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STRONG CONVERGENCE THEOREM FOR QUADRATIC MINIMIZATION PROBLEM WITH COUNTABLE CONSTRAINTS

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Abstract. In this paper, we introduce an iteration process of finding a unique solution of the quadratic minimization problem over the intersection of fixed point sets of countable nonexpansive mappings in a real Hilbert space. Then, we obtain a strong convergence theorem.

1. Introduction

The quadratic minimization problem with some constraints has been studied by many researchers. Let H be a real Hilbert space. Let $C_1, C_2, ...$ be closed convex subsets of H with $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Let u be an element of H. Then, we consider the following quadratic minimization problem:

$$\min \left\{ \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{\infty} C_n \right\},$$

where A is strongly positive. To find an optimal point of the quadratic minimization problem is connected with the *convex feasibility problem*, the problem of *image recovery* and *variational inequality problem*; see [6], [7], [10], [15], [25] and so on.

In particular, let H be a real Hilbert space. Let $T_1, T_2, ..., T_N$ be nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{N} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of T_n. Let u be an element of H. Many authors have studied the following quadratic minimization problem concerning a finite family of nonexpansive mappings:

$$\min \left\{ \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{N} F(T_n) \right\}.$$

In this setting, Yamada, Ogura, Yamashita and Sakaniwa [23] considered the following iterative scheme in a Hilbert space H:

$$x_1 = x \in H,\ x_{n+1} = \beta_n u + (I - \beta_n A) T_{n \bmod N} x_n$$

for all $n = 1, 2, ...$, where $0 \leq \beta_n \leq 1$ for every $n = 1, 2, ...$ Then, they showed that $\{x_n\}$ converges strongly to the unique solution of $\min\{(1/2)\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{N} F(T_n)\}$, where $\bigcap_{n=1}^{N} F(T_n)$ is the set of common fixed points of $T_1, T_2, ..., T_N$ satisfying

$$\bigcap_{n=1}^{N} F(T_n) = F(T_1 T_2 \cdots T_N) = F(T_N T_1 \cdots T_{N-1}) = \cdots = F(T_2 T_3 \cdots T_N T_1) \neq \emptyset,$$

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and $\{\beta_n\}$ satisfies $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+N} - \beta_n| < \infty$. Xu [20] showed a complementary result to Yamada, Ogura, Yamashita and Sakaniwa's theorem by replaced $\sum_{n=1}^{\infty} |\beta_{n+N} - \beta_n| < \infty$ with the general condtion: $\lim_{n\to\infty} \beta_n/\beta_{n+N} = 1$.

On the other hand, Takahashi [14] and Shimoji and Takahashi [10] studied a mapping, called a *W-mapping*, which was introduced for finding a common fixed point of infinite countable nonexpansive mappings; see Lemma 3.4 and Lemma 3.5.

In this paper, motivated by Takahashi [14], Shimoji and Takahashi [10], and Yamada, Ogura, Yamashita and Sakaniwa [23], we introduce an iteration process of finding a unique solution of the quadratic minimization problem over the intersection of fixed point sets of countable nonexpansive mappings in a real Hilbert space. Then, we obtain a strong convergence theorem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let I be the identity mapping on H. We also denote by $\mathbb{R}$ the set of real numbers. Let C be a nonempty closed convex subset of H. Then, a mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Using this equality, we can prove that if $T : C \to C$ is nonexpansive, then the set $F(T)$ is closed and convex; see [15]. We also know the following inequality:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$$

for all $x, y \in H$.

Let H be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper convex function. Then, we can define a multivalued mapping ∂f on H into 2^H by

$$\partial f(x) = \{z \in H : f(y) \ge \langle z, y - x\rangle + f(x), y \in H\}$$

for all $x \in H$. Such ∂f is said to be the *subdifferential* of f; see, for instance, [16].

Let C be a nonempty closed convex subset of a Hilbert space H. Then we define a function $i_C : H \to (-\infty, \infty]$ called the *indicator function* of C as follows:

$$i_C(x) = \begin{cases} 0 & (x \in C), \\ \infty & (x \notin C). \end{cases}$$

For any $x \in C$, we also define the set $N_C(x)$ as follows:

$$N_C(x) = \{z \in H : \langle z, y - x\rangle \le 0 \text{ for all } y \in C\}.$$

Such $N_C(x)$ is said to be the *normal cone* to C at $x \in C$. We know the following lemma; see, for example, [16].

Lemma 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $i_C : H \to (-\infty, \infty]$ be the indicator function of C and let $N_C(x)$ be the normal cone to C at $x \in C$. Then $\partial i_C(x) = N_C(x)$ for all $x \in C$.*

We also know the following theorem; see [15].

Theorem 2.1. *Let H be a Hilbert space and let f be a proper convex function of H into $(-\infty, \infty]$. If g is a continuous convex function of H into $(-\infty, \infty)$, then, for all $x \in H$,*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

The following lemmas [13] and [1] play important roles in the proof of our main theorem.

Lemma 2.2 ([13])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([1])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

3. Strongly positive operators and W-mappings

Let H be a real Hilbert space. Let A be a self-adjoint bounded linear operator of H into itself. Then, A is called strongly positive if there exists a real number γ with $0 < \gamma < 1$ such that

$$\langle Ax, x\rangle \geq \gamma\|x\|^2$$

for all $x \in H$. In particular, such A is called γ-strongly positive.

Remark. Since $\langle Ax, x\rangle \geq \gamma\|x\|^2$ for all $x \in H$, we have from the Schwarz inequality that for all $x \in H$,

$$\|Ax\|\|x\| \geq \langle Ax, x\rangle \geq \gamma\|x\|^2$$

and hence $\|Ax\| \geq \gamma\|x\|$. So, we have

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \geq \sup_{\|x\|=1} \gamma\|x\| \geq \gamma > 0$$

and hence

$$\|A\|^{-1} \leq \frac{1}{\gamma}.$$

If $0 < \alpha < \|A\|^{-1}$, then $0 < \alpha\gamma < \gamma\|A\|^{-1} \leq 1$.

The following lemma is in [20].

Lemma 3.1. *Let H be a Hilbert space. Let A be a γ-strongly positive self-adjoint bounded linear operator of H into itself, where $0 < \gamma < 1$. Then, for all α with $0 < \alpha < \|A\|^{-1}$, $\|I - \alpha A\| \leq 1 - \alpha\gamma$, where I is the identity mapping.*

Proof. From [12], we have

$$\|I - \alpha A\| = \sup_{\|x\|=1} \langle (I - \alpha A)x, x\rangle.$$

Hence, we have

$$\begin{aligned} \|I - \alpha A\| &= \sup_{\|x\|=1} \langle (I - \alpha A)x, x\rangle = \sup_{\|x\|=1} \{\langle x, x\rangle - \alpha\langle Ax, x\rangle\} \\ &\leq \sup_{\|x\|=1} \{\|x\|^2 - \alpha\gamma\|x\|^2\} = \sup_{\|x\|=1} (1 - \alpha\gamma)\|x\|^2 \\ &= 1 - \alpha\gamma. \end{aligned}$$

□

The following lemma is also well-known. However, for the sake of completeness, we give the proof.

Lemma 3.2. *Let H be a Hilbert space. Let A be a strongly positive self-adjoint bounded linear operator of H into itself. If f is defined by*

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle$$

for all $x \in H$, then $\partial f(x) = Ax - u$.

Proof. Since A is strongly positive and self-adjoint, we have that, for all $x, y \in H$,

$$\begin{aligned} & f(y) - f(x) - \langle Ax - u, y - x\rangle \\ &= \frac{1}{2}\langle Ay, y\rangle - \langle u, y\rangle - \frac{1}{2}\langle Ax, x\rangle + \langle u, x\rangle - \langle Ax - u, y - x\rangle \\ &= \frac{1}{2}(\langle Ay, y\rangle - 2\langle Ax, y\rangle + \langle Ax, x\rangle) \\ &= \frac{1}{2}(\langle Ay, y\rangle - \langle Ay, x\rangle - \langle Ax, y\rangle + \langle Ax, x\rangle) \\ &= \frac{1}{2}\langle A(y - x), y - x\rangle \geq 0, \end{aligned}$$

which means that $f(y) \geq \langle Ax - u, y - x\rangle + f(x)$. Hence, $Ax - u \in \partial f(x)$. Next, to show that $\partial f(x) \subset \{Ax - u\}$, let $z \in \partial f(x)$, that is,

$$\frac{1}{2}\langle Ay, y\rangle - \langle u, y\rangle \geq \langle z, y - x\rangle + \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle \text{ for all } y \in H.$$

Set $y = x + tw$ with $t > 0$ and $w \in H$. Then we have

$$\frac{1}{2}\langle A(x + tw), x + tw\rangle - \langle u, x + tw\rangle \geq \langle z, tw\rangle + \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle.$$

Since A is self-adjoint, this implies that

$$t\langle Ax - u, w\rangle + \frac{1}{2}t^2\langle Aw, w\rangle \geq t\langle z, w\rangle.$$

Dividing by t, we see that $\langle Ax-u,w\rangle+\frac{1}{2}t\langle Aw,w\rangle\geq\langle z,w\rangle$. Further as $t\downarrow 0$, we obtain

$$\langle Ax-u,w\rangle\geq\langle z,w\rangle.$$

Setting $w=z-(Ax-u)$, we have that $\|z-(Ax-u)\|^2\leq 0$, that is, $z=Ax-u$. Thus, we conclude that $\partial f(x)=Ax-u$. □

Using Theorem 2.1, we can prove the following lemma.

Lemma 3.3. *Let C be a closed convex subset of a Hilbert space H. Let A be a γ-strongly positive self-adjoint bounded linear operator of H into itself, where $0<\gamma<1$. Let g be a function of H into $(-\infty,\infty]$ defined by*

$$g(x)=\frac{1}{2}\langle Ax,x\rangle-\langle u,x\rangle+i_C(x)\text{ for all }x\in H,$$

where i_C is the indicator function of C. Let $z\in H$. Then the following are equivalent:

(1) $g(z)=\min\{g(x):x\in H\}$,
(2) $0\in\partial g(z)$,
(3) $\langle u-Az,x-z\rangle\leq 0$ *for all* $x\in C$.

In this case, $z\in C$ and such z is unique.

Proof. (1) ⇔ (2) is obvious. Further, by the definition of i_C we have $z\in C$. So, we shall show (2) ⇔ (3). Using Lemma 2.1, Lemma 3.2 and Theorem 2.1, we have

$$\partial g(z)=Az-u+N_C(z).$$

So, we have

$$\begin{aligned}&0\in\partial g(z)\\ \Leftrightarrow&0\in Az-u+N_C(z)\\ \Leftrightarrow&u-Az\in N_C(z)\\ \Leftrightarrow&\langle u-Az,x-z\rangle\leq 0\text{ for all }x\in C.\end{aligned}$$

Next, we show that such a point z is unique. Suppose that $\langle u-Az_1,x-z_1\rangle\leq 0$ and $\langle u-Az_2,y-z_2\rangle\leq 0$ for all $x,y\in C$. Putting $x=z_2$ and $y=z_1$, we have

$$\langle A(z_1-z_2),z_1-z_2\rangle\leq 0.$$

Since A is strongly positive, we have $\gamma\|z_1-z_2\|^2\leq\langle A(z_1-z_2),z_1-z_2\rangle$. Then we have $z_1=z_2$. □

Let C be a convex subset of a Hilbert space H. Let $T_1,T_2,...$ be infinite mappings of C into itself and let $\alpha_1,\alpha_2,...$ be real numbers such that $0\leq\alpha_i\leq 1$ for all $i=1,2,....$ Then, for all $n=1,2,...$, Takahashi [14] defined a mapping W_n of C into itself as follows:

$$\begin{aligned}U_{n,n+1}&=I,\\ U_{n,n}&=\alpha_nT_nU_{n,n+1}+(1-\alpha_n)I,\\ U_{n,n-1}&=\alpha_{n-1}T_{n-1}U_{n,n}+(1-\alpha_{n-1})I,\\ &\vdots\end{aligned}$$

$$\begin{aligned}U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1-\alpha_k)I,\\ U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})I,\\ &\vdots\\ U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1-\alpha_2)I,\\ W_n = U_{n,1} &= \alpha_1 T_1 U_{n,2} + (1-\alpha_1)I.\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$. We know the following lemmas by Shimoji and Takahashi [10].

Lemma 3.4 ([10])**.** *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < \alpha_n \leq b < 1$ for all $n = 1, 2, ...$ Then, for every $x \in C$ and $k = 1, 2, ...$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 3.4, for all $k = 1, 2, ...$ we define mappings $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k}x := \lim_{n\to\infty} U_{n,k}x \text{ and } Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such W is called the W-mapping generated by $T_1, T_2, ...,$ and $\alpha_1, \alpha_2, ...$

Lemma 3.5 ([10])**.** *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < \alpha_n \leq b < 1$ for all $n = 1, 2, ...$ Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

4. Main theorem

Let H be a real Hilbert space. Let $T_1, T_2, ...$ be nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let u be an element of H. Consider the following quadratic minimization problem:

$$\text{(P)} \qquad \min \left\{ \frac{1}{2}\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{\infty} F(T_n) \right\},$$

where A is strongly positive. It is known that the problem (P) has a unique solution z; see [15].

Now, we prove the following strong convergence theorem which is our main theorem in this paper:

Theorem 4.1. *Let H be a real Hilbert space. Let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, ...$ and $T_1, T_2, ...$ be nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For every $n = 1, 2, ...$, let W_n be the W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, ...$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Let u be an element of H and let A be a γ-strongly positive self-adjoint bounded linear operator of H into itself. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \beta_n u + (I - \beta_n A)W_n x_n$$

for every $n = 1, 2, \ldots$. *Then* $\{x_n\}$ *converges strongly to* z, *where* z *is a unique solution of* $\min\{(1/2)\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$.

Proof. Since $\lim_{n\to\infty} \beta_n = 0$, we may assume without loss of generality that

$$\beta_n < \|A\|^{-1}$$

for all $n = 1, 2, \ldots$. From Lemma 3.1, we have

$$\|I - \beta_n A\| \le 1 - \beta_n \gamma.$$

It follows that, for $y \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\|x_{n+1} - y\| &= \|\beta_n u + (I - \beta_n A)W_n x_n - y\| \\
&= \|\beta_n(u - Ay) + (I - \beta_n A)(W_n x_n - y)\| \\
&\le \beta_n \|u - Ay\| + (1 - \beta_n \gamma)\|x_n - y\| \\
&= \beta_n \gamma \cdot \frac{1}{\gamma}\|u - Ay\| + (1 - \beta_n \gamma)\|x_n - y\|.
\end{aligned}$$

Hence, by mathematical induction, we obtain

$$\|x_n - y\| \le \max\left\{\|x_1 - y\|, \ \frac{1}{\gamma}\|u - Ay\|\right\}.$$

This implies that $\{x_n\}$ is bounded. Then we also have that $\{W_n x_n\}$ and $\{T_n x_n\}$ are bounded. Put $K = \max\{\|u\|, \sup_{n\in\mathbb{N}} \|x_n\|, \sup_{n\in\mathbb{N}} \|T_n x_n\|, \sup_{n\in\mathbb{N}} \|A\|\|W_n x_n\|\}$. Then, we have that for every $n = 1, 2, \ldots$,

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\beta_{n+1} u + (I - \beta_{n+1} A)W_{n+1} x_{n+1} - (\beta_n u + (I - \beta_n A)W_n x_n)\| \\
&\le |\beta_{n+1} - \beta_n|\|u\| + \|(I - \beta_{n+1} A)W_{n+1} x_{n+1} - (I - \beta_{n+1} A)W_n x_{n+1} \\
&\quad + (I - \beta_{n+1} A)W_n x_{n+1} - (I - \beta_n A)W_n x_n\| \\
&\le |\beta_{n+1} - \beta_n|\|u\| + (1 - \beta_{n+1}\gamma)\|W_{n+1} x_{n+1} - W_n x_{n+1}\| \\
&\quad + (1 - \beta_{n+1}\gamma)\|W_n x_{n+1} - W_n x_n\| + |\beta_{n+1} - \beta_n|\|A\|\|W_n x_n\| \\
&\le (1 - \beta_{n+1}\gamma)\|x_{n+1} - x_n\| + 2K|\beta_{n+1} - \beta_n| \\
&\quad + (1 - \beta_{n+1}\gamma)\|W_{n+1} x_{n+1} - W_n x_{n+1}\|.
\end{aligned}$$

As in the proof of Lemma 3.4 in [10], we also have

$$\begin{aligned}
\|W_{n+1} x_{n+1} - W_n x_{n+1}\| &= \|U_{n+1,1} x_{n+1} - U_{n,1} x_{n+1}\| \\
&= \|\alpha_1 T_1 U_{n+1,2} x_{n+1} + (1 - \alpha_1) x_{n+1} \\
&\qquad - (\alpha_1 T_1 U_{n,2} x_{n+1} + (1 - \alpha_1) x_{n+1})\| \\
&= \alpha_1 \|T_1 U_{n+1,2} x_{n+1} - T_1 U_{n,2} x_{n+1}\| \\
&\le \alpha_1 \|U_{n+1,2} x_{n+1} - U_{n,2} x_{n+1}\| \\
&\le \alpha_1 \alpha_2 \|U_{n+1,3} x_{n+1} - U_{n,3} x_{n+1}\| \\
&\le \cdots \\
&\le \prod_{i=1}^{n} \alpha_i \|U_{n+1,n+1} x_{n+1} - U_{n,n+1} x_{n+1}\|
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \alpha_i \| \alpha_{n+1} T_{n+1} U_{n+1,n+2} x_{n+1} \\
&\qquad + (1-\alpha_{n+1}) x_{n+1} - x_{n+1} \| \\
&= \prod_{i=1}^{n+1} \alpha_i \| T_{n+1} x_{n+1} - x_{n+1} \| \\
&\leq 2K \left(\prod_{i=1}^{n+1} \alpha_i \right).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq (1-\beta_{n+1}\gamma)\|x_{n+1} - x_n\| + 2K|\beta_{n+1} - \beta_n| \\
&\qquad + 2K(1-\beta_{n+1}\gamma) \left(\prod_{i=1}^{n+1} \alpha_i \right) \\
&\leq (1-\beta_{n+1}\gamma)\|x_{n+1} - x_n\| + 2K|\beta_{n+1} - \beta_n| + 2K \left(\prod_{i=1}^{n+1} \alpha_i \right) \\
&= (1-\beta_{n+1}\gamma)\|x_{n+1} - x_n\| + 2K \left(|\beta_{n+1} - \beta_n| + \prod_{i=1}^{n+1} \alpha_i \right)
\end{aligned}
$$

for every $n = 1, 2, \ldots$. On the other hand, since $0 < \alpha_i \leq b < 1$, we have that $\prod_{i=1}^{n+1} \alpha_i \leq \prod_{i=1}^{n+1} b = b^{n+1}$. This implies that

$$
\sum_{n=1}^{\infty} \prod_{i=1}^{n+1} \alpha_i = \lim_{m\to\infty} \sum_{n=1}^{m} \prod_{i=1}^{n+1} \alpha_i \leq \lim_{m\to\infty} \sum_{n=1}^{m} b^{n+1} = \lim_{m\to\infty} \frac{b^2(1-b^m)}{1-b} = \frac{b^2}{1-b} < \infty.
$$

Thus $\sum_{n=1}^{\infty}(|\beta_{n+1} - \beta_n| + \prod_{i=1}^{n+1} \alpha_i) < \infty$. Therefore it follows from Lemma 2.2 that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since

$$
\begin{aligned}
\|x_{n+1} - W_n x_n\| &= \|\beta_n u + (I - \beta_n A) W_n x_n - W_n x_n\| \\
&= \beta_n \|u - A W_n x_n\| \leq 2K\beta_n
\end{aligned}
$$

for every $n = 1, 2, \ldots$, we have $\lim_{n\to\infty} \|x_{n+1} - W_n x_n\| = 0$. From

$$
\|x_n - W_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n x_n\|,
$$

we also have $\lim_{n\to\infty} \|x_n - W_n x_n\| = 0$. Let z be the unique solution of $\min\{(1/2)\langle Ax, x\rangle - \langle u, x\rangle : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$. To show $\limsup_{n\to\infty} \langle u - Az, x_n - z\rangle \leq 0$, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty} \langle u - Az, x_n - z\rangle = \lim_{i\to\infty} \langle u - Az, x_{n_i} - z\rangle.
$$

As $\{x_{n_i}\}$ is bounded, we have that a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to z_0. We may assume without loss of generality that $x_{n_i} \rightharpoonup z_0$. We show that $z_0 \in \bigcap_{n=1}^{\infty} F(T_n)$. Suppose that $z_0 \notin \bigcap_{n=1}^{\infty} F(T_n)$. By Lemma 3.5, we have $z_0 \neq W z_0$.

From Opial's theorem [9], $\lim_{n\to\infty} \|x_n - W_n x_n\| = 0$ and the definition of W, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - z_0\| &< \liminf_{i\to\infty} \|x_{n_i} - Wz_0\| \\
&\le \liminf_{i\to\infty} \{\|x_{n_i} - W_{n_i}x_{n_i}\| + \|W_{n_i}x_{n_i} - W_{n_i}z_0\| \\
&\quad +\|W_{n_i}z_0 - Wz_0\|\} \\
&\le \liminf_{i\to\infty} \{\|x_{n_i} - W_{n_i}x_{n_i}\| + \|x_{n_i} - z_0\| + \|W_{n_i}z_0 - Wz_0\|\} \\
&= \lim_{i\to\infty} \|x_{n_i} - W_{n_i}x_{n_i}\| + \liminf_{i\to\infty} \|x_{n_i} - z_0\| \\
&\quad + \lim_{i\to\infty} \|W_{n_i}z_0 - Wz_0\| \\
&= \liminf_{i\to\infty} \|x_{n_i} - z_0\|.
\end{aligned}$$

This is a contradiction. Hence, we obtain $z_0 \in \bigcap_{n=1}^{\infty} F(T_n)$. From Lemma 3.3, we have

$$\limsup_{n\to\infty} \langle u - Az, x_n - z\rangle = \langle u - Az, z_0 - z\rangle \le 0.$$

Since

$$x_{n+1} - z = (I - \beta_n A)(W_n x_n - z) + \beta_n(u - Az),$$

we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(I - \beta_n A)(W_n x_n - z) + \beta_n(u - Az)\|^2 \\
&\le \|(I - \beta_n A)(W_n x_n - z)\|^2 + 2\beta_n\langle u - Az, x_{n+1} - z\rangle \\
&\le (1 - \beta_n\gamma)\|x_n - z\|^2 + \beta_n\gamma\left(\frac{2}{\gamma}\langle u - Az, x_{n+1} - z\rangle\right).
\end{aligned}$$

Using Lemma 2.3, we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$. □

5. Applications

Let C be a nonempty closed convex subset of a Hilbert space H, and let A be a mapping C into H. The variational inequatily problem for A is to find $z \in C$ such that

$$\langle Az, x - z\rangle \ge 0$$

for all $x \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Using Theorem 4.1, we prove the following two theorems.

Theorem 5.1. *Let H be a real Hilbert space. Let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \le b < 1$ for every $i = 1, 2, \ldots$ and $T_1, T_2, \ldots$ be nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. For every $n = 1, 2, \ldots$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \le \beta_n \le 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Let A be a γ-strongly positive self-adjoint bounded linear operator of H into itself. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = (I - \beta_n A)W_n x_n$$

for every $n = 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to z, where z is the unique solution of $VI(\bigcap_{n=1}^{\infty} F(T_n), A)$.

Proof. Putting $u = 0$ in Theorem 4.1, we have $x_n \to z$, where z is the unique solution of $\min\{(1/2)\langle Ax, x\rangle : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$. From Lemma 3.3, we have that z satisfies $\langle -Az, x - z\rangle \leq 0$ for all $x \in \bigcap_{n=1}^{\infty} F(T_n)$. This implies that x_n converges to the unique solution of $VI(\bigcap_{n=1}^{\infty} F(T_n), A)$. □

Remark. Putting $T_n = P_C$ for all $n = 1, 2, \ldots$, we have from Theorem 5.1 that x_n converges strongly to the unique solution of $VI(C, A)$.

Theorem 5.2. *Let H be a real Hilbert space. Let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$ and $T_1, T_2, \ldots$ be nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For every $n = 1, 2, \ldots$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Let u be an element of H. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \beta_n u + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} u$.

Proof. Putting $A = I$ in Theorem 4.1, we have $x_n \to z$, where z is the unique solution of $\min\{(1/2)\|x\|^2 - \langle u, x\rangle : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$. Since

$$\begin{aligned} \frac{1}{2}\|x\|^2 - \langle u, x\rangle &= \frac{1}{2}\left(\|x\|^2 - 2\langle u, x\rangle + \|u\|^2 - \|u\|^2\right) \\ &= \frac{1}{2}\left(\|x - u\|^2 - \|u\|^2\right), \end{aligned}$$

we also have that z is the unique solution of $\min\{(1/2)\left(\|x - u\|^2 - \|u\|^2\right) : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$. This implies that z is the unique solution of $\min\{\|x - u\| : x \in \bigcap_{n=1}^{\infty} F(T_n)\}$. From the definition of $P_{\bigcap_{n=1}^{\infty} F(T_n)}$, we have $z = P_{\bigcap_{n=1}^{\infty} F(T_n)} u$. □

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