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TAKAHASHI'S AND FAN-BROWDER'S FIXED POINT THEOREMS IN A VECTOR LATTICE

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Abstract. The purpose of this paper is to show fixed point theorems using the topology introduced by [2]. In particular, we obtain Takahashi's fixed point theorem in the case where the whole space is a vector lattice with unit. Using Takahashi's fixed point theorem in this space, we also obtain Fan-Browder's fixed point theorem.

Dedicated to Professor Wataru Takahashi on the celebration of his retirement

1. Introduction

There are many fixed point theorems in a topological vector space, for instance, Takahashi's fixed point theorem and Fan-Browder's fixed point theorem in a topological vector space, Tychonoff's fixed point theorem in a locally convex space, Schauder's fixed point theorem in a normed space, and so on; see for example [6].

Takahashi [5] proved the following; see also [6].

Takahashi's fixed point theorem. *Let X be a Hausdorff topological vector space, Y a compact subset of X and Z a convex subset of Y. Suppose that a mapping f from Z into 2^Y satisfies*
(0) $f^{-1}(y)$ *is convex for any* $y \in Y$,
and there exists a mapping g from Z into 2^Y satisfying the following conditions:
(1) $g(z)$ *is a subset of* $f(z)$ *for any* $z \in Z$;
(2) $g^{-1}(y)$ *is non-empty for any* $y \in Y$;
(3) $g(z)$ *is an open subset of* X *for any* $z \in Z$.
Then there exists $z_0 \in Z$ *such that* $z_0 \in f(z_0)$.

In the mentioned above, $f^{-1}(y) = \{x \mid y \in f(x)\}$.

In this paper, we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [4, 7] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in the case of the vector lattice with unit.

The purpose of this paper is to show fixed point theorems using the topology introduced by [2]. In particular, we obtain Takahashi's fixed point theorem in the

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case where X is a vector lattice with unit. Using Takahashi's fixed point theorem in this space, we also obtain Fan-Browder's fixed point theorem.

2. Topology in a vector lattice

In this section we introduce a topology in a vector lattice introduced by [2].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In the case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_{\mathbf{R}}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X. Y is closed if $Y^C \in \mathcal{O}_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a,b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_{\mathbf{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of $[a,b]^e$ it is easy to see that $[a,b]^e \subset [a,b]$. A mapping from $X \times \mathcal{K}_X$ into $(0,\infty)$ is said to be a gauge. Let Δ_X be the class of gauges in X. For $x \in X$ and $\delta \in \Delta_X$, $O(x,\delta)$ is defined by

$$O(x,\delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(\xi,e)e, x + \delta(\xi,e)e]^e.$$

$O(x,\delta)$ is said to be a δ-neighborhood of x. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x,\delta)$.

Lemma 2.1. *Let X be a vector lattice with unit and Y a subset of X. Then the following are equivalent.*

(1) *Y is an open subset of X.*
(2) *There exists $\delta \in \Delta_X$ such that $O(x,\delta)$ is a subset of Y for any $x \in Y$.*
(3) *For any $x \in Y$ there exists $\delta \in \Delta_X$ such that $O(x,\delta)$ is a subset of Y.*

Proof. We first show that (1) implies (2). Suppose that $Y \in \mathcal{O}_X$. Let $x \in Y$ and $e \in \mathcal{K}_X$. Since $Y \in \mathcal{O}_X$, there exists a positive number $\delta(x,e)$ such that $[x-\delta(x,e)e, x+\delta(x,e)e] \subset Y$. Then $\delta \in \Delta_X$. Let $y \in O(x,\delta)$ arbitrary. Then there exists $e \in \mathcal{K}_X$ such that $y \in [x - \delta(x,e)e, x + \delta(x,e)e]^e$. Then it follows that

$$y \in [x - \delta(x,e)e, x + \delta(x,e)e]^e \subset [x - \delta(x,e)e, x + \delta(x,e)e] \subset Y.$$

Therefore $O(x,\delta) \subset Y$. It is obvious that (2) implies (3). So next we show that (3) implies (1). Suppose that for any $x \in Y$ there exists $\delta \in \Delta_X$ such that $O(x,\delta) \subset Y$. For any $e \in \mathcal{K}_X$ let $\delta < \delta(x,e)$. Then $[x - \delta e, x + \delta e] \subset [x - \delta(x,e)e, x + \delta(x,e)e]^e$. By the definition of $O(x,\delta)$, we have

$$[x - \delta e, x + \delta e] \subset [x - \delta(x,e)e, x + \delta(x,e)e]^e \subset O(x,\delta) \subset Y.$$

Therefore $Y \in \mathcal{O}_X$. □

For a subset Y of X we denote by $cl(Y)$ and $int(Y)$, the closure and the interior of Y, respectively. Let X and Y be vector lattices with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous in the sense of topology at x_0 if for any $V \in \mathcal{O}_Y$ with $f(x_0) \in V$ there exists $U \in \mathcal{O}_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

3. Fixed point theorems

In this section we show Takahashi's fixed point theorem and Fan-Browder's fixed point theorem using the topology introduced in Section 2.

Let X be a vector lattice with unit. X is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset Y of X is said to be compact if for any open covering of Y there exists a finite sub-covering. A subset Y of X is said to be normal if for any closed subsets F_1 and F_2 with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $F_1 \subset O_1$, $F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$. Moreover the following hold.

(1) Let X be a Hausdorff vector lattice with unit and Y a compact subset of X. Then Y is normal.

(2) Let X be a vector lattice with unit and Y a normal and closed subset of X. If $Y \subset \bigcup_{i=1}^n O_i$, where $O_i \in \mathcal{O}_X$, then there exists a continuous function β_i in the sense of topology from Y into $[0,1]$ for each i such that $\beta_i(y) = 0$ for any $y \in O_i^C \cap Y$ and $\sum_{i=1}^n \beta_i(y) = 1$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_{\mathbf{R}}$. A mapping N from $X \times \mathcal{K}_X$ to $[0, \infty]$ is defined by $N(x, e) = \sup\{r \mid r|x| \leq e\}$. Moreover we consider the following condition:

(UA) For any $e \in \mathcal{K}_X$ and for any $\{x_1, \cdots, x_m\}$ which is a linearly independent subset of X there exists $M \in \mathcal{K}_{\mathbf{R}}$ such that $N\left(\sum_{i=1}^m k_i x_i, e\right) \leq M$ for any $k_1, \cdots, k_m \in \mathbf{R}$ with $\sum_{i=1}^m k_i^2 = 1$.

Lemma 3.1. *Every Archimedean vector lattice satisfies the condition* (UA).

Proof. By [7, Theorem IV.11.1] for any Archimedean vector lattice X there exists the completion $\hat{X}$ of X. By [7, Theorem V.4.2] for the complete vector lattice $\hat{X}$ there exists an extremally disconnected compact set Ω and a vector sublattice Y of $C_\infty(\Omega)$ such that $\hat{X}$ is isomorphic to Y, where

$$C_\infty(\Omega) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is continuous from } \Omega \text{ into } [-\infty, \infty] \text{ and} \\ f^{-1}(\{\pm\infty\}) \text{ is nowhere dense} \end{array} \right\}.$$

Therefore it may be assumed that X is a vector sublattice of $C_\infty(\Omega)$. Then

$$\begin{aligned} N\left(\sum_{i=1}^m k_i x_i, e\right) &= \sup\left\{ r \;\middle|\; r\left|\sum_{i=1}^m k_i x_i(\omega)\right| \leq e(\omega) \text{ for any } \omega \in \Omega \right\} \\ &= \inf\left\{ \frac{e(\omega)}{\left|\sum_{i=1}^m k_i x_i(\omega)\right|} \;\middle|\; \omega \in \Omega \right\}. \end{aligned}$$

Let $S = \left\{(k_1, \cdots, k_m) \mid \sum_{i=1}^m k_i^2 = 1\right\}$ and E_ω a mapping from S into $[0, \infty]$ defined by $E_\omega(k_1, \cdots, k_m) = \frac{e(\omega)}{\left|\sum_{i=1}^m k_i x_i(\omega)\right|}$. Then for any $(k_1, \cdots, k_m) \in S$ there exists $\omega \in \Omega$ such that $e(\omega) \neq \infty$ and $\sum_{i=1}^m k_i x_i(\omega) \neq 0$. Actually assume that there exists $(k_1, \cdots, k_m) \in S$ such that $e(\omega) = \infty$ or $\sum_{i=1}^m k_i x_i(\omega) = 0$ for any $\omega \in \Omega$. Let $\Omega' = \{\omega \mid \sum_{i=1}^m k_i x_i(\omega) \neq 0\}$. Since each x_i is continuous, Ω' is open. On the other hand, since $\Omega' \subset \{\omega \mid e(\omega) = \infty\}$, Ω' is nowhere dense. It is a contradiction. Therefore for any $(k_1, \cdots, k_m) \in S$ there exists $\omega \in \Omega$ such that $e(\omega) \neq \infty$ and

$\sum_{i=1}^m k_i x_i(\omega) \neq 0$. Let $T_\omega = \{(k_1, \cdots, k_m) \mid (k_1, \cdots, k_m) \in S, \sum_{i=1}^m k_i x_i(\omega) \neq 0\}$. Then $\bigcup_{\omega \in \{\omega \mid e(\omega) \neq \infty\}} T_\omega = S$. Since S is compact and each T_ω is open, there exists $\omega_1, \cdots, \omega_p \in \{\omega \mid e(\omega) \neq \infty\}$ such that $\bigcup_{j=1}^p T_{\omega_j} = S$. Let $E(k_1, \cdots, k_m) = \min\{E_{\omega_j}(k_1, \cdots, k_m) \mid j = 1, \cdots, p\}$. Then E is continuous on S. Let $M = \max\{E(k_1, \cdots, k_m) \mid (k_1, \cdots, k_m) \in S\}$. Then

$$\begin{aligned} N\left(\sum_{i=1}^m k_i x_i, e\right) &= \inf\left\{ \frac{e(\omega)}{|\sum_{i=1}^m k_i x_i(\omega)|} \,\middle|\, \omega \in \Omega \right\} \\ &\leq E(k_1, \cdots, k_m) \leq M. \end{aligned}$$

Therefore X satisfies the condition (UA). □

To prove our main result, we need the following lemma.

Lemma 3.2. *Let X be an Archimedean vector lattice with unit and $\{x_1, \cdots, x_n\}$ a subset of X. Then $co\{x_1, \cdots, x_n\}$ is homeomorphic to a compact and convex subset of $\mathbf{R}^n$.*

Proof. Suppose that $\{x_1, \cdots, x_m\}$ is a linearly independent subset of $\{x_1, \cdots, x_n\}$ and $x_j = \sum_{i=1}^m a_{j,i} x_i$ for $j = m+1, \cdots, n$. Let $X_0 = Span\{x_1, \cdots, x_m\}$, $e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0) \in \mathbf{R}^m$ for any $i = 1, 2, \cdots, m$ and f a mapping from X_0 into $\mathbf{R}^m$ defined by $f\left(\sum_{i=1}^m c_i x_i\right) = \sum_{i=1}^m c_i e_i$. Then f is bijective clearly.

Since by Lemma 3.1 X satisfies the condition (UA), for any $e \in \mathcal{K}_X$ there exists $M \in \mathcal{K}_{\mathbf{R}}$ such that $|k_i| \leq M$ for any i if $|\sum_{i=1}^m k_i x_i| \leq e$. Actually it is shown as follows. It may be assumed that $\sum_{i=1}^m k_i^2 \neq 0$. Let $e \in \mathcal{K}_X$. Since X satisfies the condition (UA), there exists $M \in \mathcal{K}_{\mathbf{R}}$ such that $N\left(\sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}} x_i, e\right) \leq M$. Since

$$\sqrt{\sum_{i=1}^m k_i^2} \left| \sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}} x_i \right| = \left| \sum_{i=1}^m k_i x_i \right| \leq e,$$

by the definition of N

$$|k_i| \leq \sqrt{\sum_{i=1}^m k_i^2} \leq N\left(\sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}} x_i, e\right) \leq M$$

for any i. Take $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ arbitrary and let $V_\varepsilon = (c_1 - \varepsilon, c_1 + \varepsilon) \times \cdots \times (c_m - \varepsilon, c_m + \varepsilon)$. Take $\delta \in \Delta_X$ satisfying $\delta\left(\sum_{i=1}^m c_i x_i, e\right) \leq \frac{\varepsilon}{M}$. If $\sum_{i=1}^m (c_i + h_i) x_i \in O\left(\sum_{i=1}^m c_i x_i, \delta\right)$, then $|h_i| < \varepsilon$ for any i. Therefore

$$f\left(\sum_{i=1}^m (c_i + h_i) x_i\right) = \sum_{i=1}^m (c_i + h_i) e_i \in V_\varepsilon.$$

Let $U = int\left(O\left(\sum_{i=1}^m c_i x_i, \delta\right)\right)$. Then $f(U \cap X_0) \subset V_\varepsilon$ proving that f is continuous in the sense of topology.

Conversely f^{-1} is continuous in the sense of topology. In fact, take $U \in \mathcal{O}_X$ arbitrary. By Lemma 2.1 there exists $\delta \in \Delta_X$ such that $O\left(\sum_{i=1}^m c_i x_i, \delta\right) \subset U$. Take

$e \geq \sum_{i=1}^{m} |x_i|$ and $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ with $\varepsilon \leq \delta\left(\sum_{i=1}^{m} c_i x_i, e\right)$. If $\sum_{i=1}^{m}(c_i + h_i)e_i \in V_\varepsilon$, then $|\sum_{i=1}^{m} h_i x_i| < \varepsilon e$. Therefore

$$f^{-1}\left(\sum_{i=1}^{m}(c_i + h_i)e_i\right) = \sum_{i=1}^{m}(c_i + h_i)x_i \in O\left(\sum_{i=1}^{m} c_i x_i, \delta\right) \cap X_0 \subset U \cap X_0$$

proving that f^{-1} is continuous in the sense of topology.

Therefore X_0 is homeomorphic to $\mathbf{R}^m \subset \mathbf{R}^n$ and moreover $co\{x_1, \cdots, x_n\}$ is homeomorphic to $co\{e_1, \cdots, e_m, \sum_{i=1}^{m} a_{m+1,i}e_i, \cdots, \sum_{i=1}^{m} a_{n,i}e_i\}$. □

By the above lemma we can show the following Takahashi's fixed point theorem in a vector lattice.

Theorem 3.3. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a compact subset of X and Z a convex subset of Y. Suppose that a mapping f from Z into 2^Y satisfies*

(0) *$f^{-1}(y)$ is convex for any $y \in Y$,*

and there exists a mapping g from Z into 2^Y satisfying the following conditions:

(1) *$g(z)$ is a subset of $f(z)$ for any $z \in Z$;*

(2) *$g^{-1}(y)$ is non-empty for any $y \in Y$;*

(3) *$g(z)$ is an open subset of X for any $z \in Z$.*

Then there exists $z_0 \in Z$ such that $z_0 \in f(z_0)$.

Proof. By (2) it holds that $Y \subset \bigcup_{z \in Z} g(z)$. By (3) it holds that $g(z) \in \mathcal{O}_X$. Since Y is compact, there exists $z_1, \cdots, z_n \in Z$ such that $Y \subset \bigcup_{i=1}^{n} g(z_i)$. Since Y is normal, there exists a continuous function β_i in the sense of topology from Y into $[0, 1]$ satisfying $\beta_i(y) = 0$ for any $y \in g(z_i)^C$ and $\sum_{i=1}^{n} \beta_i(y) = 1$. Let p be a mapping from Y into Z defined by $p(y) = \sum_{i=1}^{n} \beta_i(y) z_i$. Then p is continuous in the sense of topology. Since by (1) it holds that $g^{-1}(y) \subset f^{-1}(y)$, by (0) it holds that $p(y) \in f^{-1}(y)$. Let $Z_0 = co\{z_1, \cdots, z_n\}$. By Lemma 3.2 Z_0 is homeomorphic to a compact and convex subset K of $\mathbf{R}^n$. Put a mapping h from Z_0 into K as this homeomorphism. Then $h \circ p \circ h^{-1}$ is continuous in the sense of topology from K into K. Therefore by Brouwer's fixed point theorem there exists $x_0 \in K$ such that $h(p(h^{-1}(x_0))) = x_0$. Let $z_0 = h^{-1}(x_0)$. Then $p(z_0) = z_0$. Since $p(z_0) \in f^{-1}(z_0)$, it holds that $z_0 \in f^{-1}(z_0)$ proving that $z_0 \in f(z_0)$. □

In the above theorem, putting $Z = Y$ and $g = f$, the following theorem is obtained. It is Fan-Browder's fixed point theorem in a vector lattice.

Theorem 3.4. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that a mapping f from Y into 2^Y satisfies the following conditions:*

(1) *$f^{-1}(y)$ is non-empty and convex for any $y \in Y$;*

(2) *$f(y)$ is an open subset of X for any $y \in Y$.*

Then there exists $y_0 \in Y$ such that $y_0 \in f(y_0)$.

In the above theorem, changing from f to f^{-1}, the following theorem is obtained; see [6].

Theorem 3.5. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that a mapping f from Y into 2^Y satisfies the following conditions:*

(1) *$f^{-1}(y)$ is an open subset of X for any $y \in Y$;*
(2) *$f(y)$ is non-empty and convex for any $y \in Y$.*

Then there exists $y_0 \in Y$ such that $y_0 \in f(y_0)$.

Moreover the following holds. For the sake of completeness, we show its proof.

Theorem 3.6. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a compact convex subset of X and $A \subset Y \times Y$. Suppose that A satisfies the following conditions:*

(1) *$\{x \mid (x, y) \in A\}$ is closed for any $y \in Y$;*
(2) *$\{y \mid (x, y) \notin A\}$ is convex for any $x \in Y$;*
(3) *$(x, x) \in A$ for any $x \in Y$.*

Then there exists $x_0 \in Y$ such that $\{x_0\} \times Y \subset A$.

Proof. Assume that $\{x\} \times Y \not\subset A$ for any $x \in Y$. Then there exists $y \in Y$ such that $(x, y) \notin A$. Let $f(x) = \{y \mid (x, y) \notin A\}$. Then $f(x)$ is non-empty and by (2) it is convex. Moreover by (1) $f^{-1}(y) = \{x \mid (x, y) \notin A\} \in \mathcal{O}_X$. By Theorem 3.5 there exists $x_0 \in Y$ such that $x_0 \in f(x_0)$, that is, $(x_0, x_0) \notin A$. It is a contradiction. Therefore there exists $x_0 \in Y$ such that $\{x_0\} \times Y \subset A$. □

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