem that the cardinal $|\mathbf{Qs}_F| \leq s+2$ later.

Theorem 2.8. *Let $F(\overline{x}, y)$ be a polynomial function of the form :*

$$F(\overline{x}, y) = \sum_{i=0}^{s} f_i(\overline{x})y^i, \qquad \deg_y F = s \geq 2,$$

and let $p(\overline{x})$ *be an irreducible polynomial function. If* $|\mathbf{Qs}_F|\geq s+3$*, then any solution pair* $y(\overline{x})$ *and* $h(\overline{x})$ *in* $\mathbf{Qs}_F$*, there is a* $t\in\mathbb{N}$ *such that*

$$y(\overline{x})-h(\overline{x})=dp^t(\overline{x}) \quad \textit{for some } d\in\mathbb{R}.$$

[It is remarkable that if $|\ \mathrm{Qs}_F\ |\geq s+3$, then $|\ \mathrm{Qs}_F\ |=\infty$(see Theorem 3.2 later).]

Proof. By assumption $|\mathbf{Qs}_F|\geq s+3$, we may assume that the cardinal of quasi-fixed values $|\mathbf{Qv}_F|\geq 2$ because a quasi-fixed value a corresponds at most s quasi-fixed solutions $y(\overline{x})\in\mathbf{Qs}_F$.

(i) If $|\mathbf{Qv}_F|\geq 3$, by Lemma 2.4, we have

$$y(\overline{x})-h(\overline{x})=dp^t(\overline{x}) \qquad \text{for some } d\in\mathbb{R}.$$

(ii) If $|\mathbf{Qv}_F|=2$, say $\mathbf{Qv}_F=\{a,b\}$. Since $|\mathbf{Qs}_F(\mathrm{a})|\leq s$, $|\mathbf{Qs}_F(\mathrm{b})|\leq s$ and $|\mathbf{Qs}_F|=|\mathbf{Qs}_F(\mathrm{a})|+|\mathbf{Qs}_F(\mathrm{b})|\geq s+3$, then

$$|\ \mathbf{Qs}_F(\mathrm{a})\ |\geq s+3-|\ \mathbf{Qs}_F(\mathrm{b})\ |\geq 3.$$

Similarly,

$$|\ \mathbf{Qs}_F(\mathrm{b})\ |\geq s+3-|\ \mathbf{Qs}_F(\mathrm{a})\ |\geq 3.$$

By Corollary 2.6, we obtain

$$y(\overline{x})-h(\overline{x})=dp^t(\overline{x}) \quad \text{for some } d\in\mathbb{R}.$$

The proof is completed. □

3. Main Theorems

In this section, consider $F(\overline{x},y)$ as (1.5) with $\deg_y F=s\geq 2$ and (1.4) has at least $s+1$ distinct quasi-fixed solutions, that is, $y_1(\overline{x}), y_2(\overline{x}), y_3(\overline{x}),\cdots,y_{s+1}(\overline{x}),\cdots$. Accordingly we could derive the following theorem.

Theorem 3.1. *Suppose that* $y_i(\overline{x})\in Qs_F$*,* $1\leq i\leq s+1$ *as the form:*

$$y_i(\overline{x})=y_1(\overline{x})+\lambda_i p^t(\overline{x}), \quad \lambda_i\in\mathbb{R} \tag{3.1.}$$

for some nonnegative integer t *independent of* i*. Then*

$$F(\overline{x},y)=\sum_{j=0}^{s}c_j\Big(y-y_1(\overline{x})\Big)^j(p(\overline{x}))^{m-jt}, \quad \textit{for constants } c_j\in\mathbb{R},\ 0\leq j\leq s.$$

Proof. Let $y_i(\overline{x})\in Qs_F$ be distinct quasi-fixed solutions of $F(\overline{x},y)$ corresponding to quasi-fixed values a_i, $1\leq i\leq s+1$ such that

$$F(\overline{x},y_i(\overline{x}))=a_ip^m(\overline{x}). \tag{3.2}$$

As $i=1$, $F(\overline{x},y_1(\overline{x}))=a_1p^m(\overline{x})$. Use $y-y_1(\overline{x})$ dividing the function $F(\overline{x},y)$ yields

$$F(\overline{x},y)=(y-y_1(\overline{x}))F_1(\overline{x},y)+a_1p^m(\overline{x}).$$

where $F_1(\overline{x},y)$ is the quotient and $a_1p^m(\overline{x})$ is the remainder. Continuing this process from $i=2$ to $s-1$, it follows that

$$F_i(\overline{x},y)=(y-y_{i+1}(\overline{x}))F_{i+1}(\overline{x},y)+d_{i+1}p^{m-it}(\overline{x})$$

with final step for $i=s-1$ to get

$$F_{s-1}(\overline{x},y)=(y-y_s(\overline{x}))F_s(\overline{x})+d_sp^{m-(s-1)t}(\overline{x}),$$

where $F_s(\overline{x})$ does not contain the variable y since $\deg_y F = s$. By the assumption (3.2), as $i = s$, we have

$$F(\overline{x}, y_{s+1}(\overline{x})) = a_{s+1}p^m(\overline{x}) \quad \text{and} \quad F_s(\overline{x}) = \lambda p^{m-st}(\overline{x}) \quad \text{for some } \lambda \in \mathbb{R}.$$

Consequently,

$$\begin{aligned} F(\overline{x}, y) &= (y - y_1(\overline{x}))F_1(\overline{x}, y) + a_1 p^m(\overline{x}) \\ &= (y - y_1(\overline{x}))\Big((y - y_2(\overline{x}))F_2(\overline{x}, y) + d_2 p^{m-t}(\overline{x})\Big) + a_1 p^m(\overline{x}) \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= (y - y_1(\overline{x}))\Bigg((y - y_2(\overline{x}))\Big(\cdots\Big((y - y_s(\overline{x}))F_s(\overline{x}) + d_s p^{m-(s-1)t}(\overline{x})\Big)\cdots\Big) \\ &\qquad + d_2 p^{m-t}(\overline{x})\Bigg) + a_1 p^m(\overline{x}) \\ &= (y - y_1(\overline{x}))\Bigg((y - y_2(\overline{x}))\Big(\cdots\Big((y - y_s(\overline{x}))\lambda p^{m-st} + d_s p^{m-(s-1)t}(\overline{x})\Big)\cdots\Big) \\ &\qquad + d_2 p^{m-t}(\overline{x})\Bigg) + a_1 p^m(\overline{x}). \end{aligned}$$

By (3.1), we have $y_i(\overline{x}) = y_1(\overline{x}) + \lambda_i p^t(\overline{x})$, $i = 2, 3, \ldots, s+1$. Then $F(\overline{x}, y)$ can be expanded to a power series as the following form:

$$\begin{aligned} F(\overline{x}, y) =& \Big(y - y_1(\overline{x})\Big)\Bigg((y - y_1(\overline{x}) - \lambda_2 p^t(\overline{x}))\Big(\cdots\Big((y - y_1(\overline{x}) - \lambda_s p^t(\overline{x}))\lambda p^{m-st} \\ &\quad + d_s p^{m-(s-1)t}(\overline{x})\Big)\cdots\Big) + d_2 p^{m-t}(\overline{x})\Bigg) + a_1 p^m(\overline{x}) \\ =& \sum_{j=0}^{s} c_j \Big(y - y_1(\overline{x})\Big)^j (p(\overline{x}))^{m-jt} \end{aligned}$$

for some real numbers c_j, $j = 0, 1, \ldots, s$. □

Note that in the above Theorem, the leading coeficient of $F(\overline{x}, y)$, $c_s p^{m-st}(\overline{x})$, is contained to $\mathbb{R}[\overline{x}]$. This means that $m - st \in \mathbb{N}$ and $t \leq m/s$.

Theorem 3.2. *The following three conditions are equivalent:*

(i) $|\mathbf{Qs}_F| \geq s + 3$.

(ii) $F(\overline{x}, y) = \sum_{i=0}^{s} c_i \Big(y - y_1(\overline{x})\Big)^i (p(\overline{x}))^{m-it}$ *for some* $y_1(\overline{x}) \in \mathbf{Qs}_F$, $t \in \mathbb{N}$ *and* $c_i \in \mathbb{R}$ *for* $i = 0, 1, \ldots, s$.

(iii) $|\mathbf{Qs}_F| = \infty$.

(In fact, if $|\mathbf{Qs}_F| = \infty$, then $|\mathbf{Qs}_F|$=the cardinal $|\mathbb{R}|$.)

Proof. (i)⇒(ii) Since $|\mathbf{Qs}_F| \geq s + 3$, by Theorem 2.8, for any pair of $y(\overline{x})$ and $h(\overline{x})$ in $\mathbf{Qs}_F$, there is $t \in \mathbb{N}$ such that

$$y(\overline{x}) - h(\overline{x}) = dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

By Theorem 3.1, we also get

$$F(\overline{x}, y) = \sum_{i=0}^{s} c_i \Big(y - y_1(\overline{x}) \Big)^i (p(\overline{x}))^{m-it}$$

for some $y_1(\overline{x}) \in \mathbf{Qs}_F$ and $t \in \mathbb{N}$.

(ii)⇒(iii) Suppose that (ii) holds.Then

$$\begin{aligned} F(\overline{x}, y_1(\overline{x}) + dp^t(\overline{x})) &= \sum_{i=0}^{s} c_i \Big(dp^t(\overline{x}) \Big)^i (p(\overline{x}))^{m-it} \\ &= \big(\sum_{i=0}^{s} c_i d^i \big) p^m(\overline{x}) \\ &= ap^m(\overline{x}) \quad \text{for } a = \sum_{i=0}^{s} c_i d^i \in \mathbb{R}. \end{aligned}$$

That is, $y_1(\overline{x}) + dp^t(\overline{x}) \in \mathbf{Qs}_F$ for each $d \in \mathbb{R}$. It follows that the cardinal $|\mathbf{Qs}_F| = \infty$.

(iii)⇒(i) It is naturally obtained. □

According the above theorem, if the number of solutions exist finitely many, then $|\mathbf{Qs}_F| \leq s + 2$. Thus we state the following Corollary.

Corollary 3.3. *If the number of all quasi-fixed solutions is finitely many, the number of all quasi-fixed solutions does not exceed an integer ℓ. That is,*

$$|\ \mathbf{Qs_F}\ | \leq \ell = s + 2 = \deg_y F + 2, \quad s = \deg_y F.$$

Lemma 3.4. *Suppose that*

$$F(\overline{x}, y) = \sum_{i=0}^{s} c_i \Big(y - y(\overline{x}) \Big)^i (p(\overline{x}))^{m-it} \tag{3.3}$$

for a solution $y(\overline{x}) \in \mathbf{Qs}_F$, $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, s$ and $t \in \mathbb{N}$ with $t \leq m/s$. Then $h(\overline{x}) \in \mathbb{R}[\overline{x}]$ is a quasi-fixed solution of $F(\overline{x}, y)$ if and only if

$$h(\overline{x}) = y(\overline{x}) + dp^t(\overline{x}) \quad \textit{for some } d \in \mathbb{R}.$$

Proof. By Theorem 3.2,

$$F(\overline{x}, y) = \sum_{i=0}^{s} c_i \Big(y - y(\overline{x}) \Big)^i (p(\overline{x}))^{m-it} \quad \text{if and only if} \quad |\mathbf{Qs}_F| \geq s + 3.$$

It follows from Theorem 2.8 that any quasi-fixed solution $h(\overline{x})$ can be represented as

$$h(\overline{x}) = y(\overline{x}) + dp^t(\overline{x}) \quad \text{for some } d \in \mathbb{R}.$$

The converse result follows from the proof of *(ii)* ⇒ *(iii)* in Theorem 3.2. □

Theorem 3.5. *Let a polynomial function $F(\overline{x}, y)$ be given as* (1.5)*:*

$$F(\overline{x}, y) = f_s(\overline{x}) y^s + f_{s-1}(\overline{x}) y^{s-1} + \cdots + f_0(\overline{x}),$$

and the quasi-fixed equation be given as (1.4)*:*

$$F(\overline{x}, y) = ap^m(\overline{x}).$$

If the cardinal number $|\mathbf{Qs}_F|$ is infinitely many, then for each quasi-fixed polynomial solution must be of the form

$$-\frac{f_{s-1}(\overline{x})}{sf_s(\overline{x})} + \lambda p^t(\overline{x}) \quad \text{for any } \lambda \in \mathbb{R}.$$

Proof. Assume $|\mathbf{Qs}_F| = \infty$. By Theorem 3.2, we have

$$\begin{aligned} F(\overline{x}, y) &= f_s(\overline{x})y^s + f_{s-1}(\overline{x})y^{s-1} + \cdots + f_0(\overline{x}) \\ &= \sum_{i=0}^{s} c_i \Big(y - y(\overline{x})\Big)^i (p(\overline{x}))^{m-it}, \quad c_i \in \mathbb{R},\ t \leq m/s. \end{aligned}$$

Comparing the coefficients of y^s and y^{s-1} in both sides of the above identity, we get

$$f_s(\overline{x}) = c_s p^{m-st}(\overline{x})$$

and

$$f_{s-1}(\overline{x}) = -sc_s p^{m-st}(\overline{x}) y(\overline{x}) + c_{s-1} p^{m-(s-1)t}(\overline{x}).$$

Consequently,

$$y(\overline{x}) = \frac{c_{s-1}}{sc_s} p^t(\overline{x}) - \frac{f_{s-1}(\overline{x})}{sf_s(\overline{x})} \in \mathbb{R}[\overline{x}].$$

By Lemma 3.4, for each $\lambda \in \mathbb{R}$, we have that any quasi-fixed solution can be represented by

$$\begin{aligned} y(\overline{x}) + dp^t(\overline{x}) &= \frac{c_{s-1}}{sc_s} p^t(\overline{x}) - \frac{f_{s-1}(\overline{x})}{sf_s(\overline{x})} + dp^t(\overline{x}) \\ &= -\frac{f_{s-1}(\overline{x})}{sf_s(\overline{x})} + (d + \frac{c_{s-1}}{sc_s}) p^t(\overline{x}) \\ &= -\frac{f_{s-1}(\overline{x})}{sf_s(\overline{x})} + \lambda p^t(\overline{x}). \end{aligned}$$

This completes the proof. □

Finally, we provide two examples. The Example 1 is to explain the case of $|\mathbf{Qs}_F| = s+2$ as Corollary 3.3. This shows that the finitely many number for $|\mathbf{Qs}_F|$ of quasi-fixed solutions is bounded by $s+2$.

Example 1.

$$\text{Let} \quad F(x,y) = (x^2+x+1)y^2 - x^6 - 3x^5 - 6x^4 - 7x^3 - 10x^2 - 7x - 5$$
$$\text{and} \quad p(x) = x^2 + x + 1, \quad m = 2.$$

Then

$$F(x,y) = p(x)[y^2 - p^2(x) - 4], \qquad \deg_y F = s = 2.$$

This polynomial function has exactly $4 = (s+2)$ quasi-fixed solutions as follows:

$$\begin{aligned} F(x, p(x)+2) &= 4p^2(x), \\ F(x, -p(x)-2) &= 4p^2(x), \\ F(x, p(x)-2) &= -4p^2(x), \\ F(x, -p(x)+2) &= -4p^2(x). \end{aligned}$$

□

The next example explains that the number of all quasi-fixed solutions of (1.4) is infinitely many if the number of all quasi-fixed solutions exceeds $s+2$.

Example 2. Let $\overline{x} = (x_1, x_2) \in \mathbb{R}^2$, $p(\overline{x}) = x_1 + x_2$, $m = 2$, and

$$\begin{aligned} F(\overline{x}, y) &= f_2(\overline{x})y^2 + f_1(\overline{x})y + f_0(\overline{x}) \\ &= y^2 - (2x_1x_2 - x_1 - x_2)y + (x_1^2x_2^2 - x_1^2x_2 - x_1x_2^2 + x_1^2 + 2x_1x_2 + x_2^2). \end{aligned}$$

We will solve all quasi-fixed solutions of $F(\overline{x}, y) = ap^m(\overline{x})$. This polynomial function has exactly $5(\geq s+3$, since $s = 2)$ quasi-fixed solutions as follows:

$$\begin{aligned} F(x_1, x_2, x_1x_2 - x_1 - x_2) &= 1(x_1 + x_2)^2, \\ F(x_1, x_2, x_1x_2) &= 1(x_1 + x_2)^2, \\ F(x_1, x_2, x_1x_2 + x_1 + x_2) &= 3(x_1 + x_2)^2, \\ F(x_1, x_2, x_1x_2 + 2x_1 + 2x_2) &= 3(x_1 + x_2)^2, \\ F(x_1, x_2, x_1x_2 + x_1/2 + x_2/2) &= 3/4(x_1 + x_2)^2. \end{aligned}$$

By Theorem 3.2, we have $|\mathbf{Qs}_F| = \infty$ and by Theorem 3.5, any quasi-fixed solution is written as:

$$\begin{aligned} -\frac{f_1(\overline{x})}{sf_2(\overline{x})} + \lambda p^t(\overline{x}) &= \frac{2x_1x_2 - x_1 - x_2}{2} + \lambda p(\overline{x}) \\ &= x_1x_2 + (\lambda - 1/2)p(\overline{x}) \\ &= x_1x_2 + \mu p(\overline{x}) \end{aligned}$$

where $\mu = \lambda - 1/2 \in \mathbb{R}$ is arbitrary.
This shows the quasi-fixed (polynomial) solutions have cardinal $|\mathbf{Qs}_F| = \infty$. □

We would like to provide two open problems as follows.

Problem 1.: For a real-valued polynomial function $F(x, y) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Suppose that there has only finitely many quasi-fixed (polynomial) solutions, that is, the number $\ell \leq \deg_y F + 2$. Can one solve all quasi-fixed (polynomial) solutions within reasonable time ? That means in a polynomial time of t=size F+size p+m not the exponential time e^t. (cf. Lenstra [1]).

Problem 2.: Does there exist a real-valued polynomial function $F(\overline{x}, y)$ such that
(i) $F(\overline{x}, y) = a\big(p(\overline{x})\big)^m$ has infinitely many quasi-fixed solutions,
(ii) $F(\overline{x}, y) = a\big(\widetilde{p}(\overline{x})\big)^{m'}$ has infinitely many quasi-fixed solutions
for some irreducible polynomials $p(\overline{x}) \neq \widetilde{p}(\overline{x})$ or some nonnegative integers $m \neq m'$?

References

[1] Y. C. Chen and H. C. Lai, *Quasi-fixed polynomial solutions of real-valued polynomial equation*, in Proceedings of the 9th International Conference on Fixed Point Theory and Applications, on July 16-22, 2009 held at Taiwan.

[2] H. C. Lai and Y. C. Chen, *Quasi-fixed Polynomial for Vector-valued Polynomial Functions on* $\mathbb{R}^n \times \mathbb{R}$, Fixed Point Theory (accepted), to appear in 2011.
[3] A. K. Lenstra, *Factoring multivariate polynomials over algebraic number fields*, SIAM J. Computing **16** (1987), 591–598.
[4] S. P. Tung, *Near solutions of polynomial equations*, Acta Arith. **123** (2006),163–181.

Manuscript received December 2, 2009
revised January 20, 2010

Hang-Chin Lai
Chung Yung Christian University and National Tsing Hua University, Taiwan
E-mail address: hclai@cycu.edu.tw, laihc@mx.nthu.edu.tw

Yi-Chou Chen
Chung Yung Christian University, Taiwan
E-mail address: g9101103@cycu.edu.tw