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WEAK CONVERGENCE THEOREMS FOR A COUNTABLE FAMILY OF RELATIVELY NONEXPANSIVE MAPPINGS†

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Abstract. In this paper, we establish weak convergence theorems for finding common fixed points of a countable family of relatively nonexpansive mappings in a uniformly smooth and uniformly convex Banach space. Weak convergence theorems for finding a common element of the set of fixed points and the set of solutions of a variational inequality problem are also obtained. With an appropriate setting, the corresponding results due to Nadezhkina–Takahashi [13] are deduced.

1. Introduction

Let E be a Banach space, C be a nonempty closed convex subset of E. A mapping $T : C \to E$ is said to be *Lipschitzian* if there exists a positive constant k such that

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be k-Lipschitzian. If $k = 1$, then T is known as a nonexpansive mapping. We denote by $\mathrm{F}(T)$ the set of fixed points of T, that is, $\mathrm{F}(T) = \{x \in C : x = Tx\}$. A mapping T is said to be *quasi-nonexpansive* if $\mathrm{F}(T) \neq \varnothing$ and

$$\|Tx - y\| \leq \|x - y\| \quad \text{for all } x \in C \text{ and } y \in \mathrm{F}(T).$$

It is easy to see that if T is nonexpansive with $\mathrm{F}(T) \neq \varnothing$, then it is quasi-nonexpansive. We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x. Recall that a mapping $T : C \to E$ is *demi-closed at* y, if $x_n \rightharpoonup x$ and $Tx_n \to y$, then $Tx = y$. A point p in C is said to be an *asymptotic fixed point* of T [15] if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\mathrm{F}}(T)$. It easy to see that $\mathrm{F}(T) \subset \widehat{\mathrm{F}}(T)$. Then $\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$ if and only if $I - T$ is demi-closed at zero.

Let E be a smooth Banach space and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\} \quad \text{where } x \in E.$$

The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

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We say that a mapping T is *relatively nonexpansive* [11, 12, 15] if the following conditions are satisfied:

(R1) $\mathrm{F}(T) \neq \varnothing$;
(R2) $\phi(u, Tx) \leq \phi(u, x)$ for each $x \in C$ and $u \in \mathrm{F}(T)$;
(R3) $\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$.

A relatively nonexpansive mapping T is said to be *strongly relatively nonexpansive* [15] if for each bounded sequence $\{z_n\}$ in C such that

$$\phi(u, z_n) - \phi(u, Tz_n) \to 0$$

for some $u \in \mathrm{F}(T)$, then $\phi(z_n, Tz_n) \to 0$.

Examples of relatively or strongly relatively nonexpansive mappings can be founded in Kohsaka and Takahashi [9, 10], Matsushita and Takahashi [11, 12] and Reich [15].

Several articles have appeared providing methods for approximating fixed points of relatively nonexpansive mappings [9, 10, 11, 12]. Matsushita and Takahashi [11] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTx_n) \quad \text{for all} \quad n \in \mathbb{N}, \tag{1.1}$$

where the initial guess element $x_1 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Recently, Kohsaka and Takahashi [9] extended the iteration (1.1) to obtain a weak convergence theorem for common fixed points of a finite family of relatively nonexpansive mappings $\{T_i\}_{i=1}^m$ by the following iteration:

$$x_{n+1} = \Pi_C J^{-1}\left(\sum_{i=1}^{m} w_{n,i}(\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JT_i x_n)\right) \quad \text{for all} \quad n \in \mathbb{N}, \tag{1.2}$$

where $x_1 \in C$, $\{\alpha_{n,i}\} \subset [0, 1]$ and $\{w_{n,i}\} \subset [0, 1]$ with $\sum_{i=1}^m w_{n,i} = 1$ for all $n \in \mathbb{N}$.

In this paper, we establish weak convergence theorems for finding common fixed points of a countable family of relatively nonexpansive mappings in a uniformly smooth and uniformly convex Banach space. We also establish weak convergence theorems for finding a common element of the set of fixed points and the set of solutions of a variational inequality problem. With an appropriate setting, we deduce the corresponding results due to Nadezhkina–Takahashi [13].

2. Preliminaries

Let E be a Banach space. We say that E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \text{ imply } \left\|\frac{x + y}{2}\right\| < 1.$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. Moreover, we know that the following result:

Lemma 2.1 ([17], Theorem 2)**.** *Let E be a uniformly convex Banach space and $B_r := \{x \in E : \|x\| \le r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$.

A Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) is attained uniformly for $x \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (2.1) is attained uniformly for $x, y \in S(E)$. The normalized duality mapping J from E to 2^{E^*} (see [5] for more details) is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \quad \text{where } x \in E.$$

We say that J is *weakly sequentially continuous* if for a sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $Jx_n \overset{*}{\rightharpoonup} Jx$, where denotes $\overset{*}{\rightharpoonup}$ the weak* convergence. We also know the following properties (see e.g. [16] for details):

(a) E (E^*, resp.) is uniformly convex if and only if E^* (E, resp.) is uniformly smooth.
(b) $J(x) \neq \varnothing$ for each $x \in E$.
(c) If E is reflexive, then J is a mapping of E onto E^*.
(d) If E is strictly convex, then $J(x) \cap J(y) = \varnothing$ for all $x \neq y$.
(e) If E is smooth, then J is single valued and norm-to-weak* continuous.
(f) If E has a Fréchet differentiable norm, then J is norm to norm continuous.
(g) If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \quad \text{for all } x, y \in E. \tag{2.2}$$

It is also easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $\|x_n - y_n\| \to 0$ implies that $\phi(x_n, y_n) \to 0$. The converse is also true if E is additionally assumed to be uniformly convex.

Lemma 2.2 ([8], Proposition 2)**.** *Let E be a uniform convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3 ([9], Lemma 2.5)**.** *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $h : [0, 2r] \to [0, \infty)$ such that $h(0) = 0$ and*

$$h(\|x - y\|) \le \phi(x, y)$$

for all $x, y \in B_r$.

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. It is known that [8] for any $x \in E$ there exists a unique point $\widehat{x} \in C$ such that

$$\phi(\widehat{x}, x) = \min_{y \in C} \phi(y, x).$$

Following Alber [1], we denote such an $\widehat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. Concerning the generalized projection, the following are well known.

Lemma 2.4 ([8], Proposition 4)**.** *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. Suppose that $x \in E$ and $\widehat{x} \in C$. Then*

$$\widehat{x} = \Pi_C x \iff \langle \widehat{x} - y, Jx - J\widehat{x} \rangle \ge 0 \quad \text{for each } y \in C.$$

Lemma 2.5 ([8], Proposition 5)**.** *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x) \quad \text{for each } y \in C.$$

Lemma 2.6 ([7], Lemma 2.7)**.** *Let C be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Let $\{x_n\}$ be a sequence in E such that*

$$\phi(y, x_{n+1}) \le \phi(y, x_n) \quad \text{for all } y \in C \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{\Pi_C(x_n)\}$ converges strongly to some $z \in C$.

Lemma 2.7 ([12], Proposition 2.4)**.** *Let E be a strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a relatively nonexpansive mapping from C into E. Then $\mathrm{F}(T)$ is closed and convex.*

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a reflexive, strictly convex and smooth Banach space E, let $\{T_n\}$ be a family of mappings of C into E with $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$ and $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C. $\{T_n\}$ is said to satisfy

(a) the *AKTT-condition (I)* [2] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty;$$

(b) the *AKTT-condition (II)* [3] if for each bounded closed convex subset B of C and each increasing subsequence $\{n_i\}$ of $\{n\}$, there exist a mapping $T : C \to E$ and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\lim_{j \to \infty} \sup\{\|Tz - T_{n_{i_j}} z\| : z \in B\} = 0$$

and $\widehat{\mathrm{F}}(T) = \mathrm{F}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$;

(c) the *KT-condition* [9] if for each bounded sequence $\{z_n\}$ in C such that

$$\phi(u, z_n) - \phi(u, T_n z_n) \to 0$$

for some $u \in \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$, then $\omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$.

Lemma 2.8 ([2], Lemma 3.2)**.** *Let E be a Banach space, let C be a nonempty subset of E and let $\{T_n\}$ be a family of mappings from C into E. Suppose that $\{T_n\}$ satisfies AKTT-condition (I). Then the mapping $T : C \to E$ defined by*

$$Tx = \lim_{n\to\infty} T_n x \quad \text{for all } x \in C \tag{2.3}$$

satisfies

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C. In particular, if $\widehat{\mathrm{F}}(T) = \mathrm{F}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$, then $\{T_n\}$ satisfies the AKTT-condition (II).

From now on, we will write $(\{T_n\}, T)$ satisfies AKTT-condition (I) if $\{T_n\}$ satisfies AKTT-condition (I) and T is defined by (2.3).

Lemma 2.9. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty subset of E. If T is a strongly relatively nonexpansive mapping from C into E, then $\{T_n\}$ satisfies the KT-condition, where $T_n \equiv T$.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that

$$\phi(u, z_n) - \phi(u, T z_n) \to 0$$

for some $u \in F(T)$. Since T is strongly relatively nonexpansive, $\phi(z_n, T z_n) \to 0$. By Lemma 2.2, we have $\|z_n - T z_n\| \to 0$. It follows from (R3) that $\omega_w\{z_n\} \subset \mathrm{F}(T)$. □

Lemma 2.10 ([10], Lemmas 3.1 and 3.2)**.** *Let E be a uniformly convex and uniformly smooth Banach space,and let C be a nonempty closed convex subset of E. Let T be a relatively nonexpansive mapping from C into E. Let U be the mapping defined by,*

$$U = \Pi_C J^{-1}(\alpha J + (1 - \alpha) J T)$$

where $\alpha \in (0, 1)$, then U is a strongly relatively nonexpansive mapping from C into itself and $\mathrm{F}(U) = \mathrm{F}(T)$.

Lemma 2.11. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a family of relatively nonexpansive mappings from C into E with $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$ and satisfy AKTT-condition (II). Let $\{U_n\}$ be a family of strongly relatively nonexpansive mappings from C into itself defined by,*

$$U_n = \Pi_C J^{-1}(\alpha_n J + (1 - \alpha_n) J T_n),$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then $\{U_n\}$ satisfies the KT-condition and $\bigcap_{n=1}^{\infty} \mathrm{F}(U_n) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$.

Proof. By Lemma 2.10, we have $\mathrm{F}(T_n)$ and hence $\bigcap_{n=1}^{\infty} \mathrm{F}(U_n) = \mathrm{F}(U_n) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$. To show that $\{U_n\}$ satisfies KT-condition, let $\{z_n\}$ be a bounded sequence in C such that

$$\phi(u, z_n) - \phi(u, U_n z_n) \to 0 \quad \text{for some} \quad u \in \cap_{n=1}^{\infty} \mathrm{F}(U_n). \tag{2.4}$$

Since $\{z_n\}$ is bounded and $\phi(u, T_n z_n) \leq \phi(u, z_n)$ for all $n \in \mathbb{N}$, $\{T_n z_n\}$ is bounded. Take $r > 0$ such that $\{z_n\}, \{T_n z_n\} \subset B_r$. Since E is uniformly smooth, E^* is uniformly convex. Then, by Lemma 2.1, we have a continuous strictly increasing and convex function $g^* : [0, \infty) \to [0, \infty)$ such that $g^*(0) = 0$ and

$$\|\alpha_n J z_n + (1-\alpha_n) J T_n z_n\|^2 \leq \alpha_n \|z_n\|^2 + (1-\alpha_n)\|T_n z_n\|^2 - \alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|)$$

for all $n \in \mathbb{N}$. It follows from $u \in \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ that

$$\begin{aligned}
\phi(u, U_n z_n) &\leq \phi(u, J^{-1}(\alpha_n J z_n + (1-\alpha_n) J T_n z_n)) \\
&= \|u\|^2 - 2\langle u, \alpha_n J z_n + (1-\alpha_n) J T_n z_n \rangle + \|\alpha_n J z_n + (1-\alpha_n) J T_n z_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n \langle u, J z_n \rangle - 2(1-\alpha_n)\langle u, J T_n z_n \rangle \\
&\quad + \alpha_n \|z_n\|^2 + (1-\alpha_n)\|T_n z_n\|^2 - \alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|) \\
&= \alpha_n \phi(u, z_n) + (1-\alpha_n)\phi(u, T_n z_n) - \alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|) \\
&\leq \alpha_n \phi(u, z_n) + (1-\alpha_n)\phi(u, z_n) - \alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|) \\
&= \phi(u, z_n) - \alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|),
\end{aligned}$$

that is,

$$\alpha_n(1-\alpha_n) g^*(\|J z_n - J T_n z_n\|) \leq \phi(u, z_n) - \phi(u, U_n z_n) \to 0.$$

From (2.4) and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$g^*(\|J z_n - J T_n z_n\|) \to 0.$$

This implies that

$$\|J z_n - J T_n z_n\| \to 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|J^{-1}(J z_n) - J^{-1}(J T_n z_n)\| = 0. \tag{2.5}$$

Finally, we show that $\omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \bigcap_{n=1}^{\infty} \mathrm{F}(U_n)$, let $z' \in \omega_w\{z_n\}$. Then $z_{n_i} \rightharpoonup z'$ for some subsequence $\{n_i\}$ of $\{n\}$. Since $\{T_n\}$ satisfies AKTT-condition (II), there exist a mapping $T : C \to E$ and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\lim_{j\to\infty} \sup\{\|Tz - T_{n_{i_j}} z\| : z \in \{z_n\}\} = 0$$

and $\widehat{\mathrm{F}}(T) = \mathrm{F}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$. Then $z_{n_{i_j}} \rightharpoonup z'$. From (2.5), we have

$$\begin{aligned}
\|z_{n_{i_j}} - T z_{n_{i_j}}\| &\leq \|z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| + \|T z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| \\
&\leq \|z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| + \sup\{\|Tz - T_{n_{i_j}} z\| : z \in \{z_n\}\} \to 0
\end{aligned}$$

as $j \to \infty$. This implies that $z' \in \widehat{\mathrm{F}}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ and hence $\omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$. Therefore, $\{U_n\}$ satisfies the KT-condition. □

The following lemma is proved in [9, Lemma 5.2] which can be deduced from our Lemma 2.11.

Lemma 2.12. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into E such that $\bigcap_{i=1}^m \mathrm{F}(T_i)$ is nonempty and let $\{U_n\}$ be a family of block mappings defined by*

$$U_n = \Pi_C J^{-1}\bigg(\sum_{i=1}^{m} w_{n,i}(\alpha_{n,i}J + (1-\alpha_{n,i})JT_i)\bigg),$$

for all $n \in \mathbb{N}$, where $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq m\} \subset (0,1)$ and $\{\omega_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq m\} \subset (0,1)$ are sequences such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$, $\liminf_{n\to\infty} \omega_{n,i} > 0$ for all $i \in \{1, 2, \ldots, m\}$ and $\sum_{i=1}^m w_{n,i} = 1$ for all $n \in \mathbb{N}$. Then $\{U_n\}$ satisfies the KT-condition and $\bigcap_{n=1}^\infty \mathrm{F}(U_n) = \bigcap_{i=1}^m \mathrm{F}(T_i)$.

3. Weak convergence theorems

In this section, we establish weak convergence theorems for finding common fixed points of a countable family of relatively nonexpansive mappings in a Banach space.

Theorem 3.1. *Let E be a uniformly convex and smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{U_n\}$ be a family of relatively nonexpansive mappings from C into itself such that $\mathrm{F} = \bigcap_{n=1}^\infty \mathrm{F}(U_n)$ is nonempty, and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = U_n x_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

If $\{U_n\}$ satisfies the KT-condition and J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \mathrm{F}$. Moreover, $\lim_{n\to\infty} \Pi_{\mathrm{F}}(x_n) = z$.

Proof. For each $u \in F$ and $n \in \mathbb{N}$, we have

$$\phi(u, x_{n+1}) = \phi(u, U_n x_n) \leq \phi(u, x_n). \tag{3.1}$$

This implies that $\lim_{n\to\infty} \phi(u, x_n)$ exists. It follows that $\{x_n\}$ is bounded and

$$\phi(u, x_n) - \phi(u, U_n x_n) = \phi(u, x_n) - \phi(u, x_{n+1}) \to 0.$$

Since $\{U_n\}$ satisfies the KT-condition, $\omega_w\{x_n\} \subset \mathrm{F}$. For each $n \in \mathbb{N}$, let $\widetilde{x}_n = \Pi_{\mathrm{F}}(x_n)$. By (3.1) and Lemma 2.6, there is $z \in F$ such that $\widetilde{x}_n \to z$. To prove that $x_n \rightharpoonup z$, let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z' \in \omega_w\{x_n\} \subset \mathrm{F}$. Notice that

$$\langle \widetilde{x}_n - z', Jx_n - J\widetilde{x}_n \rangle \geq 0, \qquad \text{for all } n \in \mathbb{N}.$$

In particular,

$$\langle \widetilde{x}_{n_i} - z', Jx_{n_i} - J\widetilde{x}_{n_i} \rangle \geq 0.$$

Since $\widetilde{x}_n \to z$ and J is weakly sequentially continuous,

$$\langle z - z', Jz' - Jz \rangle \geq 0.$$

On the other hand, from the monotonicity of J, we have

$$\langle z' - z, Jz' - Jz \rangle \geq 0.$$

Thus, we have

$$\langle z' - z, Jz' - Jz \rangle = 0.$$

Using the strict convexity of E, we obtain $z' = z$. This implies that $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_{\mathrm{F}}(x_n)$. This completes the proof. □

Using Lemma 2.11 and Theorem 3.1, we have the following result.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a family of relatively nonexpansive mappings from C into E such that $\mathrm{F} = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n) \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. If $\{T_n\}$ satisfies the AKTT-condition (II) and J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_{\mathrm{F}}(x_n)\}$.

Using Lemma 2.12 and Theorem 3.1, we have the following result.

Corollary 3.3 ([9], Theorem 5.3)**.** *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into E such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}\left(\sum_{i=1}^m w_{n,i}(\alpha_{n,i} Jx_n + (1-\alpha_{n,i})JT_i x_n)\right) \quad \text{for all} \quad n \in \mathbb{N},$$

where $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq m\} \subset (0,1)$ and $\{\omega_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq m\} \subset (0,1)$ are sequences such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$, $\liminf_{n\to\infty} \omega_{n,i} > 0$ for all $i \in \{1, 2, \ldots, m\}$ and $\sum_{i=1}^m w_{n,i} = 1$ for all $n \in \mathbb{N}$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_{\mathrm{F}}(x_n)\}$.

4. Common solutions of a fixed point problem and a variational inequality problem

In this section, we present several related results which can be deduced by corresponding convergence theorems obtained in Section 3. Let C be a nonempty closed convex subset of a Hilbert space H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C. We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C \tag{4.1}$$

and

$$\|P_C x - y\|^2 \leq \|x - y\|^2 - \|P_C x - x\|^2 \quad \text{for all } x \in H, y \in C. \tag{4.2}$$

In Hilbert spaces, we have

(1) T is relatively nonexpansive if and only if T is quasi-nonexpansive with $I - T$ is demi-closed at zero;

(2) $\Pi_C = P_C$;
(3) J is an identity operator.

It is also known that H is uniformly convex and uniformly smooth.

Using Theorem 3.2, we obtain the following result:

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a family of quasi-nonexpansive mappings from C into H such that $\mathrm{F} = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$ and $I - T_n$ is demi-closed at zero for all $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)T_n x_n) \quad \text{for all} \quad n \in \mathbb{N}, \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. If $\{T_n\}$ satisfies the AKTT-condition (II), then $\{x_n\}$ converges weakly to the strong limit of $\{P_{\mathrm{F}}(x_n)\}$.

Lemma 4.2 ([6], Theorem 10.4)**.** *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into H. Then $I - T$ is demi-closed at zero.*

Corollary 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ be a family of nonexpansive mappings from C into H such that $\mathrm{F} = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \varnothing$. Let $\{x_n\}$ be a sequence defined by (4.3), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. If $\{T_n\}$ satisfies the AKTT-condition (II), then $\{x_n\}$ converges weakly to the strong limit of $\{P_{\mathrm{F}}(x_n)\}$.*

Let C be a nonempty closed convex subset of a Hilbert space H and A be a mapping of C into H. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of classical variational inequality problem is denoted by $\mathrm{VI}(C, A)$. A mapping A of C into H is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad \text{for all } x, y \in C;$$

(2) *α-inverse-strongly-monotone*, where $\alpha > 0$, if

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

Note that every α-inverse-strongly-monotone mapping is monotone and $(1/\alpha)$- Lipschitzian.

We need the following lemmas.

Lemma 4.4 ([4], Corollaries 15, 17)**.** *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitzian mapping of C into H and S be a nonexpansive mapping from C into H such that $\mathrm{F}(S) \cap \mathrm{VI}(C, A) \neq \varnothing$. Let T be a mapping of C into H defined by*

$$T = SP_C(I - \lambda A(P_C(I - \lambda A))),$$

where $\lambda \in (0, 1/k)$. Then

(i) *T is quasi-nonexpansive and $\mathrm{F}(T) = \mathrm{F}(S) \cap \mathrm{VI}(C, A)$,*

(ii) *$I - T$ is demi-closed at zero.*

Lemma 4.5. *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitzian mapping of C into H and $\{S_n\}$ be a family of nonexpansive mappings of C into H such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap \mathrm{VI}(C, A) \neq \varnothing$. Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of C into H defined by*

$$T_n = S_n P_C(I - \lambda_n A(P_C(I - \lambda_n A))),$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 1/k)$. If $(\{S_n\}, S)$ satisfies AKTT-condition (I) and $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$, then $\{T_n\}$ satisfies AKTT-condition (II).

Proof. By Lemma 4.4, we have $\mathrm{F}(T_n) = \mathrm{F}(S_n) \cap \mathrm{VI}(C, A)$ and hence

$$\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap \mathrm{VI}(C, A) = \mathrm{F}(S) \cap \mathrm{VI}(C, A) \neq \varnothing.$$

Let $\{n_i\}$ be a subsequence of $\{n\}$. Since $\{\lambda_{n_i}\}$ is a sequence in $[c, d]$, there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\lambda_{n_{i_j}} \to \lambda \in [c, d]$. Put

$$T = SP_C(I - \lambda A(P_C(I - \lambda A))).$$

Then T is a quasi-nonexpansive mapping of C into H and $I - T$ is demi-closed at zero. So, we get

$$\widehat{\mathrm{F}}(T) = \mathrm{F}(T) = \mathrm{F}(S) \cap \mathrm{VI}(C, A) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n).$$

Let $W_n = P_C(I - \lambda_n A(P_C(I - \lambda_n A))$ and $W = P_C(I - \lambda A(P_C(I - \lambda A)))$. Since P_C is nonexpansive and A is k-Lipschitzian,

$$\begin{aligned}
\|W_{n_{i_j}} z - Wz\| &\leq \|(I - \lambda_{n_{i_j}} A(P_C(I - \lambda_{n_{i_j}} A)))z - (I - \lambda A(P_C(I - \lambda A)))z\| \\
&= \left|\lambda_{n_{i_j}} - \lambda\right| \|A(P_C(I - \lambda_{n_{i_j}} A))z - A(P_C(I - \lambda A))z\| \\
&\leq k\left|\lambda_{n_{i_j}} - \lambda\right| \|P_C(I - \lambda_{n_{i_j}} A)z - P_C(I - \lambda A)z\| \\
&\leq k\left|\lambda_{n_{i_j}} - \lambda\right| \|(I - \lambda_{n_{i_j}} A)z - (I - \lambda A)z\| \\
&= k\left|\lambda_{n_{i_j}} - \lambda\right|^2 \|Az\| \qquad (4.4)
\end{aligned}$$

for all $z \in C$ and $j \in \mathbb{N}$. Let B be a bounded subset of C. Then $\{Az : z \in B\}$ and $\{Wz : z \in B\}$ are bounded. From (4.4) and Lemma 2.8, we obtain

$$\lim_{j \to \infty} \sup\{\|Wz - W_{n_{i_j}} z\| : z \in B\} = 0 \qquad (4.5)$$

and

$$\lim_{j \to \infty} \sup\{\|SWz - S_{n_{i_j}} Wz\| : z \in B\} = 0, \qquad (4.6)$$

respectively. From (4.5) and (4.6), we get

$$\begin{aligned}
&\sup\{\|Tz - T_{n_{i_j}} z\| : z \in B\} \\
&\quad = \sup\{\|SWz - S_{n_{i_j}} W_{n_{i_j}} z\| : z \in B\}
\end{aligned}$$

$$\leq \sup\{\|SWz - S_{n_{i_j}}Wz\| + \|S_{n_{i_j}}Wz - S_{n_{i_j}}W_{n_{i_j}}z\| : z \in B\}$$
$$\leq \sup\{\|SWz - S_{n_{i_j}}Wz\| + \|Wz - W_{n_{i_j}}z\| : z \in B\}$$
$$\leq \sup\{\|SWz - S_{n_{i_j}}Wz\| : z \in B\} + \sup\{\|Wz - W_{n_{i_j}}z\| : z \in B\} \to 0$$

as $j \to \infty$. This implies that $\{T_n\}$ satisfies the AKTT-condition (II). □

Using Lemma 4.5 and Theorem 4.1, we have the following theorem.

Theorem 4.6. *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitzian mapping of C into H and $\{S_n\}$ be a family of nonexpansive mappings of C into H such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap \mathrm{VI}(C, A) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$y_n = P_C(x_n - \lambda_n Ax_n),$$
$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)S_n P_C(x_n - \lambda_n Ay_n)),$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[a, b] \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 1/k)$. If $(\{S_n\}, S)$ satisfies AKTT-condition (I) and $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$, then $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap\mathrm{VI}(C,A)}(x_n)$.

Setting $S_n \equiv S$ in Theorem 4.6, we have the following result.

Corollary 4.7. *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitzian mapping of C into H and S be a nonexpansive mapping of C into H such that $\mathrm{F}(S) \cap \mathrm{VI}(C, A) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$y_n = P_C(x_n - \lambda_n Ax_n),$$
$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n)),$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[a, b] \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 1/k)$. Then $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap\mathrm{VI}(C,A)}(x_n)$.

Remark 4.8. Corollary 4.7 includes Theorem 3.1 of [13] as a special case.

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