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LAGRANGIAN AND HAMILTONIAN NECESSARY CONDITIONS FOR THE GENERALIZED BOLZA PROBLEM AND APPLICATIONS

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Abstract. Our aim in this paper is to refine the well-known necessary optimality conditions for the general Bolza problem under a calmness assumption. We prove Lagrangian and Hamiltonian necessary optimality conditions without standard convexity assumptions. Our refinements consist in the utilization of a small subdifferential and in the presence of the maximum condition without convexity assumption on the velocities. Our approach lies in reducing the generalized Bolza problem in an optimal control problem governed by bounded and measurably Lipschitz differential inclusions. Our results allow us to simplify enough the proof of the maximum principle, to obtain a new Euler-Lagrange inclusion for optimal control problems of Mayer type and to develop Lagrangian and Hamiltonian necessary conditions for optimal control problems governed by nonconvex unbounded differential inclusions.

1. Introduction

The general Bolza problem (GBP) concerns the minimization of a Bolza functional whose form is identical to that in the Calculus of Variations :

$$B(x) := \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))dt.$$

The domain over which the minimization occurs is typically one of the functions $W^{1,1}([a,b], \mathbf{R}^n)$ (abbreviated $W^{1,1}$), consisting of all absolutely continuous functions $x\colon [a,b] \mapsto \mathbf{R}^n$ ($\dot{x}$ denotes the derivative (almost everywhere) of x). An *arc* is a function in $W^{1,1}$.

The generalized Bolza problem is distinguished from its classical precursor by the extremality mild hypotheses imposed on the endpoint cost ℓ and the integrand L. Both are allowed to take the value $+\infty$, for example, so that a variety of endpoint and differential constraints can be handled (e.g, Lagrange problems, differential inclusions problems, optimal control problems, etc.).

The classical theory of necessary conditions for the Bolza problem suggests that if an arc z minimizes B, then there should be an arc p satisfying the Euler-Lagrange equation, and the transversality condition

$$(\dot{p}(t), p(t)) = \nabla L(t, z(t), \dot{z}(t)) \quad a.e.\ t \in [a,b] \tag{1.1}$$

$$(p(a), -p(b)) = \nabla \ell(z(a), z(b)). \tag{1.2}$$

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Now in the general case, all the gradients in (1.1) and (1.2) could fail to exist. The first result concerning the nondifferentiable framework has been provided by Rockafellar in [23] and [22] under the convexity of ℓ and $L(t,\cdot,\cdot)$ with the use of the Fenchel subdifferential. Later, Clarke [3] established similar conditions with his subdifferential when the functions generalized these conditions to the locally Lipschitz case by using its subdifferential when the functions ℓ and $L(t,\cdot,\cdot)$ are locally Lipschitz, i.e. he obtained

$$(\dot{p}(t), p(t)) \in \overline{\partial} L(t, z(t), \dot{z}(t)) \quad a.e.t \in [a,b] \tag{1.3}$$

$$(p(a), -p(b)) \in \overline{\partial}\ell(z(a), z(b)). \tag{1.4}$$

where $\overline{\partial} f(x)$ stands for the Clarke's subdifferential of f at x. In [13]-[14], Loewen and Rockafellar obtain the following new conditions which are weaker than the Clarke's one by assuming the *convexity* of $L(t,x,\cdot)$

$$\dot{p}(t) \in \mathrm{co}\{q : \quad (q, p(t)) \in \partial L(t, z(t), \dot{z}(t))\} \quad a.e.t \in [a,b] \tag{1.5}$$

$$(p(a), -p(b)) \in \partial\ell(z(a), z(b)). \tag{1.6}$$

Here "co" stands for the convex hull and $\partial f(x)$ denotes the limiting proximal subdifferential which coincides with the Mordukhovich subdifferential [16]-[19] in finite dimensional spaces and which is smaller than the Clarke generalized gradient. Note that, since $L(t,x,\cdot)$ is convex, relation (1.5) automatically implies the following maximum condition

$$\langle p(t), \dot{z}(t)\rangle - L(t, z(t), \dot{z}(t)) = \max\{\langle p(t), v\rangle - L(t, z(t), v) : v \in \mathbf{R}^n\}. \tag{1.7}$$

In [4], Clarke established Hamiltonian necessary conditions for the generalized Bolza problem (GBP), i.e., he replaced the inclusion (1.3) by the following one

$$(-\dot{p}(t), \dot{z}(t)) \in \overline{\partial} H(t, z(t), p(t)) \ a.e. \tag{1.8}$$

where $H(t,x,p) = \sup\limits_{y}\{\langle p, y\rangle - L(t,x,y)\}$. He assumed that $L(t,x,\cdot)$ is convex and that H satisfies a strong Lipschitz condition.

Other necessary conditions for this problem already exist in the literature (see for example the papers [13]-[15], [8], etc. and references therein for comparisons with other existence results and the papers [7], [24], etc. and references therein for the relation between the Hamiltonian and the Lagrangien multipliers).

The case of nonconvex velocities turns out to be more complicated, unless strong regularity assumptions are imposed. First note that the convexity hypothesis seems to be rather restrictive and does not hold in many important applications. For example, in the case of the following optimal control problem

minimize $\ell(x(a), x(b))$

over the couples (x, u) satisfying :

$\dot{x}(t) = f(t, x(t), u(t)) \quad u(t) \in U(t) \quad a.e.t \in [a,b].$

This assumption is close to the linearity of f with respect to u and the convexity of the control set $U(t)$.

To our knowledge, the first result for nonconvex velocities is due to Mordukhovich [21] in which he provides necessary optimality conditions for a general variational problem for which the dynamic constraint is a nonconvex-valued (Lipschitz) differential inclusion. Inspired by the paper [21], Ioffe-Rockafellar [8] establish these necessary conditions by assuming that L is finite valued and satisfies the following assumption : for any $N > 0$ there are $\varepsilon_N > 0$ and a summable function $k_N(t)$ such that for almost all $t \in [a, b]$, for all $x, x' \in z(t) + \varepsilon_N \mathbf{B}$, and $y \in \dot{z}(t) + N\mathbf{B}$ one has

$$L(t, x, y) - L(t, x', y) \leq k_N(t)\|x - x'\|, \text{ and } L(t, x, y) \geq -k_N(t).$$

In [7], Ioffe used the result established in [8] to derive necessary optimality conditions for optimal control problems of unbounded integrably sub-Lipschitz (in the sense of Loewen-Rockafellar) differential inclusions. Based on the result by Ioffe-Rockafellar, Vinter and Zheng [25] provide necessary optimality conditions for problems involving an integral functional and a Lipschitz differential inclusion. Mordukhovich [20]-[21] established the same result for nonconvex bounded differential inclusions. It is worth to mention that the proof proposed in [20] is based on reducing (by the method of discrete approximations) the nonsmooth optimal control problem under consideration to a sequence of nonsmooth optimization problems.

A big step was exceeded by Clarke [5] who obtained Hamiltonian conditions for optimal control problems governed by bounded differential inclusions. These conditions are expressed in terms of the Clarke's subdifferential, i.e., inclusion (1.8). The (uniform) boundedness of the inclusion allows Clarke to work in $W^{1,2}$ instead of $W^{1,1}$ and to apply Stegall's variational principle, which holds in spaces having the Radon-Nikodym property. Note that this is not the case for $W^{1,1}$.

Our goal in this paper is to establish the necessary optimality conditions (1.5)-(1.7) and to refine the inclusion (1.8) with *nonconvexity* on the velocities. Our assumption on the Lagrangian L is more general (in some sense) than that of Ioffe-Rockafellar. Indeed we assume that L, which may be extended-valued, is epi-Lipschitzian in the sense of Clarke [4]. Our approach consists in reducing the generalized Bolza problem in an optimal control problem governed by a *bounded and measurably Lipschitz* (in the sense by Clarke [4]) differential inclusion. The key of our refined and generalized results is the use of a result by Ioffe [7]. Here we use only a variant of this result, i.e., necessary optimality of optimal control problems of nonconvex bounded and Lipschitz differential inclusions. Note that our result may be deduced (not directly) from [7] without reduction in an optimal control problem of bounded differential inclusions. We use intentionally this approach. Indeed, as we can easily deduce from [10] that necessary optimality conditions for optimal control problems of bounded Lipschitz (resp. unbounded integrably sub-Lipschitz) differential inclusions are equivalent. In other words, we can show that (the local version of) Mordukhovich's theorem [20] and Ioffe's theorem [7] are equivalent. It can be shown that both results are equivalent to those of Vinter-Zheng [25]. Our results facilitate enough the proof of the maximum principle, since any relaxation is needed to obtain the maximum condition and to obtain a new Euler-Lagrange inclusion for optimal control problems of Mayer type. Under standard assumptions, we also investigate Lagrangian and Hamiltonian necessary conditions for the following general variational problem for which the dynamic is an unbounded nonconvex differential inclusion

$$\text{minimize } \ell(x(a), x(b)) + \int_a^b f(t, x(t), \dot{x}(t))dt$$
$$\text{over the arcs } x \text{ satisfying :}$$
$$\dot{x}(t) \in F(t, x(t)) \quad a.e.\ t \in [a,b], (x(a), x(b)) \in S.$$

where $\ell : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ and $f : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ are given functions, $F : [a,b] \times \mathbf{R}^n \mapsto \mathbf{R}^n$ is a given multivalued mapping and $S \subset \mathbf{R}^n \times \mathbf{R}^n$ is a given nonempty set.

2. Notations and preliminaries

We begin by stating basic tools of generalized differentiation that are more appropriate for our main purpose. Details may be found in [16]-[19].

Let C be a closed subset of $\mathbf{R}^n$ containing some point c. The $\varepsilon-$normal cone to C at c is the set

$$\hat{N}_\varepsilon(C; c) := \{\xi \in \mathbf{R}^n : \liminf_{x \in C \to c} \frac{\langle -\xi, x - c \rangle}{\|x - c\|} \geq -\varepsilon\}.$$

The normal cone to C at c is the set

$$N(C; c) := \limsup_{\substack{x \in C \to c \\ \varepsilon \to 0^+}} \hat{N}_\varepsilon(C, x).$$

Now let $f : \mathbf{R}^n \mapsto \mathbf{R} \cup \{\infty\}$ be a lower semicontinuous (l.s.c.) function and let $c \in \mathbf{R}^n$ be such that $f(c) < \infty$. The limiting Fréchet subdifferential of f at c is the set

$$\partial f(c) := \{\xi \in \mathbf{R}^n : (\xi, -1) \in N(epi f; (c, f(c)))\}$$

where $epi f$ denotes the epigraph of f. We have the following analytic characterization of $\partial f(c)$:

$$\partial f(c) = \limsup_{\substack{x \to c \\ f(x) \to f(c) \\ \varepsilon \to 0^+}} \partial_\varepsilon f(x)$$

where

$$\partial_\varepsilon f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq -\varepsilon\}.$$

Next we consider a multivalued mapping G from $\mathbf{R}^n$ to $\mathbf{R}^m$ of the closed graph

$$GrG := \{(x, y) : y \in G(x)\}.$$

The multivalued mapping $D^*G(x, y) : \mathbf{R}^m \mapsto \mathbf{R}^n$ defined by

$$D^*G(x, y)(y^*) := \{x^* \in \mathbf{R}^n : (x^*, -y^*) \in N(GrG; (x, y))\}$$

is called the coderivative of G at the point $(x, y) \in GrG$.

Here, and throughout the paper, we will use $\| \cdot \|$ to denote both the euclidien norm of $\mathbf{R}^n$ and the norm of $W^{1,1}$ ($\|x\| = \|x(a)\| + \int_a^b \|\dot{x}(t)\| dt$), $\mathbf{B}$ to denote the closed unit ball of $\mathbf{R}^k$, and $B(z, r)$ to designate the closed ball in $W^{1,1}$ of center z and of radius r.

Now we state the result by Ioffe [7] which will be used later. We consider the following problem which we call (P_0):

minimize $\ell(x(a), x(b))$

over the arcs x satisfying :

$\dot{x}(t) \in F(t, x(t)) \quad a.e. t \in [a, b], \quad (x(a), x(b)) \in S.$

We recall that $F(t, x)$ is *measurably Lipschitzian* [4] at an arc z if it is locally Lipschitzian in z with a summable modulus and measurable in t.

Theorem 2.1. *Let z be a local optimal solution to the problem (P_0) (in $W^{1,1}$). Assume that F is closed-valued and measurably Lipschitzian at z and bounded by a summable function around $z(t)$ a.e. in $[a, b]$, and that ℓ is locally Lipschitzian around $(z(a), z(b))$ while S is closed. Then there are $\lambda \geq 0$ and an arc $p : [a, b] \mapsto \mathbf{R}^n$, not both zero, such that one has :*

$$\dot{p}(t) \in coD^*F(t, z(t), \dot{z}(t))(-p(t)) \quad a.e.\ t \in [a, b]$$

$$(p(a), -p(b)) \in \lambda\partial\ell(z(a), z(b)) + N(S; (z(a), z(b))$$

$$\langle p(t), \dot{z}(t)\rangle = \max_{v \in F(t,z(t))} \langle p(t), v\rangle \quad a.e.\ t \in [a, b].$$

Note that Ioffe [7] established this result for unbounded integrably sub-Lipschitz (in the sense of Loewen-Rockafellar) differential inclusions. As it can be deduced from [10] (which is a carrying of the present paper) those necessary optimality conditions for optimal control problems of bounded Lipschitz (resp. unbounded integrably sub-Lipschitz) differential inclusions are equivalent, whence the approach that we propose in Section 4.

In [21] (see also [20]), Mordukhovich established these necessary conditions, with global minimum instead of local minimum, by using discrete approximations and doesn't require any relaxation procedure.

3. Lagrangian conditions for the problem (GBP)

In this section we state our main result on necessary optimality conditions for the problem (GBP). Here, and throughout the paper we impose the following:

(H) BASIC HYPOTHESES. The functions $L : [a, b] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ and $\ell : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ are such that for each $t \in [a, b]$, the functions $L(t, \cdot, \cdot)$ and ℓ are l.s.c. on $\mathbf{R}^n \times \mathbf{R}^n$.

In this section and the next ones we describe necessary conditions, for which some additional hypotheses are required. But before proceeding to state our main result, let us give some definitions.

We begin by the following definition of Clarke [3] on epi-Lipschitzness.

Definition 3.1. The function L is epi-Lipschitzian at an arc z if there exist an integrable function $k : [a, b] \mapsto \mathbf{R}$ and a positive ε satisfying the following conditions : for almost all $t \in [a, b]$, given two points z_1 and z_2 within ε of $z(t)$ and $u_1 \in \mathbf{R}^n$ such that $L(t, z_1, u_1)$ is finite, there exist a point $u_2 \in \mathbf{R}^n$ and $\delta \geq 0$ such that $L(t, z_2, u_2)$ is finite and

$$\|u_1 - u_2\| + |L(t, z_1, u_1) - L(t, z_2, u_2) - \delta| \leq k(t)\|z_1 - z_2\|.$$

The above definition is equivalent to saying that the multivalued mapping

$$E(t,x) = \{(u,r) \in \mathbf{R}^n \times \mathbf{R} : L(t,x,u) \le r\}$$

is Lipschitzian in x on $z(t) + \varepsilon\mathbf{B}$.

Examples of such functions are given in the Clarke's paper [3]. We shall adopt the following conventions. If for a given absolutely continuous function x the integral or the sum in (GBP) is not defined, we set the functional in (GBP) equal to $+\infty$. To say that z solves locally (GBP) will mean the following : for $x = z$ the integral in (GBP) is defined and finite and $\ell(x(a), x(b))$ is finite ; for another absolutely continuous function in some $W^{1,1}$-neighbourhood of z for which $\ell(x(a), x(b))$ is finite and the integral in (GBP) defined, the value of the problem is no less than its value at z. We do not rule out the possibility that the integral in (GBP) equal $-\infty$ for some absolutely continuous function x.

With these conventions, for any ε in $(0, \infty)$ and $s \in \mathbf{R}^n$, we define

$$\varphi^0_\varepsilon(s) = \inf\{\ell(x(a)+s, x(b)) + \int_a^b L(t, x(t), \dot{x}(t))dt : x \in W^{1,1},\ \|x - z\| \le \varepsilon\}$$

and we define $\varphi^1_\varepsilon(s)$ similary for $\ell(x(a)+s, x(b))$ replaced by $\ell(x(a), x(b)+s)$.

Definition 3.2 ([3])**.** The problem (GBP) is calm at z if for some $\varepsilon > 0$ and for $i = 0$ or 1 we have $\varphi^i_\varepsilon(0) \in \mathbf{R}$ and

$$\liminf_{s\to 0} \quad \frac{\varphi^i_\varepsilon(s) - \varphi^i_\varepsilon(0)}{\|s\|} > -\infty.$$

If either of the following is satisfied, then [3] the generalized Bolza problem is calm at z :

$$\ell(z_0, z_1) = \ell_0(z_0) + \ell_1(z_0, z_1),$$

where ℓ_1 is finite and Lipschitzian in z_1 in a neighbourhood of $(z(a), z(b))$;

$$\ell(z_0, z_1) = \ell_1(z_1) + \ell_0(z_0, z_1),$$

where ℓ_0 is finite and Lipschitzian in z_0 in a neighbourhood of $(z(a), z(b))$.

Definition 3.3 ([3])**.** L is said to be epi-measurable (in t) if for each $s \in \mathbf{R}^n$, the multivalued mapping $E(t,s) = epiL(t,s,.)$ is Lebesgue measurable in t.

From now and in the rest of the paper we will assume that local minimum and calmness are formulated with the same real, i.e., if this real is, for example, equal to ε then the minimum is attained on some ball of radius ε and the calmness is expressed with the same ε.

The notation ∂L will denote the limiting Fréchet subdifferential of the function $L(t,.,.)$.

Now we are in position to state necessary optimality conditions for the generalized Bolza problem.

Theorem 3.4. *Let z solves locally the generalized Bolza problem (GBP) (in $W^{1,1}$), where the problem is calm at z. Suppose that $L(t,x,u)$ is epi-measurable in t, and epi-Lipschitzian at z. Then there exists an arc p such that one has :*

the Euler-Lagrange inclusion (1.5)

the transversality inclusion (1.6)
the maximum condition (1.7).

We begin with the following simplification : we shall assume that the calmness condition of Definition 3.2 holds for $i = 1$. Where this is not the case, we could return to this situation by replacing t by $(a + b) - t$ throughout, and $L(t, x, y)$ by $L((a + b) - t, x, -y)$. The equivalent transformation would satisfy the calmness condition at 1.

We set σ equal to the following finite number :

$$\sigma = -\min\{0, \liminf_{s \to 0} \frac{\varphi^1_\varepsilon(s) - \varphi^1_\varepsilon(0)}{\|s\|}\}.$$

Reformulation of the problem (GBP)

We adopt the following convention : s^* will refer to a point of the form (s^1, s^2, s^3, s^4) in $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$. Similary, an arc x^* has component arcs x^1, x^3 in $W^{1,1}$ and x^2, x^4 in $W^{1,1}([a, b], \mathbf{R})$. We define ([3]) the multivalued mapping E by

$$E(t, s^*) = \{(v, r, 0, 0) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} : L(t, s^1, v) \leq r\}$$

for $\|s^1 - z(t)\| \leq \frac{\varepsilon}{2}$, and $E(t, s^*)$ empty otherwise. We also define

$$z^*(t) = (z(t), \int_a^t L(s, z(s), \dot{z}(s))ds, z(b), \ell(z(a), z(b)))$$

$$C_a = \{s^* : l(s^1, s^3) \leq s^4, \, s^2 = 0\}, \quad C_b = \{s^* : s^1 = s^3\}.$$

We set $m = 2(\sigma + 1)$. We have the following Clarke's result.

Proposition 3.5 ([3]). *Let the assumptions of Theorem 3.4 be satisfied. Then, for some positive δ, the arc z^* minimizes*

$$x^2(b) + x^4(b) + md(x^*(b), C_b)$$

over the arcs x^* satisfying :
$x^*(a) \in C_a$, $\dot{x}^*(t) \in E(t, x^*(t))$ *a.e.* $\|x^1 - z\| \leq \delta$, $\|x^3 - z(b)\| \leq \delta$.

4. Proof of Theorem 3.4

We begin this section by establishing necessary optimality conditions for problem (GBP) with some additional assumptions on the integrand L. We have the following theorem on which our main result is based.

Theorem 4.1. *Suppose in addition to the assumptions of Theorem 3.4 that there exist $\varepsilon > 0$, $K > 0$ and integrable functions k_1, $k_2 : [a, b] \mapsto \mathbf{R}$ such that for almost all $t \in [a, b]$, $L(t, \cdot, \cdot)$ is finite-valued on $(z(t) + \varepsilon\mathbf{B}) \times \mathbf{R}^n$, for all $s_1, s_2 \in z(t) + \varepsilon\mathbf{B}$, $u_1, u_2 \in \dot{z}(t) + k_2(t)\mathbf{B}$ and all $u \in \mathbf{R}^n$, one has :*

i) $|L(t, s_1, u_1) - L(t, s_2, u_2)| \leq k_1(t)(\|s_1 - s_2\| + \|u_1 - u_2\|)$;
ii) $|L(t, s_1, u) - L(t, s_2, u)| \leq k_1(t)\|s_1 - s_2\|$;
iii) $|L(t, s_1, u) - L(t, s_1, \dot{z}(t))| \leq K\|u - \dot{z}(t)\|$.

Then the conclusion of Theorem 3.4 holds.

For each integer j, we consider the multivalued mapping G_j defined by

$$G_j(t, s^*) = E(t, s^*) \cap B_j(t)$$

where

$$\begin{aligned} B_j(t) &= \{\dot{z}(t) + (k_2(t) + j)\overline{\mathbf{B}}\} \times \{L(t, z(t), \dot{z}(t)) + \alpha_j(t)[-1, 1]\} \\ &\quad \times \{z(b) + (\|z(b)\| + 1)\overline{\mathbf{B}}\} \times \{\ell(z(a), z(b)) + (|\ell(z(a), z(b))| + 1)[-1, 1]\} \\ \alpha_j(t) &= K(k_2(t) + j) + 4k_1(t)\varepsilon. \end{aligned}$$

Here $\overline{\mathbf{B}}$ denotes the closed unit ball of $\mathbf{R}^n$. We set

$$\gamma_j(t) = 2\|\dot{z}(t)\| + k_2(t) + j + |L(t, z(t), \dot{z}(t))| + \alpha_j(t) + 2\|z(b)\| + 2|\ell(z(a), z(b))| + 3.$$

With the above notations, we have the following lemmas.

Lemma 4.2. *G_j is closed-valued and measurably Lipschitzian (in the sense of Clarke [4]) of constant $k_1(t)$ and bounded by the summable function γ_j around $z(t)$ a.e. in $[a, b]$.*

Proof. We only need to show that $G_j(t, \cdot)$ is Lipschitzian of constant $k_1(t)$. So let (s^*, q^*) be such that $s^1, q^1 \in z(t) + \varepsilon\mathbf{B}$ and let $(v^1, r^1, 0, 0) \in G_j(t, s^*)$. Then $L(t, s^1, v^1) \le r^1$. If $L(t, q^1, v^1) \le r^1$, then $(v^1, r^1, 0, 0) \in G_j(t, q^*)$. If $r^1 < L(t, q^1, v^1)$, for $v^2 = v^1$ we can write (via *ii)*)

$$r^1 < L(t, q^1, v^2) \le L(t, s^1, v^2) + k_1(t)\|s^1 - q^1\|.$$

Hence, if we put $r^2 = L(t, s^1, v^2) + k_1(t)\|s^1 - q^1\|$, we obtain

$$0 < r^2 - r^1 \le k_1(t)\|s^1 - q^1\|.$$

Further, by *ii)* and *iii)*, we have

$$|r^2 - L(t, z(t), \dot{z}(t))| \le \alpha_j(t).$$

Thus $(v^2, r^2, 0, 0) \in G_j(t, q^*)$ and hence

$$(v^1, r^1, 0, 0) \in G_j(t, q^*) + k_1(t)\|s^* - q^*\|(\overline{\mathbf{B}} \times [-1, 1] \times \overline{\mathbf{B}} \times [-1, 1]).$$

□

Lemma 4.3. *For some positive δ, the arc z^* minimizes*

$$x^2(b) + x^4(b) + md(x^*(b), C_b)$$

over the arcs x^* satisfying :
$x^*(a) \in C_a$, $\dot{x}^*(t) \in G_j(t, x^*(t))$ *a.e.*, $\|x^1 - z\| \le \delta$, $\|x^3 - z(b)\| \le \delta$.

Proof of Theorem 4.1. By Lemma 4.3 and Theorem 2.1 there are $\lambda \ge 0$ and an arc $p^* : [a, b] \mapsto \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$, not both zero, such that

$$\dot{p}^*(t) \in \mathrm{co}D^*G_j(t, z^*(t), \dot{z}^*(t))(-p^*(t)) \quad a.e.\ t \in [a, b] \tag{4.1}$$

$$\begin{aligned} (p^*(a), -p^*(b)) \in\ & N(C_a; z^*(a)) \times \{0\} + \{0\} \\ & \times \lambda[(0, 1, 0, 1) + m\partial d(z^*(b); C_b)] \end{aligned} \tag{4.2}$$

$$\langle p^*(t), \dot{z}^*(t)\rangle = \max_{v^* \in G_j(t,z^*(t))} \langle p^*(t), v^*\rangle \quad a.e.\ t \in [a,b]. \tag{4.3}$$

We deduce from (4.1) that $\dot{p}^2$, $\dot{p}^3$, $\dot{p}^4$ are 0 a.e., because G_j depends only on s^1 and from (4.2)

$$p^2(b) = p^4(b) = -\lambda, \quad p^1(b) = -p^3(b) \quad \text{and} \quad \|p^1(b)\| \le m\lambda.$$

Note that $\lambda > 0$, otherwise $(\lambda, p^*) = (0,0)$. Indeed, if $\lambda = 0$ then $p^2 = p^4 = 0$, $p^3 = 0$ and $p^1(b) = 0$. By relation (4.1) and the Lipschitz property of G_j we get (Lemma 4.2)

$$\|\dot{p}^1(t)\| \le k_1(t)\|p^1(t)\|\ a.e.$$

and, by Gronwall Lemma, we get $p^1 = 0$ and this contradicts the fact that $(\lambda, p^*) \neq (0,0)$. So we may assume that $\lambda = 1$ and hence $p^2(t) = -1$ for all t. By the definition of G_j, we get

$$N(\mathrm{Gr}E(t,\cdot),(z^*(t),\dot{z}^*(t)) = \{(w,0,0,0,v,\alpha,u,\beta) :\\ \alpha,\beta \in \mathbf{R},\ u \in \mathbf{R}^n,\ (w,v,\alpha) \in N(\mathrm{epi}L(t,\cdot,\cdot),(z(t),\dot{z}(t),L(t,z(t),\dot{z}(t)))\}.$$

The last equality together with $p^2(t) = -1$ and relation (4.1) ensures that

$$\dot{p}^1(t) \in \mathrm{co}\{q : \quad (q,p^1(t)) \in \partial L(t,z(t),\dot{z}(t))\} \quad a.e. t \in [a,b] \tag{4.4}$$

$$(p^1(a), -p^1(b)) \in\ \partial\ell(z(a),z(b)) \tag{4.5}$$

It is now time to recall that all the above has been obtained for any j, so that in the key relations (4.3)-(4.5), the quantity p^1 depends on j, which we denote p_j. Note that, by (4.4) and the assumption $ii)$

$$\|\dot{p}_j(t)\| \le k_1(t)\, a.e.\ t \in [a,b]$$

and hence

$$\|p_j(a)\| \le m + \int_a^b k_1(t)dt.$$

So, because of $i)$, the multivalued mapping Γ defined by

$$\Gamma(t,p) = \mathrm{co}\{q : \quad (q,p) \in \partial L(t,z(t),\dot{z}(t))\}$$

satisfies all the assumptions of Theorem 3.1.7 in [4] and then there is a subsequence of (p_j) which converges uniformly to an arc p satisfying

$$\dot{p}(t) \in \Gamma(t,p(t))\, a.e.\ t \in [a,b].$$

Thus, by (4.5) and (4.3), one has

$$(p(a), -p(b)) \in \partial\ell(z(a),z(b))$$

and for almost every $t \in [a,b]$

$$\langle p(t),\dot{z}(t)\rangle - L(t,z(t),\dot{z}(t)) = \max\{\langle p(t),v\rangle - L(t,z(t),v) :\ v \in \mathbf{R}^n\}.$$

□

Proof of Theorem 3.4 with the general assumptions. First we state the following lemma which can be deduced from Theorem 4.1.

Lemma 4.4 (compare with Lemma 2, pp. 124 in [4])**.** *Let C be a closed set in $\mathbf{R}^n$ and let $F : [a,b] \times \mathbf{R}^n \mapsto \mathbf{R}^n$ be a closed-valued multivalued mapping which is measurably Lipschitzian at z. Let S be the set of solutions of the system*

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e.}, \quad x(a) \in C.$$

If $z \in S$, then there are $r > 0$ and $\alpha > 0$ such that for all $x \in B(z, r)$

$$d(x, S) \leq \alpha[d(x(a), C) + \int_a^b d(\dot{x}(t), F(t, x(t)))dt].$$

Now it remains to remove the Interim Hypotheses. By Proposition 3.5, for some positive δ, the arc z^* minimizes

$$x^2(b) + x^4(b) + md(x^*(b), C_b)$$

over the arcs x^* satisfying :
$x^*(a) \in C_a$, $\dot{x}^*(t) \in E(t, x^*(t))$ *a.e.* $\|x^1 - z\| \leq \delta$, $\|x^3 - z(b)\| \leq \delta$.
Thus by Lemma 4.4 there exists a positive number $K > 2$ such that z^* minimizes locally the function

$$x^* \mapsto x^2(b) + x^4(b) + md(x^*(b), C_b) + Kd(x^*(a), C_a) + K\int_a^b d(\dot{x}^*(t), E(t, x^*(t)))dt. \tag{4.6}$$

Now we are in the situation of Theorem 4.1 with L and ℓ replaced by $\overline{L}$ and $\overline{\ell}$ defined by

$$\overline{L}(t, s^*, q^*) = Kd(q^*, E(t, s^*)$$

and

$$\overline{\ell}(u^*, v^*) = v^2 + v^4 + md(v^*, C_b) + Kd(u^*, C_a).$$

Taking into account the inequality (for some constant $\overline{K}(t)$ depending only on t in a neighbourhood of $(z^*(t), \dot{z}^*(t))$)

$$\overline{L}(t, s^*, q^*) \leq \overline{K}(t)[d(epiL(t, \cdot, \cdot); (s^1, q^1, q^2)) + \|q^3\| + \|q^4\|]$$

we obtain

$$\partial\overline{L}(t, z^*(t), \dot{z}^*(t)) \subset \{(\xi^*, \pi^*) : (\xi^1, \pi^1, \pi^2) \in N(epiL(t, \cdot, \cdot); ((z(t), \dot{z}(t)), L(t, (z(t), \dot{z}(t))); \; \xi^2 = \xi^4 = 0, \; \xi^3 = 0\}. \tag{4.7}$$

So the proof is terminated by applying Theorem 4.1. □

We may obtain the following corollary.

Corollary 4.5. *Let z solves locally the generalized Bolza problem (GBP) (in $W^{1,1}$), where the problem is calm at z. Suppose that*

i) *for any $N > 0$ there are $\varepsilon > 0$ and a summable function $k_N(t)$ such that for almost all $t \in [a,b]$, for all $x, x' \in z(t) + \varepsilon\mathbf{B}$, and $y \in \dot{z}(t) + N\mathbf{B}$, with $L(t, x, y)$ finite, there exist $y' \in \dot{z}(t) + N\mathbf{B}$, with $L(t, x', y')$ finite, and $\delta \geq 0$ such that*

$$\|y - y'\| + |L(t, x, y) - L(t, x', y') - \delta| \leq k_N(t)\|x - x'\|;$$

ii) *there exists an integrable function* k *such that for almost all* $t \in [a,b]$ *the partial subdifferential* $\partial_x L(t, z(t), \dot{z}(t))$ *(*[11], [9]*) of* L *in* x *at* $(z(t), \dot{z}(t))$ *is included in* $k(t)\mathbf{B}$.

iii) *there exists a set* $I \subset [a,b]$ *of positive measure such that for almost all* $t \in I$,

$$N(co[domL(t, z(t), \cdot)], \dot{z}(t)) = \{0\}.$$

Then the conclusion of Theorem 3.4 holds.

Proof. Consider the function $L_N : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ defined by

$$L_N(t,x,y) = \begin{cases} L(t,x,y) & \text{if } (x,y) \in A(t), \\ +\infty & \text{othewise.} \end{cases}$$

where $A(t) = (z(t) + \varepsilon\mathbf{B}) \times \dot{z}(t) + N\mathbf{B}$. Then L_N satisfies all the assumptions of Theorem 3.4 which may be applied to produce an arc $p_N \in W^{1,1}$ satisfying in addition to (1.5) and (1.6), the following maximum condition

$$\langle p_N(t), \dot{z}(t)\rangle - L_N(t, z(t), \dot{z}(t)) = \max\{\langle p_N(t), v\rangle - L_N(t, z(t), v) : \ v \in \mathbf{R}^n\} \ a.e.$$

We will show that the sequence $(p_N(a))$ is bounded. To do this we consider, for each N, the set

$$I_N = \{t \in I : \ N(\text{co}[\text{dom}L(t, z(t), \cdot)], \dot{z}(t)) = \{0\}, \\ \langle p_N(t), \dot{z}(t)\rangle - L_N(t, z(t), \dot{z}(t)) = \max\{\langle p_N(t), v\rangle - L_N(t, z(t), v) : \ v \in \mathbf{R}^n\} \}$$

and put $I_\infty = \bigcap_N I_N$.

We claim that for each $t \in I_\infty$, the sequence $(p_N(t))_N$ is bounded. Suppose the contrary. Then for some $t \in I_\infty$, $(p_N(t))_N$ is unbounded. Without loss of generality, we may assume that $\|p_N(t)\| \to +\infty$ and $w_N := \frac{p_N(t)}{\|p_N(t)\|} \to w$, with $\|w\| = 1$. The maximum condition ensures that for each $v \in \text{dom}L(t, z(t), \cdot)$ and $N \geq \|v - \dot{z}(t)\|$

$$\langle w_N(t), \dot{z}(t)\rangle - \frac{1}{\|p_N(t)\|} L(t, z(t), \dot{z}(t)) \geq \langle w_N(t), v\rangle - \frac{1}{\|p_N(t)\|} L(t, z(t), v).$$

Thus we get $w \in N(\text{co}[\text{dom}L(t, z(t), \cdot)], \dot{z}(t))$ and this contradicts *iii)* and establishes our claim.

Now relation (1.5) and *ii)* imply that

$$\|p_N(a) - p_N(t)\| \leq \int_a^t \|\dot{p}_N(\tau)\| d\tau \leq \int_a^t k(\tau) d\tau$$

which asserts that our sequence $(p_N(a))$ is bounded. So that the proof is terminated by applying Theorem 3.1.7 in [4]. □

5. Hamiltonian conditions for the problem (GBP)

We define the Hamiltonian $H : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ associated to the Lagrangian L by

$$H(t,x,p) = \sup\{\langle p, y\rangle - L(t,x,y) : \ y \in \mathbf{R}^n\}.$$

Based on Theorem 3.4, on the Rockafellar's dualization result [24] and on the relaxation theorem by Clarke [3] we may establish the following Hamiltonian necessary optimality result.

Theorem 5.1. *Under assumptions of Theorem 3.4 there exists an arc $p \in W^{1,1}$ satisfying the transversality inclusion (1.6), the maximum condition (1.7) and the following Hamiltonian condition*

$$\dot{p}(t) \in \ co\ \{q:\ (-q, \dot{z}(t)) \in \partial H(t, z(t), p(t))\}\ a.e.\ t \in [a,b]$$

Proof. The arc z^* continues to minimize locally the function in relation (4.6) by replacing $E(t,x^*)$ by its convex closure cl co$E(t,x^*)$ (this is due to the relaxation theorem [3]). We observe further that cl co$E(t,x^*)$ satisfies the measurability assumption and the Lipschitz property. So by Theorem 3.4 there exists an arc $p^* \in W^{1,1}([a,b], \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R})$ such that

$$\dot{p}^*(t) \in \text{co}\ \{q^*:\ (q^*, p^*) \in K\partial d(\cdot; \text{cl co}E(t,\cdot))(z^*(t), \dot{z}^*(t))\}\ a.e.$$

$$(p^*(a), -p^*(b)) \in N(C_a; z^*(a)) \times \{0\} + \{0\} \times [(0,1,0,1) + m\partial d(z^*(b); C_b)]$$

$$\langle p^*(t), \dot{z}^*(t)\rangle = \sup_{y^*}\{\langle p^*(t), y^*\rangle - Kd(y^*; \text{cl co}E(t, z^*(t))\}\ a.e.$$

So that $\dot{p}^2$, $\dot{p}^3$, $\dot{p}^4$ are 0 because cl co$E$ depends only on x^1 and hence

$$p^2(b) = p^4(b) = -1, \quad p^1(b) = -p^3(b).$$

Now by Rockafellar result [24] we get

$$\dot{p}^*(t) \in \text{co}\ \{q^*:\ (-q^*, \dot{z}^*(t)) \in H^*(t, z^*(t), p^*(t))\}\ a.e.$$

where H^* is the Hamiltonian defined by

$$H^*(t,x^*,p^*) = \sup\{\langle p^*, y^*\rangle - Kd(y^*; \text{cl co}E(t,x^*)) :\ y^* \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}\}.$$

Note that $\max_{t\in[a,b]} \|p^*(t)\| \le K$. For each p^*, with $\|p^*\| \le K$, we get $H^*(t,x^*,p^*) = \sup_{y^* \in E(t,x^*))} \langle p^*, y^*\rangle$.

The proof of the theorem is terminated by using the following lemma.

Lemma 5.2. *Let $\bar{x}^*$, $\bar{p}^* \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$, with $\bar{p}^2 = -1$, and let $(u^*, v^*) \in \partial H^*(t, \bar{x}^*, \bar{p}^*)$. Then $u^{*,2} = u^{*,4} = v^{*,4} = 0$, $u^{*,3} = v^{*,3} = 0$ and*

$$(u^{*,1}, v^{*,1}) \in \partial H(t, \bar{x}^1, \bar{p}^1).$$

Proof. Consider the Hamiltonian $\bar{H}$ defined by

$$\bar{H}(t,x,p,\lambda) = \sup\{\langle p, y\rangle + \lambda r:\ (y,r) \in \text{epi}L(t,x,\cdot)\}.$$

Then for each p^*, with $\|p^*\| \le K$, we have

$$H^*(t,x^*,p^*) = \bar{H}(t,x^1,p^1,p^2).$$

So if $(u^*,v^*) \in \partial H^*(t,\bar{x}^*,\bar{p}^*)$ then $u^{*,2} = u^{*,4} = v^{*,4} = 0$, $u^{*,3} = v^{*,3} = 0$ and $(u^{*,1}, v^{*,1}, v^{*,2}) \in \partial \bar{H}(t, \bar{x}^1, \bar{p}^1, \bar{p}^2)$. Thus, since $\bar{p}^2 = -1$, we use the definition of the limiting Fréchet subdifferential to obtain

$$(u^{*,1}, v^{*,1}) \in \partial H(t, \bar{x}^1, \bar{p}^1)$$

and the proof is complete. □

Remark 5.3. Our theorem refines and generalizes Theorem 4.2.2 in [4]. Indeed in [4] it is assumed that $L(t,x,\cdot)$ is convex and H satisfies a strong Lipschitz condition.

Corollary 5.4. *Consider the generalized Bolza problem (GBP) with* $\ell(x,y) = \ell_1(x) + \ell_2(x,y)$ *where we suppose that* ℓ_2 *is finite and Lipschitz in* y *in some neighbourhood of* $(z(a), z(b))$ *and* L *is epi-measurable and epi-Lipschitzian at the local solution* z *of (GBP). Then the conclusion of Theorem 5.1 remains valid.*

6. Applications

In this section we give several applications of our results. The first one is the maximum principle and the second and the third ones concern Lagrangian and Hamiltonian necessary optimality conditions for optimal control of unbounded nonconvex differential inclusions. The last one concerns a general variational problem for which the dynamic is an unbounded nonconvex differential inclusion.

6.1. **The maximum principle.**

Our goal in this subsection is to give a simple proof of the maximum principle using Theorem 3.4 and to obtain a new Euler-Lagrange inclusion for optimal control problem of Mayer type.

To every integrable map $u(t)$ taking values in a given set $U(t)$ of $\mathbf{R}^m$ we associate the solution x to the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)).$$

The problem : minimize over all such pairs (x,u) the functional

$$g_0(x(a), x(b)) + \int_a^b g(t, x(t), u(t))dt$$

where $f : [a,b] \times \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ is a mapping, $g_0 : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ and $g : [a,b] \times \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R} \cup \{+\infty\}$ are functions, and $U : [a,b] \mapsto \mathbf{R}^m$ is a multivalued mapping.

The calmness of this problem is defined in a maner completely analogous to Definition 3.2. We shall assume the following assumptions :

1) $f(t,x,u)$ is measurable in t and continuous in u
2) $g(t,x,u)$ is l.s.c. in u and for each x, $g(.,x,.)$ is measurable with respect to the σ-field generated by Lebesgue sets in $[a,b]$ and Borel sets in $\mathbf{R}^m$.
3) g_0 is l.s.c.
4) U is measurable and closed-valued.

To this problem we associate the Hamiltonian H defined by

$$H(t,x,p,q) = \sup_{u \in U(t)} \{\langle p, f(t,x,u)\rangle + \langle q, u\rangle - g(t,x,u)\}.$$

Theorem 6.1. *(The maximum principle). Suppose* (z,v) *solves locally the above control problem, where*

(i) *the problem is calm at* z.

(ii) *there exist $\varepsilon > 0$ and an integrable function $k : [a,b] \to \mathbf{R}$ such that, for almost all $t \in [a,b]$, given two points z_1 and z_2 within ε of $z(t)$ and $u \in U(t)$ we have*

$$\|f(t,z_1,u) - f(t,z_2,u)\| \le k(t)\|z_1 - z_2\|$$
$$|g(t,z_1,u) - g(t,z_2,u)| \le k(t)\|z_1 - z_2\|.$$

Then there exists an arc p such that

$\dot p(t) \in co\{q : \ (-q, \dot z(t), v(t)) \in \partial H(t, z(t), p(t), 0)\}$ *a.e.*

(*Hamiltonian inclusion*)

$(resp. \dot p(t) \in co\, \partial_x[g(t,.,.) - < p(t), f(t,.,.) >](z(t), v(t))$ *a.e.*)

(*Euler-Lagrange inclusion*))

$(p(a), -p(b)) \in \partial g_0(z(a), z(b))$

(*Transversality inclusion*)

$\langle p(t), f(t, z(t), v(t))\rangle - g(t, z(t), v(t)) = H(t, z(t), p(t), 0)$ *a.e.*

If in addition $f(t,\cdot,\cdot)$ and $g(t,\cdot,\cdot)$ are locally Lipschitzian around $(z(t), v(t))$ then the Euler-Lagrange inclusion may be replaced by the following one :

$\dot p(t) \in co\{q : \ (q, 0) \in \ \partial[g(t,\cdot) - \langle p(t), f(t,\cdot)\rangle + \Psi_{U(t)}(\cdot)](z(t), v(t))$ *a.e.*

Here $\partial_x h(z,v)$ denotes the partial limiting Fréchet subdifferential of h in z with respect to v (see [11]) and Ψ_C denotes the indicator function of C.

Proof. We reframe ([3]) the problem to one on $\mathbf{R}^{n+m}$ by the following definitions ((s,s^*) represents a point in $\mathbf{R}^n \times \mathbf{R}^m$)

$$\ell(s_0, s_0^*, s_1, s_1^*) = g_0(s_0, s_1)$$

$$L(t, s, s^*, u, u^*) = \begin{cases} g(t,s,u^*) & \text{if } u^* \in U(t),\ u = f(t,s,u^*), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the pair (z, v) solves the optimal control problem iff (z, w) solves the above generalized Bolza problem , where $w(t) = \displaystyle\int_a^t v(r)dr$. By Propositions 3 and 7 in [3], L is epi-measurable and epi-Lipschitzian at (z, w). So we apply Theorem 3.4 to the last problem. The proof of the last condition, which does not appear in previous works, follows from the computation of $\partial L(t, z(t), w(t), \dot z(t), v(t))$ because if $(\xi_1^*, \xi_2^*, \xi_3^*, \xi_4^*) \in \partial L(t, z(t), w(t), \dot z(t), v(t))$, then

$$\xi_2^* = 0 \ \text{ and } \ (\xi_1^*, \xi_4^*) \in \partial[g(t,\cdot,\cdot) - \langle \xi_3^*, f(t,\cdot,\cdot)\rangle + \Psi_{U(t)}(\cdot)](z(t), v(t)).$$

For the proof of the Hamiltonian condition we consider in (4.6) the arc

$$z^*(t) = (z(t), w(t), \int_a^t g(t, z(t), v(t))dt, z(b), w(b), g_0(z(a), z(b)))$$

and the multivalued mapping E defined by

$$E(t, x^*) = \text{epi } \tilde L(t, x^1, \cdot, \cdot) \times \{0\}$$

where $x^* = (x^1, w^1, r^1, x^2, w^2, r^2) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ and

$$\tilde L(t, s, u, u^*) = \begin{cases} g(t,s,u^*) & \text{if } u^* \in U(t),\ u = f(t,s,u^*), \\ +\infty & \text{otherwise.} \end{cases}$$

By remarking that

$$d((y^1,v^1,r^1,y^2,v^2,r^2);\mathrm{co}E(t,x^*)) = \\ d((y^1,v^1,r^1);\mathrm{co\ epi}\ \tilde{L}(t,x^1,\cdot,\cdot)) + |y^2| + |v^2| + |r^2|$$

we get, as in the proof of Theorem 5.1, the existence of an arc p satisfying, in addition to the transversality condition and the maximum condition, the following Euler-Lagrange inclusion

$$\dot{p}(t) \in \mathrm{co}\ \{q:\ (q,p(t),0,-1) \in \\ \partial K(t)d((z(t),\dot{z},v(t),g(t,z(t),v(t));\mathrm{co\ epi}\tilde{L}(t,\cdot))\}.$$

for some $K(t) > 0$. Thus by Theorem 3.3 in [24] we have

$$\dot{p}(t) \in \mathrm{co}\ \{q:\ (-q,\dot{z},v(t),g(t,z(t),v(t))) \in \partial\tilde{H}(t,z(t),p(t),0,-1)\}$$

where $\tilde{H}(t,x,p,q,r) = \sup_{(y,u,s)\in\mathrm{epi}\ \tilde{L}(t,x,\cdot)} \{\langle p,y\rangle + \langle q,u\rangle + rs\}$. The proof is terminated by using the following lemma.

Lemma 6.2. *If* $(p_1,p_2,p_3,p_4) \in \partial\tilde{H}(t,z(t),p(t),0,-1)$ *then* $(p_1,p_2,p_3) \in \partial H(t,z(t),p(t),0)$.

Proof. This follows from the definition of the limiting Fréchet subdifferential and the fact that $\tilde{H}(t,x,p,q,r) = -rH(t,x,\frac{p}{-r},\frac{q}{-r})$ for r near -1. □

6.2. **Problems with dynamic and endpoint constraints.** Our goal in this subsection is to give Lagrangian and Hamiltonian necessary conditions for the following general variational problem for which the dynamic is an unbounded nonconvex differential inclusion

$$\text{(6.1)} \qquad \text{minimize } \ell(x(a),x(b)) + \int_a^b f(t,x(t),\dot{x}(t))dt$$

$$\text{over the arcs } x \text{ satisfying :}$$

$$\dot{x}(t) \in F(t,x(t)) \quad a.e.\ t \in [a,b], (x(a),x(b)) \in S.$$

where $\ell : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ is locally Lipschitzian at $(z(a),z(b))$, $F:[a,b]\times\mathbf{R}^n\times\mathbf{R}^n \mapsto \mathbf{R}^n$ is a multivalued mapping which is closed-valued and measurably Lipschitzian ([4]) at z, $S \subset \mathbf{R}^n \times \mathbf{R}^n$ is a closed set and $f:[a,b]\times\mathbf{R}^n\times\mathbf{R}^n \mapsto \mathbf{R}$ is a function satisfying the following assumption:

H_f) $f(t,x,y)$ is measurable in t for fixed (x,y) and there exist an integrable function $k:[a,b]\mapsto\mathbf{R}_+$ and $\varepsilon >$ such that

$$|f(t,x,y) - f(t,x',y')| \le k(t)[\|x-x'\| + \|y-y'\|]$$

for all (x,y), $(x',y') \in [z(t)+\varepsilon\mathbf{B}] \times \mathbf{R}^n$.

This problem has been considered by several authors including Loewen-Rockafellar [12]-[15], Mordukhovich [21], Vinter-Zheng [25], $\cdots$. In [12]-[15], Loewen and Rockafellar have considered this problem (with additional state constraints) where they assume that $f(t,x,\cdot)$ is convex while the set $F(t,x)$ is convex. Using discrete approximations, Mordukhovich [21] has established Lagrangian necessary optimality conditions for this problem without convexity and under some "closedness assumption". Using the result by Ioffe-Rockafellar [8], Vinter-Zheng [25] gave necessry optimality conditions for this problem without convexity nor closedness assumptions. All these results have been proved without a calmness assumption.

Our aim in this subsection is to prove that, under a calmness assumption, Lagrangian and Hamiltonian necessary optimality conditions for this problem hold without convexity. This follows immediately from our main Theorems 3.4 and 5.1.

Consider the Hamiltonian $\tilde{H}$ defined by

$$\tilde{H}(t,x,p) = \sup_{y \in F(t,x)} \{\langle p, y\rangle - f(t,x,y)\}$$

Theorem 6.3. *Let z solves locally the problem (6.1), where the problem is calm at z. Then there exists an arc $p \in W^{1,1}$ such that*

$$\dot{p}(t) \in co\{q : \ (q, p(t)) \in \partial f(t, z(t), \dot{z}(t)) + N(GrF(t,\cdot); (z(t), \dot{z}(t)))\} \ a.e.$$

$$(resp.\ \dot{p}(t) \in co\{q : \quad (-q, \dot{z}(t)) \in \partial \tilde{H}(t, z(t), p(t))\}\ a.e.)$$

$$(p(a), -p(b)) \in \partial \ell(z(a), z(b)).$$

$$\langle p(t), \dot{z}(t)\rangle - f(t, z(t), \dot{z}(t)) = \tilde{H}(t, z(t), \dot{z}(t)) \quad a.e.$$

Proof. Consider the Lagrangian $\tilde{L} : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{L}(t,x,y) = \begin{cases} f(t,x,y) & \text{if } y \in F(t,x) \\ +\infty & \text{otherwise} \end{cases}$$

and apply Theorem 5.1. □

We have to note that general necessary optimality conditions (whose proof is based on Theorem 3.4) have been discovred in [2] and [10] for multiobjective optimal control of nonconvex unbounded differential inclusions with endpoint constraints involving a general preference.

In the following examples we present sufficient conditions for calmness.

Example 1. Let $S \subset \mathbf{R}^n \times \mathbf{R}^n$ be a closed set. Suppose that ℓ is locally Lipschitzian at $(z(a), z(b))$, where z is a local solution to the problem

$$\text{minimize } \ell(x(a), x(b)) \tag{6.2}$$

over the arcs x satisfying :

$$(x(a), x(b)) \in S, \quad \dot{x}(t) \in F(t, x(t)) \quad a.e.\ t \in [a,b] \tag{6.3}$$

Problem (6.2) is calm at z provided that the system (6.3) is semi-normal ([10]) at z. As a result, the conclusion of Theorem 6.3 remains valid.

In [7], Ioffe established Hamiltonian necessary optimality conditions for problem (6.2) under a normality assumption. It is not difficult to see that this assumption implies the calmness of (6.2). In the following example we give a (nonconvex) optimal control problem which is calm and the corresponding system is not normal at a solution point.

Example 2. Consider the multivalued mapping $F : \mathbf{R}^3 \mapsto \mathbf{R}^3$ defined by

$$F(x,y,z) = \{(u,v,w) \in \mathbf{R}^3 : \ |v| \le 7, \ w = x^2 v^2\}.$$

Then F is nonconvex-valued and measurably Lipschitzian ([4]) at z. Now we consider the following optimal control problem

$$\min(x_1(1) - x_1(0))^2 + (x_2(1) - x_2(0))^2$$
$$\text{subject to}$$
$$(\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t)) \in F(x_1(t), x_2(t), x_3(t)) \ a.e. \tag{6.4}$$
$$x(0) \in \{0\} \times [-1, 1] \times \{2\},\ x(1) \in \{0\} \times [2, 4] \times \{2\} \tag{6.5}$$

The arc z, given by $z(t) = (0, t + 1, 2)$, is a solution to this problem which is calm at z, but the system (6.4)-(6.5) is not normal at this point.

Example 3. Consider the following optimal control problem

$$\min \int_0^1 ((\dot{x}_1(t))^2 + (\dot{x}_2(t))^2)dt$$
$$\text{subject to}$$
$$(\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t)) \in F(x_1(t), x_2(t), x_3(t)) \ a.e. \tag{6.6}$$
$$x(0) \in \{0\} \times [-1, 1] \times \{2\},\ x(1) \in \{0\} \times [2, 4] \times \{2\} \tag{6.7}$$

where F is as in Example 2. Then the arc z, given by $z(t) = (0, t+1, 2)$, is a solution to this problem which is calm at z, but the system (6.6)-(6.7) is not normal at this point.

References

[1] S. Bellaassali and A. Jourani,New Euler-Lagrange inclusion with applications to isoperimetric problems and to Ramsey model, Université de Bourgogne, France 2000.

[2] S. Bellaassali and A. Jourani, *Necessary optimality conditions in multiobjective dynamic optimization*, SIAM Journal on Control and Optimization **42** (2003), 2043–2061.

[3] F. H. Clarke, *The generalized problem of Bolza*, SIAM J. Control Optim. **14** (1976), 682–699.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New-York, 1983.

[5] F. H. Clarke, *Necessary conditions in dynamic programming*, Memoirs AMS **73** (2005), x+113.

[6] A. D. Ioffe, *Approximate subdifferentials and applications 3 : Metric theory*, Mathematika **36** (1989), 1–38.

[7] A. D. Ioffe, *Euler-Lagrange and Hamiltonian formalisms in dynamic optimization*, Trans. Amer. Math. Soc. **349** (1997), 2871–2900.

[8] A. D. Ioffe and R. T. Rockafellar, *The Euler and Wierstrass conditions for nonsmooth variational problems*, Calculus of Variations and PDE **4** (1996), 59–87.

[9] A. Jourani, *Qualification conditions for multivalued functions in Banach spaces with applications to nonsmooth vector optimization problems*, Math. Program. **66** (1994), 1–23.

[10] A. Jourani, *Normality, local controllability and NOC in multiobjective optimal control problems*, in Contemporary Trends in Nonlinear Geometric Control Theory and its Applications, A. Anzaldo-Meneses, B. Bonnard, J.-P. Gauthier and F. Monroy (eds.), World Scientific Publishers, Singapore, 2002, pp. 359–380.

[11] A. Jourani and L. Thibault, *Approximate subdifferential and metric regularity : Finite dimensional case*, Math Program. **47** (1990), 203–218.

[12] P. D. Loewen and R. T. Rockafellar, *The adjoint arc in nonsmooth optimization*, Trans. Amer. Math. Soc. **325** (1991), 39–72.

[13] P. D. Loewen and R. T. Rockafellar, *Optimal control of unbounded differential inclusions*, SIAM J. Control Optim. **32** (1994), 442–470.

[14] P. D. Loewen and R. T. Rockafellar, *New necessary conditions for the generalized problem of Bolza*, SIAM J. Control Optim. **34** (1996), 1496–1511.

[15] P. D. Loewen and R. T. Rockafellar, *Bolza problems with general time constraints*, SIAM J. Control Optim. **35** (1997), 2050–2069.
[16] B. S. Mordukhovich, *Maximum principle in problems of time optimal control with nonsmooth constraints*, J. Appl. Math. Mech. **40** (1976), 960–969.
[17] B. S. Mordukhovich, *Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems*, Soviet Math. Dokl. **22** (1980), 526–530.
[18] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Grundlehren Series (Fundamental Principles of Mathematical Sciences) **330**, Springer-Verlag, Berlin, 2006.
[19] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. II: Applications*, Grundlehren Series (Fundamental Principles of Mathematical Sciences) **331**, Springer-Verlag, Berlin, 2006.
[20] B. S. Mordukhovich, *Optimal control of difference, differential and differential-difference inclusions*, J. Math. Sci., **100** (2000), 2613–2632.
[21] B. S. Mordukhovich, *Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions*, SIAM J. Control Optim. **33** (1995), 882–915.
[22] R. T. Rockafellar, *Existence and duality theorems for convex problems of Bolza*, Trans. AMS, **159** (1971), 1–40.
[23] R. T. Rockafellar, *Saddle points of Hamiltonian systems in convex problems of Lagrange*, J. Optim. Th. Appl. **12** (1973), 367–390.
[24] R. T. Rockafellar, *Equivalent subgradient versions of Hamiltonian and Euler-Lagrange equations in variational analysis*, SIAM J. Control Optim. **34** (1996), 1300–1315.
[25] R. Vinter and H. Zheng, *The extended Euler-Lagrange condition for nonconvex variational problems*, SIAM J. Control Optim. **35** (1997), 56–77.

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