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STRONG CONVERGENCE THEOREMS BY MONOTONE HYBRID METHOD FOR A FAMILY OF HEMI-RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove a strong convergence theorem by using monotone hybrid method for a family of hemi-relatively nonexpansive mappings. Using this theorem, we get some new results for a hemi-relatively nonexpansive mapping or a family of hemi-relatively nonexpansive mappings in a Banach space. Consequently, we obtain strong convergence theorems for a nonexpansive mapping or a family of nonexpansive mappings in a Hilbert space.

1. Introduction

Let E be a real Banach space with $\|\cdot\|$ and let C be a nonempty closed convex subset of E. Then a mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T of C into itself is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive.

The theory of nonexpansive mappings is an important subject which can be applied widely in applied areas, in particular, in image recovery and signal processing [3-4]. However, the Picard's sequence $\{T^n x\}_{n=1}^{\infty}$ of iterates of a nonexpansive mapping T at a point $x \in C$ may not converge even in the weak topology. In 1953, Mann [13] introduced an iterative scheme which is now known as Mann's iteration process. This iteration is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$. However, we note that Mann's iteration has only weak convergence even in a Hilbert space.

In 2003, Nakajo and Takahashi [18] proposed the following modification of Mann's iteration process (1.1), by using the following hybrid method in mathematical programming, for a single nonexpansive mapping $T : C \to C$ in a Hilbert space H: Let

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$x_1 = x \in C$ and define $\{x_n\}$ by

$$
(1.2) \qquad \begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of T under an appropriate control condition on the sequence $\{\alpha_n\}$.

In 2008, Takahashi, Takeuchi and Kubota [28] proposed the following modification of the iteration method (1.2) for a family of nonexpansive mappings $T_n : C \to C$ in a Hilbert space H: Let $x_1 = x \in C$ and define $\{x_n\}$ by

$$
(1.3) \qquad \begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. They proved strong convergence of the sequence $\{x_n\}$ generated by (1.3) under an appropriate control condition on the sequence $\{\alpha_n\}$ and under the condition that the families $\{T_n\}_{n=1}^{\infty}$ satisfies the NST-condition.

In 2008, Qin and Su [19] proposed the following modification of the iteration (1.2) called the *monotone hybrid method* for a nonexpansive mapping T in a Hilbert space as follows: Define $\{x_n\}$ by

$$
(1.4) \qquad \begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. By using this method, they proved a strong convergence theorem under a control condition on the sequence $\{\alpha_n\}$ but the technic they used in this paper is different from Nakajo and Takahashi [18]. More precisely, they can show that the sequence $\{x_n\}$ generated by (1.4) is a Cauchy sequence, without the use of demiclosedness principle, Opial's condition and the Kadec-Klee property.

Recently, Su, Wang and Shang [25] proposed the following monotone hybrid method with generalized projection for a hemi-relatively noexpansive mapping T in a Banach space: Define $\{x_n\}$ by

$$
(1.5) \qquad \begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n)JTx_n\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}
$$

where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. They proved that if $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Employing the ideas of Qin and Su [19] and of Takahashi et al. [28] and Su et al. [25], we modify iterations (1.3), (1.4) and (1.5) for a countable family of hemi-relatively nonexpansive mappings in a Banach space and prove a strong convergence theorem in a Banach space. Using this theorem, we obtain some strong convergence theorems for a countable family of hemi-relatively nonexpansive mappings in a Banach space. Consequently, we obtain strong convergence theorems for a nonexpansive mapping or a family of nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following; see [26]:

(i) If E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Throughout this paper, define the function $\phi : E\times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \ \forall y, x \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. It is obvious from the definition of the function ϕ that, for all $x, y \in E$,

(1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$,
(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Following Alber [1], the generalized projection Π_C from E onto C is a map that assigns to an arbitrary point $x \in E$ the minimum point $\bar{x}$ of the functional $\phi(y, x)$, that is, $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In a Hilbert space, Π_C is the metric projection of H onto C.

Let C be a closed convex subset of a Banach space E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a self-mapping $T : C \to C$ is *hemi-relatively nonexpansive* if $\phi(u, Tx) \leq \phi(u, x)$ for all $x \in C$ and $u \in F(T)$.

A point $u \in C$ is said to be an *asymptotic* fixed point of T [21] if C contains a sequence $\{x_n\}$ which converges weakly to u and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A hemi-relatively nonexpansive mapping $T : C \to C$ is said to be *relatively nonexpansive* if $\widehat{F}(T) = F(T) \neq \emptyset$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [2].

We need the following lemmas for the proofs of our main results.

Lemma 2.1 (Kamimura and Takahashi [8]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 (Matsushita and Takahashi [16]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.3 (Alber [1], Kamimura and Takahashi [8]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space, $x \in E$ and let $z \in C$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$.*

Lemma 2.4 (Alber [1], Kamimura and Takahashi [8]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \ \forall x \in C, \ y \in E.$$

Lemma 2.5 (Kamimura and Takahashi [8]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.6 (Zalinescu [29]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.7 (Kohsaka and Takahashi [11]). *Let E be a reflexive, strictly convex and smooth Banach space, let $z \in E$ and let $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in E such that*

$$\phi\big(z, J^{-1}(\sum_{i=1}^m t_i Jx_i)\big) = \sum_{i=1}^m t_i \phi(z, x_i),$$

then $x_1 = x_2 = \cdots = x_m$.

3. NST-condition

Let E be a real Banach space and let C be a closed convex subset of E. Motivated by Nakajo, Shimoji and Takahashi [17], we give the following definitions: Let $\{T_n\}$ and $\mathcal{T}$ be two families of hemi-relatively noexpansive mappings of C into E such that $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T}) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and $\mathcal{T}$ is the set of all common fixed points of $\mathcal{T}$. Then , $\{T_n\}$ is said to satisfy the *NST-condition* with $\mathcal{T}$ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Rightarrow \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \ \ \textit{for all}\ T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the NST-condition with T. It is obvious that $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies the NST-condition with $\mathcal{T} = \{T\}$.

Lemma 3.1. *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let T be a hemi-relatively nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define a mapping T_n from C into E by*

$$T_n x = J^{-1}\big(\beta_n Jx + (1-\beta_n)JTx\big)$$

for all $x \in C$, where J is the duality mapping on E. Then, $\{T_n\}$ is a countable family of hemi-relatively nonexpansive mappings satisfying the NST-condition with T.

Proof. First, we can show that $F(T_n) = F(T)$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^\infty F(T_n) = F(T)$ for all $n \in \mathbb{N}$ and T_n is a hemi-relatively nonexpansive mapping. Indeed, for $u \in F(T_n)$ and $x \in C$, we obtain that

$$\begin{aligned}
\phi(u, T_n x) &= \phi(u, J^{-1}\big(\beta_n Jx + (1-\beta_n)JTx\big)) \\
&= \|u\|^2 - 2\langle u, \beta_n Jx + (1-\beta_n)JTx\rangle + \|\beta_n Jx + (1-\beta_n)JTx\|^2 \\
&\le \|u\|^2 - 2\beta_n\langle u, Jx\rangle - 2(1-\beta_n)\langle u, JTx\rangle + \beta_n\|x\|^2 + (1-\beta_n)\|Tx\|^2 \\
&= \beta_n\phi(u, x) + (1-\beta_n)\phi(u, Tx) \\
&\le \beta_n\phi(u, x) + (1-\beta_n)\phi(u, x) = \phi(u, x)
\end{aligned}$$

for all $x \in C$. Hence T_n is hemi-relatively nonexpansive.

Next, we show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. To show this, suppose that $\{x_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $\{x_n\}$ is bounded, we obtain that $\{Jx_n\}$ and $\{JTx_n\}$ are also bounded. Put $r = \max\{\sup_n \|x_n\|, \sup_n \|Jx_n\|, \sup_n \|JTx_n\|\}$. Then there exists $r > 0$ such that

$\{x_n\}, \{Jx_n\}, \{JTy_n\} \subset B_r(0)$. Therefore Lemma 2.6 is applicable and we observe that for $u \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\phi(u, T_n x_n) &= \phi(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n)) \\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)JTx_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JTx_n\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, JTx_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|Tx_n\|^2 \\
&\quad - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \\
&= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, Tx_n) - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \\
&\leq \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|).
\end{aligned}$$

That is, we have

$$\beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \leq \phi(u, x_n) - \phi(u, T_n x_n). \tag{3.1}$$

Let $\{\|x_{n_k} - Tx_{n_k}\|\}$ be any subsequence of $\{\|x_n - Tx_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \phi(u, x_{n'_j}) = \limsup_{k\to\infty} \phi(u, x_{n_k}) = a,$$

where $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Using properties (2) and (3) of ϕ, we have

$$\begin{aligned}
\phi(u, x_{n'_j}) &= \phi(u, T_{n'_j}x_{n'_j}) + \phi(T_{n'_j}x_{n'_j}, x_{n'_j}) + 2\langle u - T_{n'_j}x_{n'_j}, JT_{n'_j}x_{n'_j} - Jx_{n'_j}\rangle \\
&\leq \phi(u, T_{n'_j}x_{n'_j}) + \|T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Jx_{n'_j}\| + \|T_{n'_j}x_{n'_j} - x_{n'_j}\|\|x_{n'_j}\| \\
&\quad + 2\|u - T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Jx_{n'_j}\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$ and E is uniformly smooth, we have

$$\lim_{n\to\infty} \|Jx_n - JT_n x_n\| = 0.$$

So, it follows that

$$a = \liminf_{j\to\infty} \phi(u, x_{n'_j}) \leq \liminf_{j\to\infty} \phi(u, T_{n'_j}x_{n'_j}).$$

Since $\phi(u, T_n x_n) \leq \phi(u, x_n)$, we have

$$\limsup_{j\to\infty} \phi(u, T_{n'j}x_{n'_j}) \leq \limsup_{j\to\infty} \phi(u, x_{n'_j}) = a.$$

Hence

$$\lim_{j\to\infty} \phi(u, x_{n'_j}) = \lim_{j\to\infty} \phi(u, T_{n'_j}x_{n'_j}) = a.$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.1) that

$$\lim_{n\to\infty} g(\|Jx_{n_j} - JTx_{n_j}\|) = 0.$$

By properties of the function g, we have $\lim_{j\to\infty} \|Jx_{n'_j} - JTx_{n'_j}\| = 0$.
Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain $\lim_{j\to\infty} \|x_{n'_j} - Tx_{n'_j}\| = 0$ and then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. □

Lemma 3.2. *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let S and T be hemi-relatively nonexpansive mappings from C into E with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define a mapping T_n from C into E by*

$$T_n x = J^{-1}\big(\beta_n JSx + (1-\beta_n)JTx\big)$$

for all $x \in C$, where J is the duality mapping on E. Then, $\{T_n\}$ is a countable family of hemi-relatively nonexpansive mappings satisfying the NST-condition with $\mathcal{T} = \{S, T\}$.

Proof. First, we can easily show that $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T})$ for all $n \in \mathbb{N}$ and T_n is a hemi-relatively nonexpansive mapping. Indeed, note that $F(\mathcal{T}) = F(S) \cap F(T) \subset \bigcap_{n=1}^\infty F(T_n)$ for all $n \in \mathbb{N}$. For $u \in F(S) \cap F(T)$ and $x \in C$, we obtain that

$$\begin{aligned}
\phi(u, T_n x) &= \phi(u, J^{-1}\big(\beta_n JSx + (1-\beta_n)JTx\big)) \\
&= \|u\|^2 - 2\langle u, \beta_n JSx + (1-\beta_n)JTx\rangle + \|\beta_n JSx + (1-\beta_n)JTx\|^2 \\
&\le \|u\|^2 - 2\beta_n\langle u, JSx\rangle - 2(1-\beta_n)\langle u, JTx\rangle + \beta_n\|Sx\|^2 + (1-\beta_n)\|Tx\|^2 \\
&= \beta_n\phi(u, Sx) + (1-\beta_n)\phi(u, Tx) \\
&\le \beta_n\phi(u, x) + (1-\beta_n)\phi(u, x) \\
&= \phi(u, x).
\end{aligned}$$

Then, for all $v \in F(T_n)$, we have

$$\begin{aligned}
\phi(u, v) = \phi(u, T_n v) &= \phi(u, J^{-1}\big(\beta_n JSv + (1-\beta_n)JTv\big)) \\
&= \|u\|^2 - 2\langle u, \beta_n JSv + (1-\beta_n)JTv\rangle + \|\beta_n JSv + (1-\beta_n)JTv\|^2 \\
&\le \|u\|^2 - 2\beta_n\langle u, JSx\rangle - 2(1-\beta_n)\langle u, JTv\rangle + \beta_n\|Sv\|^2 + (1-\beta_n)\|Tv\|^2 \\
&= \beta_n\phi(u, Sv) + (1-\beta_n)\phi(u, Tv) \\
&\le \beta_n\phi(u, v) + (1-\beta_n)\phi(u, v) = \phi(u, v).
\end{aligned}$$

That is, we have

$$\phi(u, J^{-1}\big(\beta n JSv + (1-\beta_n)JTv\big)) = \beta_n\phi(u, Sv) + (1-\beta_n)\phi(u, Tv) = \phi(u, v).$$

By Lemma 2.7, we have $v = Sv = Tv$. So $F(T_n) \subset F(S) \cap F(T)$ for all $n \in \mathbb{N}$. This implies that $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T})$ for all $n \in \mathbb{N}$.

Next, we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Sx_n\| = 0$ if $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. By Lemma 2.6, we have that for $u \in \bigcap_{n=1}^\infty F(T_n)$,

$$\begin{aligned}
\phi(u, T_n x_n) &= \phi(u, J^{-1}\big(\beta_n JSx_n + (1-\beta_n)JTx_n\big)) \\
&= \|u\|^2 - 2\langle u, \beta_n JSx_n + (1-\beta_n)JTx_n\rangle + \|\beta_n JSx_n + (1-\beta_n)JTx_n\|^2 \\
&\le \|u\|^2 - 2\beta_n\langle u, JSx_n\rangle - 2(1-\beta_n)\langle u, JTx_n\rangle + \beta_n\|Sx_n\|^2 \\
&\quad + (1-\beta_n)\|Tx_n\|^2 - \beta_n(1-\beta_n)g(\|JSx_n - JTx_n\|) \\
&= \beta_n\phi(u, Sx_n) + (1-\beta_n)\phi(u, Tx_n) - \beta_n(1-\beta_n)g(\|JSx_n - JTx_n\|) \\
&\le \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, x_n) - \beta_n(1-\beta_n)g(\|JSx_n - JTx_n\|) \\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|JSx_n - JTx_n\|),
\end{aligned}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing and convex function with $g(0) = 0$. That is

$$\beta_n(1-\beta_n)g(\|JSx_n - JTx_n\|) \le \phi(u,x_n) - \phi(u,T_nx_n). \tag{3.2}$$

Let $\{\|Sx_{n_k} - Tx_{n_k}\|\}$ be any subsequence of $\{\|Sx_n - Tx_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \phi(u, x_{n'_j}) = \limsup_{k\to\infty} \phi(u, x_{n_k}) = a,$$

where $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Using properties (2) and (3) of ϕ, we have

$$\begin{aligned}\phi(u,x_{n'_j}) &= \phi(u,T_{n'_j}x_{n'_j}) + \phi(T_{n'_j}x_{n'_j}, x_{n'_j}) + 2\langle u - T_{n'_j}x_{n'_j}, JT_{n'_j}x_{n'_j} - Jx_{n'_j}\rangle \\ &\le \phi(u', T_{n'_j}x_{n'_j}) + \|T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Jx_{n'_j}\| + \|T_{n'_j}x_{n'_j} - x_{n'_j}\|\|x_{n'_j}\| \\ &\quad + 2\|u - T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Jx_{n'_j}\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$ and E is uniformly smooth, we have

$$\lim_{n\to\infty}\|Jx_n - JT_nx_n\| = 0.$$

So, it follows that

$$a = \liminf_{j\to\infty} \phi(u, x_{n'_j}) \le \liminf_{j\to\infty} \phi(u, T_{n'_j}x_{n'_j}).$$

Since $\phi(u, T_nx_n) \le \phi(u,x_n)$, we have

$$\limsup_{j\to\infty} \phi(u, T_{n'_j}x_{n'_j}) \le \limsup_{j\to\infty} \phi(u, x_{n'_j}) = a.$$

It follows that

$$\lim_{j\to\infty} \phi(u, x_{n'_j}) = \lim_{j\to\infty} \phi(u, T_{n'_j}x_{n'_j}) = a.$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.2) that

$$\lim_{n\to\infty} g(\|JSx_{n'_j} - JTx_{n'_j}\|) = 0.$$

By properties of the function g, we have $\lim_{j\to\infty}\|JSx_{n'_j} - JTx_{n'_j}\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain $\lim_{j\to\infty}\|Sx_{n'_j} - Tx_{n'_j}\| = 0$ and then $\lim_{n\to\infty}\|Sx_n - Tx_n\| = 0$. Since

$$\begin{aligned}\|Jx_n - JSx_n\| &\le \|Jx_n - JT_nx_n\| + \|JT_nx_n - JSx_n\| \\ &= \|Jx_n - JT_nx_n\| + (1-\beta_n)\|JSx_n - JTx_n\|,\end{aligned}$$

we obtain $\lim_{n\to\infty}\|Jx_n - JSx_n\| = 0$. Hence, we have $\lim_{n\to\infty}\|Jx_n - JTx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and hence, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. □

4. Strong convergence theorem

In this section, we prove a strong convergence theorem for a family of hemi-relatively nonexpansive mappings in a Banach space by using the monotone hybrid method. Recall that an operator T in a Banach space is called *closed*, if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a countable family of hemi-relatively nonexpansive mappings from C into E and let $\mathcal{T}$ be a family of closed hemi-relatively nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(\mathcal{T})}x$, where $\Pi_{F(\mathcal{T})}$ is the generalized projection from C onto $F(\mathcal{T})$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \in \mathbb{N}$. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N}$. Next, we prove that C_n is convex. This follows since $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Ju_n \rangle,$$

which is affine in z, and hence C_n is convex. So, $C_n \cap Q_n$ is a closed and convex subset of E for all $n \in \mathbb{N}$. It is clear that $F(\mathcal{T}) \subset C = C_0 \cap Q_0$. Next, we show that $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Suppose that $F(\mathcal{T}) \subset C_{k-1} \cap Q_{k-1}$ for $k \in \mathbb{N}$. Let $u \in F(\mathcal{T})$. Since T_n are hemi-relatively nonexpansive mappings for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \phi(u, u_k) &= \phi(u, J^{-1}\big(\alpha_k Jx_k + (1-\alpha_k)JT_k x_k\big)) \\ &= \|u\|^2 - 2\langle u, \alpha_k Jx_k + (1-\alpha_n)JT_k x_k \rangle + \|\alpha_k Jx_k + (1-\alpha_n)JT_k x_k\|^2 \\ &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_k \rangle - 2(1-\alpha_k)\langle u, JT_k x_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_k x_k\|^2 \\ &= \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, T_k x_k) \\ &\leq \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, x_k) \\ &= \phi(u, x_k). \end{aligned}$$

This implies that $F(\mathcal{T}) \subset C_k$. Since x_k is the projection of x onto $C_{k-1} \cap Q_{k-1}$, by Lemma 2.3 we have

$$\langle x_k - z, Jx - Jx_k \rangle \geq 0, \quad \forall z \in C_{k-1} \cap Q_{k-1}.$$

Since $F(\mathcal{T}) \subset C_{k-1} \cap Q_{k-1}$, we have

$$\langle x_k - z, Jx - Jx_k\rangle \geq 0, \qquad \forall z \in F(\mathcal{T}).$$

This together with definition of Q_k implies that $F(\mathcal{T}) \subset Q_k$ and hence $F(\mathcal{T}) \subset C_k \cap Q_k$. By induction, we obtain $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. From $x_n = \Pi_{Q_n} x$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n \cap Q_n \subset Q_n$, we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \qquad \forall n \geq 0.$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. It follows from Lemma 2.4 and $x_n = \Pi_{Q_n} x$ that

$$\phi(x_n, x) = \phi(\Pi_{Q_n} x, x) \leq \phi(u, x) - \phi(u, \Pi_{Q_n} x) \leq \phi(u, x)$$

for all $u \in F(\mathcal{T}) \subset Q_n$. Therefore, $\{\phi(x_n, x)\}$ is bounded. Moreover, by the definition of ϕ, we know that $\{x_n\}$ is bounded. So, the limit of $\{\phi(x_n, x)\}$ exists. For any positive integer k, we have from $x_n = \Pi_{Q_n} x$ that

$$\phi(x_{n+k}, x_n) = \phi(x_{n+k}, \Pi_{Q_n} x) \leq \phi(x_{n+k}, x) - \phi(\Pi_{Q_n} x, x) = \phi(x_{n+k}, x) - \phi(x_n, x).$$

This implies that $\lim_{n\to\infty} \phi(x_{n+k}, x_n) = 0$. Using Lemma 2.5, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x) - \phi(x_n, x),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence in C, so there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n$ and the definition of C_n, we also have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = \lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

So, we have $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{4.1}$$

On the other hand, we have

$$\begin{aligned} \|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n) JT_n x_n\| \\ &= \|\alpha_n (Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JT_n x_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JT_n x_n) - \alpha_n (Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JT_n x_n\| - \alpha_n \|Jx_n - Jx_{n+1}\|. \end{aligned}$$

This means that

$$\|Jx_{n+1} - JT_n x_n\| \leq \frac{1}{1-\alpha_n}\big(\|Jx_{n+1} - Ju_n\| + \alpha_n \|Jx_n - Jx_{n+1}\|\big).$$

From (4.1) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we obtain that $\lim_{n\to\infty} \|Jx_{n+1} - JT_n x_n\| = 0$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0.$$

From

$$\|x_n - T_n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$, we have that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0$$

for all $T \in \mathcal{T}$. Since $x_n \to w$ and T is closed, w is a fixed point of $\mathcal{T}$. From Lemma 2.4, we have

$$\phi(w, \Pi_{F(\mathcal{T})}x) + \phi(\Pi_{F(\mathcal{T})}x, x) \le \phi(w, x).$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x$ and $w \in F(\mathcal{T}) \subset C_n \cap Q_n$, we get from Lemma 2.4 that

$$\phi(\Pi_{F(\mathcal{T})}x, x_{n+1}) + \phi(x_{n+1}, x) \le \phi(\Pi_{F(\mathcal{T})}x, x).$$

By the definition of ϕ, it follows that $\phi(w, x) \le \phi(\Pi_{F(\mathcal{T})}x, x)$ and $\phi(w, x) \ge \phi(\Pi_{F(\mathcal{T})}x, x)$, hence $\phi(w, x) = \phi(\Pi_{F(\mathcal{T})}x, x)$. Therefore, it follows from the uniqueness of the $\Pi_{F(\mathcal{T})}x$ that $w = \Pi_{F(\mathcal{T})}x$. This completes the proof. □

5. Deduced results

In this section, using Theorem 4.1, we obtain some new strong convergence theorems for hemi-relatively nonexpansive mappings and a family of hemi-relatively nonexpansive mappings in a Banach space.

Theorem 5.1 (Su, Wang and Shang [25, Theorem 3.1]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let T be a closed hemi-relatively nonexpansive mapping of C into E such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T x_n\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Define $T_n = T$ for all $n \in \mathbb{N}$. It obvious that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result by using Theorem 4.1. □

Theorem 5.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let T be a closed hemi-relatively nonexpansive mapping of C into E such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JTx_n)\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Define $T_n x = J^{-1}\big(\alpha_n Jx + (1-\alpha_n)JTx\big)$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 3.1, we know that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result by using Theorem 4.1. □

Theorem 5.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let S and T be closed hemi-relatively nonexpansive mappings of C into E such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)(\beta_n JSx_n + (1-\beta_n)JTx_n)\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)}x$, where $\Pi_{F(S)\cap F(T)}$ is the generalized projection from C onto $F(S) \cap F(T)$.

Proof. Define $T_n x = J^{-1}\big(\alpha_n JSx + (1-\alpha_n)JTx\big)$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 3.2, we know that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T} = \{S, T\}$. So, we obtain the desired result by using Theorem 4.1. □

6. Applications

In this section, we prove strong convergence theorems for families of nonexpansive mappings in Hilbert spaces. In a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$ and every nonexpansive mapping with a fixed point is hemi-relatively nonexpansive and closed. The following two lemmas are directly obtained by Lemma 3.1 and Lemma 3.2, respectively.

Lemma 6.1 ([28, Lemma 2.1]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define a mapping T_n of C into itself by*

$$T_n x = \beta_n x + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of nonexpansive mappings satisfying the NST-condition with T.

Lemma 6.2 ([28, Lemma 2.3]). *Let C be a closed convex subset of a Hilbert space H and let S and T be nonexpansive mappings from C into itself with $F(S)\cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define a mapping T_n of C into itself by*

$$T_n x = \beta_n Sx + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of nonexpansive mappings satisfying the NST-condition with $\{S,T\}$.

Theorem 6.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C_{n-1}\cap Q_{n-1} : \|z-u_n\| \leq \|z-x_n\|\}, \\ Q_n = \{z \in C_{n-1}\cap Q_{n-1} : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(\mathcal{T})}x$, where $\Pi_{F(\mathcal{T})}$ is the metric projection from C onto $F(\mathcal{T})$.

Proof. In a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$. We also know that every nonexpansive mapping with a fixed point is hemi-relatively nonexpansive and closed. By using Theorem 4.1, we are easily able to obtain the desired conclusion. □

Theorem 6.4 (Su and Qin [19]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ C_n = \{z \in C_{n-1}\cap Q_{n-1} : \|z-u_n\| \leq \|z-x_n\|\}, \\ Q_n = \{z \in C_{n-1}\cap Q_{n-1} : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the metric projection from C onto $F(T)$.

Proof. Define $T_n = T$ for all $n \in \mathbb{N}$. It obvious that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result by using Theorem 6.3. □

Theorem 6.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the metric projection from C onto $F(T)$.

Proof. Define $T_n x = \alpha_n x + (1-\alpha_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 6.1, we know that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result by using Theorem 6.3. □

Theorem 6.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)(\beta_n S x_n + (1-\beta_n)Tx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)}x$, where $\Pi_{F(S)\cap F(T)}$ is the metric projection from C onto $F(S) \cap F(T)$.

Proof. Define $T_n x = \alpha_n Sx + (1-\alpha_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 6.2, we know that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T} = \{S, T\}$. So, we obtain the desired result by using Theorem 6.3. □

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