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APPLICATION OF A DISCRETE FIXED POINT THEOREM TO THE COURNOT MODEL

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ABSTRACT. In this paper, we apply a discrete fixed point theorem, which is based on monotonicity of a set-valued mapping, due to the author and Kawasaki (Taiwanese J. Math., 2009) to the Cournot model, giving us a Cournot model in which the production of the firms assumes integral values, 0, 1, 2 and so on. To handle this, we define a discrete Cournot-Nash equilibrium, and prove its existence.

1. Introduction

The Cournot model is a well-known market competition model. In this duopoly market model, each firm sets its production levels and the market then decides the price accordingly. This model and the concept of a non-cooperative equilibrium were introduced by Cournot [2]; See also Aubin [1, Section 7.9] for more details on the original Cournot model. However, in some situations, such as when the production is on a small scale, this model is somewhat inappropriate since the equilibrium, which is the optimal production of each firm, is real-valued. We therefore introduce a discretized Cournot model in which each firm's production is an integer value, and show the existence of an equilibrium for such a model in this paper.

For the original Cournot model, classical fixed point theorems such as Kakutani's and Brouwer's ensure the existence of an equilibrium. According to this relationship, we expect that discrete fixed point theorems should play a central role in demonstrating the existence of an equilibrium for a discretized Cournot model. In fact, in economics, Vives [12] applied Tarski's fixed point theorem [11] to an extended Cournot model. As a result, he showed the existence of an equilibrium; the equilibrium set is a complete lattice in the model. However, in his model, each firm's production was a continuous variable, so the equilibrium is also allowed to be real-valued.

Therefore, in this paper, we consider exactly the situation of integer valued production. More precisely, we apply the discrete fixed point theorem from [9] to the Cournot model, and this enables us to consider integer valued production. Note that this discrete fixed point theorem is based on the monotonicity of a certain set-valued mapping.

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This paper is organized as follows: In Section 2, we introduce a discretized Cournot model and present our main result. In Section 3, we present some examples for which our required assumptions are satisfied. In Section 4, we prove our main theorem.

2. The model and the main result

Throughout this paper, i is an element of $\{1,2\}$, $\{-i\}$ means $\{1,2\} \setminus \{i\}$, C_i ($i = 1,2$) are firms entered into the market, and $q_i \in Q_i$ is the production of C_i, where $Q_i := \{0, 1, \ldots, m_i\}$ ($m_i \in \mathbb{N}$). We denote by $\overline{Q}_i$ the interval $[0, m_i]$. Furthermore, we denote by $p(q_1 + q_2)$ the price of the production lot as follows:

$$p(q_1 + q_2) := \max\{r(q_1 + q_2), 0\},$$

where r is a smooth function defined on some open interval containing $[0, m_1 + m_2]$. Note that, for example, if we take $r = a - b(q_1 + q_2)$ $(a, b > 0)$, then the model reflects the classical situation. $c_i \in \mathbb{N}$ is the average value of the cost, that is, it costs firm C_i, c_i to produce one unit. Firm C_i aims to maximize its profit function

$$\pi_i(q_i, q_{-i}) := \max\{(p(q_1 + q_2) - c_i)q_i, 0\}.$$

Definition 2.1. We call a pair $(q_1^*, q_2^*) \in Q_1 \times Q_2$ a *discrete Cournot-Nash equilibrium*, if for each $i \in \{1, 2\}$

$$\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*), \quad \forall q_i \in Q_i.$$

Here we denote by $\phi_i(q_{-i})$, the set of best responses of firm C_i to q_{-i}, restricted to integers, that is,

$$\phi_i(q_{-i}) = \left\{ q_i \in Q_i \colon \pi_i(q_i, q_{-i}) = \max_{q_i \in Q_i} \pi_i(q_i, q_{-i}) \right\}.$$

Further, we set $\phi(q_1, q_2) := \phi_1(q_2) \times \phi_2(q_1)$. Then $q^* = (q_1^*, q_2^*)$ is a discrete Cournot-Nash equilibrium if and only if $q^* \in \phi(q^*)$. Similarly, we define $\varphi(q_1, q_2) := \varphi_1(q_2) \times \varphi_2(q_1)$ for $(q_1, q_2) \in \overline{Q}_1 \times \overline{Q}_2$, where

$$\varphi_i(q_{-i}) = \left\{ q_i \in \overline{Q}_i \colon \pi_i(q_i, q_{-i}) = \max_{q_i \in \overline{Q}_i} \pi_i(q_i, q_{-i}) \right\}.$$

In this paper, we assume the following (H1)–(H5):

- (H1): $r(0) > 0$;
- (H2): $r(q_1 + q_2)$ is twice continuously differentiable;
- (H3): $r(q_1 + q_2)$ is monotone decreasing on $[0, m_1 + m_2]$, and strictly monotone decreasing on the interval $\{q_1 + q_2 \in [0, m_1 + m_2] \colon r(q_1 + q_2) > 0\}$;
- (H4): $\pi_i(q_i, q_{-i})$ is unimodal with respect to q_i;
- (H5): if firm C_i produces its upper bound m_i for some $q_{-i} \in \overline{Q}_i$, then $m_i \in \varphi_i(q_{-i} - \varepsilon)$ for any $\varepsilon > 0$.

Note that by (H1), $\{q_1 + q_2 \in [0, m_1 + m_2] \colon r(q_1 + q_2) > 0\}$ is not empty, and by (H4), the mapping φ_i is single-valued.

The main result of this paper is stated in the next theorem:

Theorem 2.2 (Main theorem)**.** *Assume* (H1)–(H5). *Further, we assume that for each* $i \in \{1,2\}$ *and given* $q_{-i} \in [0, m_{-i}]$, *if* $q_i \in \operatorname{argmax}_{t \in \overline{Q}_i} \pi_i(t, q_{-i})$ *is an element of* $(0, m_i)$, *then* q_i *satisfies one of the following three conditions:*

(i) $r''(q_1+q_2) < 0$ *and* $q_i \neq \dfrac{-2r'(q_1+q_2)}{r''(q_1+q_2)}$,

(ii) $r''(q_1+q_2) = 0$,

(iii) $r''(q_1+q_2) > 0$ *and* $0 \leq q_i \leq \dfrac{-r'(q_1+q_2)}{r''(q_1+q_2)}$,

where $r'(q_1+q_2)$ *and* $r''(q_1+q_2)$ *stand for the first and second derivatives of* r *at* q_1+q_2, *respectively.*

Then there exists a discrete Cournot-Nash equilibrium. In other words, there exists $q^* := (q_1^*, q_2^*) \in Q$ *such that* $q^* \in \phi(q^*)$.

Remark 2.3. We do not require that the price function be convex. Furthermore, the theorem includes the classical situation where the price function is concave; see Examples 3.1 and 3.2 below.

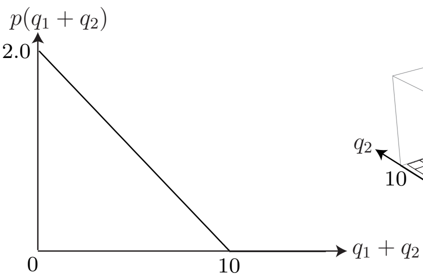
3. Examples

In this section, we give a few examples of price functions that satisfy the assumption of the main theorem. In these examples, $m_1 = m_2 = 10$.

Example 3.1. Let $r(q_1+q_2) = -1/5(q_1+q_2)^2 + 2$ and $c_1 = c_2 = 1$. Then

$$p(q_1+q_2) = \begin{cases} -\dfrac{1}{5}(q_1+q_2) + 2, & \text{if } \; 0 \leq q_1+q_2 \leq 10 \\ 0, & \text{if } \; 10 < q_1+q_2, \end{cases}$$

This example satisfies assumptions (ii) and (H1)–(H5); see Figure 1 left. Figure 1 right is a graph of C_i's profit function. It is clear that $\pi_1(q_1, q_2)$ is unimodal with respect to q_1.



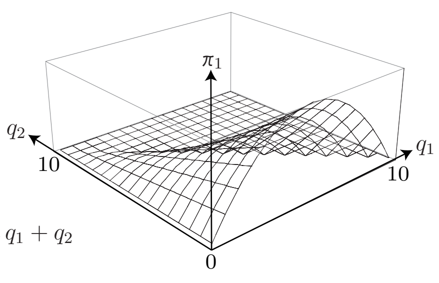


Figure 1. The price function and C_1's profit function

Example 3.2. Let $r(q_1+q_2) = -(q_1+q_2)^2 + 100$ and $c_1 + c_2 = 10$. Then

$$p(q_1+q_2) = \begin{cases} -(q_1+q_2)^2 + 100, & \text{if } \; 0 \leq q_1+q_2 \leq 10 \\ 0, & \text{if } \; 10 < q_1+q_2, \end{cases}$$

This example satisfies assumptions (i) and (H1)–(H5).

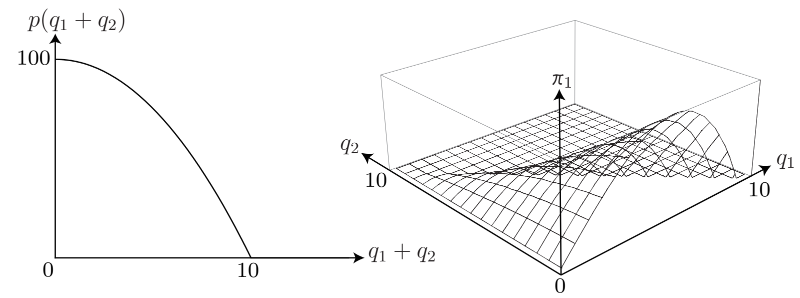


FIGURE 2. The price function and C_1's profit function

Example 3.3. Let $r(q_1+q_2) = -\arctan(q_1+q_2-10)+2$ and $c_1 = c_2 = 2$. Then $p(q_1+q_2) = r(q_1+q_2)$ on $[0, 20]$. This example satisfies assumptions (i)–(iii) and (H1)–(H5).

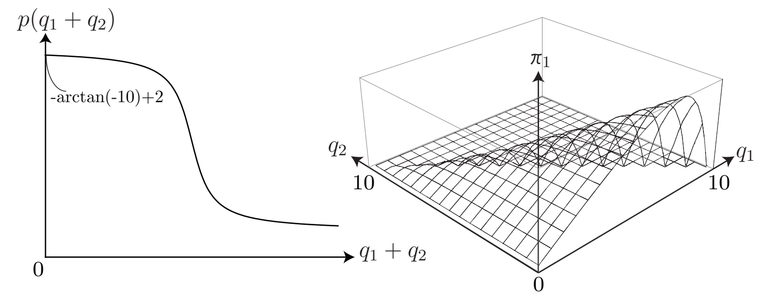


FIGURE 3. Neither concave nor convex price function and C_1's profit function

4. PROOF OF THEOREM 2.2

We prove this theorem in three steps. First, we analyze the behavior of the best responses for the case where each firm's production is continuously variable. Second, we discretize the results obtained in Step 1 to handle the case where each firm's production is integral. Finally, using the discrete fixed point theorem from [9] and the results of Step 2, we demonstrate the existence of a discrete Cournot-Nash equilibrium.

Step 1: Analysis of the behavior of $\varphi_i(q_{-i})$. We first note that the domain of $\varphi_i(\cdot)$ is continuous. In this step, we prove the following lemma.

Lemma 4.1. $\varphi_i(q_{-i})$ $(i = 1, 2)$ *is monotone decreasing with respect to* q_{-i}.

In order to prove the lemma above, we require the following lemma, which will first be proven.

Lemma 4.2. *Assume that there exists* $q^0_{-i} \in [0, m_{-i}]$ *such that* $0 \in \varphi_i(q^0_{-i})$. *Then we have* $\varphi_i(q^0_{-i} + \varepsilon) = \{0\}$ *for every positive* ε.

Proof. Since by $0 \in \varphi_i(q^0_{-i})$ and (H3), we get

$$c_i \geq p(0 + q^0_{-i}) \geq r(0 + q^0_{-i}) \geq r(0 + q^0_{-i} + \varepsilon_i),$$

that is, $r(0 + q^0_{-i} + \varepsilon) \leq c_i$. Therefore, C_i does not produce anything, that is, $\varphi_i(q^0_{-i} + \varepsilon) = \{0\}$. □

Proof of Lemma 4.1. We first temporarily set the domain of q_{-i} to $(-\delta, m_{-i} + \delta)$, where δ is a sufficiently small positive constant. We define the following sets:

$$R^-_{-i} := \left\{ q_{-i} \in (-\delta, m_{-i} + \delta) \colon r''(q_1 + q_2) < 0,\ q_i \in \operatorname*{argmax}_{t \in [0, m_i]} \pi_i(t, q_{-i}) \right\},$$

$$R^0_{-i} := \left\{ q_{-i} \in (-\delta, m_{-i} + \delta) \colon r''(q_1 + q_2) = 0,\ q_i \in \operatorname*{argmax}_{t \in [0, m_i]} \pi_i(t, q_{-i}) \right\},$$

$$R^+_{-i} := \left\{ q_{-i} \in (-\delta, m_{-i} + \delta) \colon r''(q_1 + q_2) > 0,\ q_i \in \operatorname*{argmax}_{t \in [0, m_i]} \pi_i(t, q_{-i}) \right\},$$

and note that these sets can be empty and that they depend on q_i.

For any $q_{-i} \in (-\delta, m_{-i} + \delta)$, if $q_i \in \varphi_i(q_{-i})$, then one of (I)–(III) holds: (I) $q_i \in (0, m_i)$, (II) $q_i = 0$, or (III) $q_i = m_i$, because $q_i \in [0, m_i]$. We hence distinguish these three cases.

(I) The case where $q_i \in (0, m_i)$: Since $q_i \neq 0$, we have $p(q_1 + q_2) > c_i > 0$. Then

$$\pi_i(q_1, q_2) = (p(q_1 + q_2) - c_i) q_i = (r(q_1 + q_2) - c_i) q_i.$$

This implies that $\pi_i(q_1, q_2)$ is twice differentiable in both variables. Thus, q_i is a solution of the following equation:

$$\frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = r'(q_1 + q_2) q_i + r(q_1 + q_2) - c_i = 0. \tag{4.1}$$

We set $h(q_i) := r'(q_1 + q_2) q_i + r(q_1 + q_2) - c_i$, then

$$\frac{\partial^2 \pi_i(q_1, q_2)}{\partial q_i^2} = h'(q_{-i}) = r''(q_1 + q_2) q_i + 2r'(q_1 + q_2). \tag{4.2}$$

Further, we distinguish three cases: (a) $q_{-i} \in R^-_{-i}$, (b) $q_{-i} \in R^0_{-i}$, or (c) $q_{-i} \in R^+_{-i}$.

(I-a) The case where $q_{-i} \in R^-_{-i}$: By assumption (i), we have $r''(q_1 + q_2) q_i + 2r'(q_1 + q_2) \neq 0$, that is, $h'(q_i) \neq 0$. Therefore, by the implicit function theorem, (4.1) is uniquely solved in a neighborhood of (q_{-i}, c_i), say, $U(q_{-i})$, such that $q_i = q_i(q_{-i})$. In other words,

$$r'(q_i(q_{-i}) + q_{-i}) q_i(q_{-i}) + r(q_i(q_{-i}) + q_{-i}) - c_i = 0. \tag{4.3}$$

Differentiating (4.3) with respect to q_{-i}, we obtain

$$r'' \cdot \left(\frac{\partial q_i}{\partial q_{-i}} + 1 \right) q_i + r' \cdot \frac{\partial q_i}{\partial q_{-i}} + r' \cdot \left(\frac{\partial q_i}{\partial q_{-i}} + 1 \right) = 0,$$

which is equivalent to

$$\frac{\partial q_i}{\partial q_{-i}} = -\frac{r'' \cdot q_i + r'}{r'' \cdot q_i + 2r'}. \tag{4.4}$$

Since $r'' \cdot q_i + 2r' < r'' \cdot q_i + r' < 0$ is satisfied, it follows that

$$r'' \cdot q_i + r' < 0 \quad \text{and} \quad r'' \cdot q_i + 2r' < 0. \tag{4.5}$$

From (4.4), (4.5) and (4.2), we obtain $\partial q_i/\partial q_{-i} < 0$ in $U(q_{-i})$ and $\partial^2\pi_i/\partial q_i^2 < 0$.

(I-b) The case where $q_{-i} \in R^0_{-i}$: From assumption (ii), (H2) and (4.2), we have $2r' < r' < 0$ and $h' \neq 0$, and $\partial^2\pi_i/\partial q_i^2 < 0$ also holds. Thus, by a discussion similar to the case where $q \in R^-_{-i}$, we obtain $\partial q_i/\partial q_{-i} < 0$ in $U(q_{-i})$.

(I-c) The case where $q_{-i} \in R^+_{-i}$: From assumption (iii) and (H2), it follows that $r'' \cdot q_i + 2r' < r'' \cdot q_i + r' \leq 0$. Thus, by (4.2), we get $h' \neq 0$ and $\partial^2\pi_i/\partial q_i^2 < 0$. Therefore, by similar discussion as for the case where $q \in R^-_{-i}$, we obtain $\partial q_i/\partial q_{-i} \leq 0$ in $U(q_{-i})$.

Since $\{U(q_{-i})\colon 0 \leq q_{-i} \leq m_{-i}\}$ is an open covering of $[0, m_{-i}]$, there exists a finite subcovering $\{U_j\}_{j=1}^m$, where $U_j := U(q^j_{-i})$ for some $q^j_{-i} \in [0, m_{-i}]$ $(j = 1, \ldots, m)$. Also, from the uniqueness of the implicit function in the neighborhood U_j, the implicit functions q_i are smoothly connected and $\partial q_i/\partial q_{-i} \leq 0$ in $[0, m_{-i}]$. Finally, by the unimodality of π_i with respect to q_i and the fact that $\partial^2\pi_i/\partial q_i^2 < 0$, we get $q_i(q_{-i}) = \varphi_i(q_{-i})$.

(II) The case where $q_i = 0$: There exists $q^0_{-i} \in [0, m_{-i}]$ such that $\varphi_i(q^0_{-i}) = 0$. Therefore, by Lemma 4.2, we get $\varphi_i(q_{-i}) = 0$ for any $q_{-i} \in [q^0_{-i}, m_{-i}]$.

(III) The case where $q_i = m_i$: There exists $q^M_{-i} \in [0, m_{-i}]$ such that $\varphi_i(q^M_{-i}) = m_i$. Therefore, by (H5), we get $\varphi_i(q_{-i}) = m_i$ for any $q_{-i} \in [0, q^M_{-i}]$.

By (I)–(III), $\varphi_i(q_{-i})$ is monotone decreasing with respect to q_{-i}, which proves the claim. □

Step 2: Discretization of the result of Step 1. In this step, we prove that if $\varphi_i(q_{-i})$ is monotone decreasing, then $\phi_i(q_{-i})$ also has a monotone decreasing-like property.

Lemma 4.3. *Assume that $\varphi_i(q_{-i}) \geq \varphi_i(q_{-i} + 1)$ for some $q_{-i} \in Q_{-i}$. Then there exist $\alpha \in \phi_i(q_{-i})$ and $\beta \in \phi_i(q_{-i} + 1)$ such that $\alpha \geq \beta$.*

Proof. By the unimodality of π_i with respect to q_i,

$$\phi_i(q_{-i}) = \begin{cases} \{\lceil\varphi_i(q_{-i})\rceil\}, & \text{if } \pi_i(\lceil\varphi_i(q_{-i})\rceil, q_{-i}) > \pi_i(\lfloor\varphi_i(q_{-i})\rfloor, q_{-i}) \\ \{\lceil\varphi_i(q_{-i})\rceil, \lfloor\varphi_i(q_{-i})\rfloor\}, & \text{if } \pi_i(\lceil\varphi_i(q_{-i})\rceil, q_{-i}) = \pi_i(\lfloor\varphi_i(q_{-i})\rfloor, q_{-i}) \\ \{\lfloor\varphi_i(q_{-i})\rfloor\}, & \text{if } \pi_i(\lceil\varphi_i(q_{-i})\rceil, q_{-i}) < \pi_i(\lfloor\varphi_i(q_{-i})\rfloor, q_{-i}), \end{cases}$$

for any $q_{-i} \in Q_{-i}$, where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ represent rounding up and down to the nearest integer, respectively. Here we distinguish two cases: $\lceil\varphi_i(q_{-i}+1)\rceil \leqq \lfloor\varphi_i(q_{-i})\rfloor$ holds or $\lceil\varphi_i(q_{-i}+1)\rceil > \lfloor\varphi_i(q_{-i})\rfloor$ holds.

Case 1: $\lceil\varphi_i(q_{-i}+1)\rceil \leqq \lfloor\varphi_i(q_{-i})\rfloor$. Then we have

$$\lfloor\varphi_i(q_{-i}+1)\rfloor \leqq \lceil\varphi_i(q_{-i}+1)\rceil \leqq \lfloor\varphi_i(q_{-i})\rfloor \leqq \lceil\varphi_i(q_{-i})\rceil. \tag{4.6}$$

Since $\phi_i(q_{-i})$ consists of $\lfloor\varphi_i(q_{-i})\rfloor$ or $\lceil\varphi_i(q_{-i})\rceil$, and $\phi_i(q_{-i}+1)$ consists of $\lfloor\varphi_i(q_{-i}+1)\rfloor$ or $\lceil\varphi_i(q_{-i}+1)\rceil$, the claim is evident from (4.6).

Case 2: $\lceil \varphi_i(q_{-i}+1)\rceil > \lfloor \varphi_i(q_{-i})\rfloor$. By the assumption of this lemma, we get

$$\lfloor \varphi_i(q_{-i}+1)\rfloor \leqq \lfloor \varphi_i(q_{-i})\rfloor < \lceil \varphi_i(q_{-i}+1)\rceil \leqq \lceil \varphi_i(q_{-i})\rceil. \tag{4.7}$$

Denying the claim, we have $\alpha < \beta$ for any $\alpha \in \phi_i(q_{-i})$ and $\beta \in \phi_i(q_{-i}+1)$. Hence we see from (4.7) that $\phi_i(q_{-i}) = \{\lfloor \varphi_i(q_{-i})\rfloor\} = d$ and $\phi_i(q_{-i}+1) = \{\lceil \varphi_i(q_{-i}+1)\rceil\} = d+1$ for some integer d.

Since $\phi_i(q_{-i}+1) = d+1$ is the best response, we get

$$\pi_i(d+1, q_{-i}+1) \geq \pi_i(d, q_{-i}+1). \tag{4.8}$$

Moreover, since $d+1 \geq 1$, we see $p(d+1+q_{-i}+1) > c_i > 0$. Thus, we have $p(d+1+q_{-i}+1) = r(d+1+q_{-i}+1)$. This implies that

$$\pi_i(d+1, q_{-i}+1) = (r(d+1+q_{-i}+1) - c_i)(d+1).$$

Finally, by (H3),

$$\begin{aligned}(r(d+1+q_{-i}+1) - c_i)(d+1) &< (r(d+q_{-i}+1) - c_i)(d+1)\\ &= \max\{(p(d+q_{-i}+1) - c_i)(d+1), 0\}\\ &= \pi_i(d, q_{-i}+1),\end{aligned}$$

which contradicts (4.8). □

Step 3: Conclusion (Application of the discrete fixed point theorem). In this step, we first quote the discrete fixed point theorem from [9]. Throughout this step, $V \subset \mathbb{Z}^n$, $(V, \trianglelefteqq)$ is a partially ordered set and $F : V \to V$ is a nonempty set-valued mapping. Also, the symbol $x \trianglelefteq y$ means $x \trianglelefteqq y$ and $x \neq y$.

Proposition 4.4 ([9, Theorem 2.2]). *Assume that there exists a sequence $\{x^k\}_{k\geq 0}$ in V such that $x^k \trianglelefteqq x^{k+1} \in F(x^k)$ for any $k \geq 0$, and $\{x \in V : x^0 \trianglelefteqq x\}$ is finite. Then F has a fixed point $x^* \in F(x^*)$.*

We here modify $\varphi_1(q_2)$ and $\varphi_2(q_1)$ to be integers, and define sequences $\{q_i^k\}_{k=0}^{m_{-i}} \subset Q_i$ $(i = 1, 2)$:

$$q_i^k := \begin{cases} \lceil \varphi_i(k)\rceil, & \text{if } \pi_i(\lceil \varphi_i(k)\rceil, k) > \pi_i(\lfloor \varphi_i(k)\rfloor, k)\\ \lfloor \varphi_i(k)\rfloor, & \text{if } \pi_i(\lceil \varphi_i(k)\rceil, k) \leq \pi_i(\lfloor \varphi_i(k)\rfloor, k).\end{cases}$$

Then the following hold:
(a) $q_1^k \in \phi_1(k)$ for all $k \in Q_2$;
(b) $q_2^l \in \phi_2(l)$ for all $l \in Q_1$.

Further, by Lemmas 4.1 and 4.3, we get $q_1^k \geq q_1^{k+1}$ for all $k = 0, \ldots, m_1 - 1$ and $q_2^l \geq q_2^{l+1}$ for all $l = 0, \ldots, m_2 - 1$. We here define the partial order $(q_1, q_2) \trianglelefteqq (\hat{q}_1, \hat{q}_2)$ for any $(q_1, q_2), (\hat{q}_1, \hat{q}_2) \in Q := Q_1 \times Q_2$ by $q_1 \leq \hat{q}_1$ and $q_2 \geq \hat{q}_2$. Moreover, starting with the minimum point $q^0 := (0, m_2)$, we define $q^1 := (q_1^{m_2}, q_2^0)$, $q^2 := (q_1^{q_2^0}, q_2^{q_1^{m_2}})$ and so on. Then we have

$$q^0 \trianglelefteqq q^1 \trianglelefteqq q^2 \trianglelefteqq \cdots \tag{4.9}$$

and $q^{k+1} \in \phi(q^k)$ for all $k \geq 0$. Indeed, the first inequality of (4.9) is evident and the others follow from (a) and (b). Finally, applying Proposition 4.4 to $(V, f) = (Q, \phi)$, we conclude that ϕ has a discrete fixed point, which is a discrete Cournot-Nash equilibrium. This proves the main theorem, Theorem 2.2. □

5. Concluding remarks

It is not hard to calculate a discrete Cournot-Nash equilibrium for each example in Section 3. Indeed, $(1,2)$, $(3,4)$ and $(3,5)$ are the discrete Cournot-Nash equilibria for Examples 3.2, 3.1 and 3.3, respectively. Although Theorem 2.2 proves the existence of a discrete Cournot-Nash equilibrium, it does not define how it should be computed, which is another important research question.

As an aside, several studies have focused on discretized market competition models. We conclude this paper by introducing two of these studies. The first is based on discrete convex analysis as proposed by Murota [8], while the other is based on discrete fixed point theorems.

The economic model proposed by Danilov *et al.* [4] was the first based on discrete convex analysis, and was derived from an Arrow-Debreu type model. The model was extended by Danilov *et al.* [3]. Lehmann *et al.* [7] considered auctions, while Tamura [10] expounded on discretized market competition models.

On the other hand, Iimura [5] showed the existence of a Walrasian equilibrium with indivisible commodities as an application of the discrete fixed point theorem of Iimura *et al.* [6]. His theorem is based on Brouwer's fixed point theorem and relies on an integrally convex set, while Proposition 4.4 is based on monotonicity of a set-valued mapping and is valid for any finite set. Therefore, these discrete fixed point theorems are of different types.

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