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GLOBAL BIFURCATION OF A NEUMANN PROBLEM

IN-SOOK KIM AND YUN-HO KIM

Abstract. We are concerned with global bifurcation of a nonlinear Neumann problem

$$-\mathrm{div}\,(w(x)|\nabla u|^{p-2}\nabla u) = (\mu m(x) - 1)|u|^{p-2}u + f(\lambda, x, u, \nabla u)$$

when μ is not an eigenvalue of the corresponding problem with $f = 0$ in the weak formulation.

1. Introduction

Some bifurcation problems for a nonlinear equation of the form

$$-\mathrm{div}\,(w(x)|\nabla u|^{p-2}\nabla u) = \lambda m(x)|u|^{p-2}u + f(\lambda, x, u)$$

subject to Dirichlet boundary conditions are given in [1]. Recently, a global bifurcation result of nonlinear Neumann problem

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda m(x)|u|^{p-2}u + f(\lambda, x, u)$$

was obtained in [4], based on the work of Huang [3]. While bifurcation from the first eigenvalue was dealt with in [1,4], global bifurcation of the p-Laplacian with Dirichlet boundary conditions

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \mu|u|^{p-2}u + f(\lambda, x, u, \nabla u)$$

was investigated in [7] when μ is not an eigenvalue in some sense, by applying nonlinear spectral theory for homogeneous operators.

In the present paper, we study the following boundary value problem

$$\text{(B)}\qquad \begin{cases} -\mathrm{div}\,(w(x)|\nabla u|^{p-2}\nabla u) = (\mu m(x) - 1)|u|^{p-2}u + f(\lambda, x, u, \nabla u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

when μ is not an eigenvalue of

$$\text{(E)}\qquad \begin{cases} -\mathrm{div}\,(w(x)|\nabla u|^{p-2}\nabla u) = (\mu m(x) - 1)|u|^{p-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

in the weak formulation. Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $p > 1$, w is a weight function, m belongs to $L^\infty(\Omega)$, $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies a Carathéodory condition, and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$.

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Let w be a positive measurable function in Ω that satisfies

$$w^{-\frac{1}{p-1}} \in L_{1,\,\mathrm{loc}}(\Omega), \quad w^{-s} \in L_1(\Omega) \ \text{ for some } s \in \big(\frac{N}{p}, \infty\big) \cap \big[\frac{1}{p-1}, \infty\big).$$

Let $X = W^{1,p}(w, \Omega)$ be the weighted Sobolev space endowed with the norm

$$\|u\|_X = \Big(\int_\Omega w|\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \Big)^{\frac{1}{p}},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$ and $\mathbb{R}^1$, respectively. Let $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ be the usual pairing of X and its dual X^*. Define three operators $J : X \to X^*, G : X \to X^*$, and $F : \mathbb{R} \times X \to X^*$ by setting

$$\begin{aligned}
\langle J(u), \varphi \rangle &= \int_\Omega (w|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi + |u|^{p-2} u \, \varphi) \, dx, \\
\langle G(u), \varphi \rangle &= \int_\Omega m|u|^{p-2} u \, \varphi \, dx, \\
\langle F(\lambda, u), \varphi \rangle &= \int_\Omega f(\lambda, x, u, \nabla u) \, \varphi \, dx.
\end{aligned}$$

A pair (λ, u) in $\mathbb{R} \times X$ is said to be a *weak solution* of (B) if u is a solution of the operator equation

$$J(u) - \mu G(u) = F(\lambda, u).$$

A real number μ is said to be an *eigenvalue* of (E) if the equation $J(u) = \mu G(u)$ has a nontrivial solution.

2. Main result

Let $1 < p < \infty$, $p' = p/(p-1)$, and $p_s = ps/(s+1)$. Assume that

(f1) $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\lambda, \cdot, u, v)$ is measurable for all $(\lambda, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and $f(\cdot, x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$.

(f2) For each bounded interval $I \subset \mathbb{R}$, there are a function $a_I \in L_q(\Omega)$ and a nonnegative constant b_I such that

$$|f(\lambda, x, u, v)| \le a_I(x) + b_I(|u|^{\frac{p}{q}} + |v|^{\frac{p_s}{q}})$$

for almost all $x \in \Omega$ and for all $(\lambda, u, v) \in I \times \mathbb{R} \times \mathbb{R}^N$, where the conjugate exponent of $q > 1$ is strictly less than p^*.

(f3) There exist a function $a \in L_{p'}(\Omega)$ and a locally bounded function $b : [0, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} b(r)/r^{p-1} = 0$ such that

$$|f(0, x, u, v)| \le a(x) + b(|u| + |v|)^{\frac{s}{s+1}}$$

for almost all $x \in \Omega$ and for all $(u, v) \in \mathbb{R} \times \mathbb{R}^N$.

The following key tool for achieving the main result is proved in [5].

Lemma 2.1. *Let X be a Banach space and Y a normed space. Suppose that $J : X \to Y$ is a homeomorphism and $G : X \to Y$ is a compact continuous map, and*

$F : \mathbb{R} \times X \to Y$ is a compact continuous map such that the composition $J^{-1} \circ G$ is odd. If the set

$$\bigcup_{t\in[0,1]} \{u \in X : J(u) - G(u) = tF(0,u)\}$$

is bounded, then the set

$$\{(\lambda, u) \in \mathbb{R} \times X : J(u) - G(u) = F(\lambda, u)\}$$

has a noncompact connected set C which intersects $\{0\} \times X$.

From now on we give some of fundamental properties of the integral operators J, F and G defined in Section 1.

Lemma 2.2. *The operator $J : X \to X^*$ is a homeomorphism.*

Proof. The continuity of J follows from the continuity of superposition operators, via the inequality

$$\|J(v) - J(u)\|_{X^*} \le \Big(\int_\Omega | w^{\frac{1}{p'}} |\nabla v|^{p-2}\nabla v - w^{\frac{1}{p'}} |\nabla u|^{p-2}\nabla u |^{p'} \, dx\Big)^{\frac{1}{p'}} + \Big(\int_\Omega | |v|^{p-2}v - |u|^{p-2}u |^{p'} \, dx\Big)^{\frac{1}{p'}}.$$

We now claim that the following estimate holds:

$$\langle J(u) - J(v), u - v\rangle \ge (\|u\|_X^{p-1} - \|v\|_X^{p-1})(\|u\|_X - \|v\|_X)$$

for all $u, v \in X$. Indeed, from Hölder's inequality and the inequality

$$a^{\frac{1}{p'}} c^{\frac{1}{p}} + b^{\frac{1}{p'}} d^{\frac{1}{p}} \le (a+b)^{\frac{1}{p'}} (c+d)^{\frac{1}{p}}$$

for any positive numbers a, b, c, d, it follows that

$$\begin{aligned}
\langle J(u), v\rangle &= \int_\Omega (w|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)\, dx \\
&\le \Big(\int_\Omega w|\nabla u|^p \, dx\Big)^{\frac{1}{p'}} \Big(\int_\Omega w|\nabla v|^p \, dx\Big)^{\frac{1}{p}} + \Big(\int_\Omega |u|^p \, dx\Big)^{\frac{1}{p'}} \Big(\int_\Omega |v|^p \, dx\Big)^{\frac{1}{p}} \\
&\le \Big(\int_\Omega w|\nabla u|^p \, dx + \int_\Omega |u|^p \, dx\Big)^{\frac{1}{p'}} \Big(\int_\Omega w|\nabla v|^p \, dx + \int_\Omega |v|^p \, dx\Big)^{\frac{1}{p}} \\
&= \|u\|_X^{p-1}\|v\|_X
\end{aligned}$$

and in a similar manner

$$\langle J(v), u\rangle \le \|v\|_X^{p-1}\|u\|_X,$$

which imply

$$\begin{aligned}
\langle J(u) - J(v), u - v\rangle &= \|u\|_X^p + \|v\|_X^p - \langle J(u), v\rangle - \langle J(v), u\rangle \\
&\ge \|u\|_X^p + \|v\|_X^p - \|u\|_X^{p-1}\|v\|_X - \|v\|_X^{p-1}\|u\|_X.
\end{aligned}$$

Since J is thus strictly monotone and coercive on the reflexive Banach space X, Browder's theorem says that it is bijective; see e.g. [9]. By the uniform convexity of X and the above inequality claimed, an analogous argument to the proof of Lemma 3.3 in [1] establishes that J^{-1} is continuous on X^*. This completes the proof. □

In proving the following result, a basic idea is to use a continuity result on superposition operators due to Väth [8], as in the proof of Theorem 4.1 in [6]

Lemma 2.3. *Under assumptions (f1) and (f2), the operator $F : \mathbb{R} \times X \to X^*$ is continuous and compact. The operator $G : X \to X^*$ is continuous and compact.*

Proof. A linear operator $I_1 : \mathbb{R} \times X \to \mathbb{R} \times L_p(\Omega) \times L_{p_s}(\Omega, \mathbb{R}^N)$ defined by

$$I_1(\lambda, u) := (\lambda, u, \nabla u)$$

is bounded. In fact, it follows from Hölder's inequality that

$$\begin{aligned} \int_\Omega |\nabla u|^{\frac{ps}{s+1}}\, dx &= \int_\Omega |\nabla u|^{\frac{ps}{s+1}} w^{\frac{s}{s+1}}(x) w^{-\frac{s}{s+1}}(x)\, dx \\ &\le \Big(\int_\Omega w(x)|\nabla u|^p\, dx\Big)^{\frac{s}{s+1}} \Big(\int_\Omega w^{-s}(x)\, dx\Big)^{\frac{1}{s+1}} \\ &\le \|u\|_X^{\frac{ps}{s+1}} \|w^{-s}\|_{L_1(\Omega)}^{\frac{1}{s+1}}. \end{aligned}$$

Let $\Psi : Y = \mathbb{R} \times L_p(\Omega) \times L_{p_s}(\Omega, \mathbb{R}^N) \to L_q(\Omega)$ be defined by

$$\Psi(\lambda, u, v)(x) := f(\lambda, x, u(x), v(x)).$$

If I is a bounded interval in $\mathbb{R}$ and $a_I \in L_q(\Omega)$ and $b_I \in [0, \infty)$ are chosen from (f2), then we have

$$\begin{aligned} \|\Psi(\lambda, u, v)\|_{L_q(\Omega)}^q &\le \int_\Omega (3 \max\{|a_I|, b_I|u|^{\frac{p}{q}}, b_I|v|^{\frac{p_s}{q}}\})^q dx \\ &\le 3^q \big(\|a_I\|_{L_q(\Omega)}^q + (b_I)^q \|u\|_{L_p(\Omega)}^p + (b_I)^q \|v\|_{L_{p_s}(\Omega, \mathbb{R}^N)}^{p_s}\big). \end{aligned}$$

Thus, Ψ is bounded. Since Y is a generalized ideal space and $L_q(\Omega)$ is a regular ideal space, Theorem 6.4 of [8] implies that Ψ is continuous on Y. The inclusion $I_2 : X \hookrightarrow L_{q'}(\Omega)$ is continuous and compact (see [1]) and so is the adjoint operator $I_2^* : L_q(\Omega) \to X^*$ given by

$$\langle I_2^*(u), \varphi\rangle = \int_\Omega u\, \varphi\, dx.$$

From the relation $F = I_2^* \circ \Psi \circ I_1$ it follows that F is continuous and compact. In particular, if we set $f(\lambda, x, u, v) = m(x)|u|^{p-2}u$, then G is continuous and compact. This completes the proof. ☐

We will now observe the behavior of $F(0, \cdot)$ at infinity, as in [5,7].

Lemma 2.4. *Under assumptions (f1) and (f3), the operator $F(0, \cdot) : X \to X^*$ has the following property:*

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0, u)\|_{X^*}}{\|u\|_X^{p-1}} = 0.$$

Proof. Let $\varepsilon > 0$. Choose a positive constant R such that $|b(r)| \le \varepsilon r^{p-1}$ for all $r \ge R$. Since b is locally bounded, there is a nonnegative constant C_R such that

$|b(r)| \le C_R$ for all $r \in [0, R]$. Let $u \in X$. If we set $\Omega_R = \{x \in \Omega : |u(x)| + |\nabla u(x)| \le R\}$, we have by Minkowski's and Jensen's inequalities

$$\begin{aligned}
&\Big(\int_\Omega |f(0,x,u(x),\nabla u(x))|^{p'} dx\Big)^{\frac{1}{p'}} \\
&\quad \le \|a\|_{L_{p'}(\Omega)} + \Big(\int_\Omega |b\big(|u(x)| + |\nabla u(x)|\big)|^{p'}\, dx\Big)^{\frac{1}{p'}} \\
&\quad \le \|a\|_{L_{p'}(\Omega)} + \Big(\int_{\Omega_R} (C_R)^{p'}\, dx\Big)^{\frac{1}{p'}} + \Big(\int_{\Omega\setminus\Omega_R} \varepsilon^{\frac{p's}{s+1}} \big(|u(x)| + |\nabla u(x)|\big)^{p_s}\, dx\Big)^{\frac{1}{p'}}.
\end{aligned}$$

From the proof of Lemma 2.3 and Minkowski's inequality we know that

$$\begin{aligned}
\Big(\int_{\Omega\setminus\Omega_R} \varepsilon^{\frac{p's}{s+1}} \big(|u(x)| + |\nabla u(x)|\big)^{p_s}\, dx\Big)^{\frac{1}{p'}} &\le \varepsilon^{\frac{s}{s+1}} \big(\|u\|_{L_{p_s}(\Omega)} + c_1\|u\|_X\big)^{\frac{p_s}{p'}} \\
&\le \varepsilon^{\frac{s}{s+1}} \big(c_2\|u\|_{L_p(\Omega)} + c_1\|u\|_X\big)^{\frac{(p-1)s}{s+1}} \\
&\le \varepsilon^{\frac{s}{s+1}} c_3\, \|u\|_X^{p-1}
\end{aligned}$$

for all $u \in X$ with $\|u\|_X \ge 1$, where c_1, c_2, and c_3 are some positive constants. Hence we obtain

$$\begin{aligned}
&\Big(\int_\Omega |f(0,x,u(x),\nabla u(x))|^{p'} dx\Big)^{\frac{1}{p'}} \\
&\quad \le \|a\|_{L_{p'}(\Omega)} + C_R(\operatorname{meas}\Omega)^{\frac{1}{p'}} + \varepsilon^{\frac{s}{s+1}} c_3\, \|u\|_X^{p-1}
\end{aligned}$$

for all $u \in X$ with $\|u\|_X \ge 1$. It follows from Hölder's inequality that

$$\begin{aligned}
|\langle F(0,u),\, \varphi\rangle| &\le \Big(\int_\Omega |f(0,x,u(x),\nabla u(x))|^{p'} dx\Big)^{\frac{1}{p'}} \|\varphi\|_{L_p(\Omega)} \\
&\le \Big(\|a\|_{L_{p'}(\Omega)} + C_R(\operatorname{meas}\Omega)^{\frac{1}{p'}} + \varepsilon^{\frac{s}{s+1}} c_3\, \|u\|_X^{p-1}\Big) \|\varphi\|_X
\end{aligned}$$

for all $u, \varphi \in X$ with $\|u\|_X \ge 1$. Consequently, we get

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0,u)\|_{X^*}}{\|u\|_X^{p-1}} = 0.$$

□

Lemma 2.5. *If μ is not an eigenvalue of (E), we have*

$$\liminf_{\|u\|_X \to \infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p-1}} > 0.$$

Proof. By Lemmas 2.2 and 2.3, $J : X \to X^*$ is a homeomorphism and $G : X \to X^*$ is compact and they are odd and positively homogeneous of order $p - 1$. Applying nonlinear spectral theory for homogeneous operators given in [2], we arrive at the conclusion. □

Finally, we can prove a global bifurcation result for the above Neumann problem (B).

Theorem 2.6. *Suppose that conditions (f1)–(f3) are satisfied. If* μ *is not an eigenvalue of (E), then there exists a noncompact connected set* C *intersecting* $\{0\} \times X$ *such that each point* (λ, u) *in* C *is a weak solution of the above problem (B). In the case where* $f(0, \cdot, \cdot) = 0$*, we have* $(0, 0) \in C$.

Proof. Note that J is a homeomorphism, G and F are compact continuous operators, and $J^{-1} \circ (\mu G)$ is odd. Since μ is not an eigenvalue of (E), Lemmas 2.4 and 2.5 imply that for some $\delta > 0$, there is a positive constant R such that

$$\|J(u) - \mu G(u)\|_{X^*} > \delta \|u\|_X^{p-1} > \|F(0, u)\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and hence

$$\|J(u) - \mu G(u)\|_{X^*} > \|tF(0, u)\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and for all $t \in [0, 1]$. Hence, the set

$$\bigcup_{t \in [0,1]} \{u \in X : J(u) - \mu G(u) = tF(0, u)\}$$

is bounded. By Lemma 2.1, we conclude that the solution set

$$\{(\lambda, u) \in \mathbb{R} \times X : J(u) - \mu G(u) = F(\lambda, u)\}$$

contains a noncompact connected set C which intersects $\{0\} \times X$. If $f(0, \cdot, \cdot) = 0$, we have

$$\{u \in X : J(u) - \mu G(u) = 0\} = \{0\} \quad \text{and thus} \quad (0, 0) \in C$$

because μ is not an eigenvalue of (E). This completes the proof. □

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In-Sook Kim
Department of Mathematics, Sungkyunkwan University, Suwon 440–746, Korea
E-mail address: iskim@skku.edu

Yun-Ho Kim
Department of Mathematics, Sungkyunkwan University, Suwon 440–746, Korea
E-mail address: kyh1213@skku.edu