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INDEX AND FIXED POINT THEORY FOR COMPACT ABSORBING CONTRACTIVE PERMISSIBLE MAPS

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Abstract. An index theory is presented for compact absorbing contractive permissible maps and several new fixed point theorems are given for such maps.

1. Introduction

In Section 2 we present a slight generalization of the fixed point index for permissible maps [7] and in Section 3 we provide an alternative approach to establishing fixed point theory using projective limits. These results improve those in the literature; see [1-6, 8, 10-11, 13-15] and the references therein.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$.

Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q.

It is well known [9, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

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Let X and Y be Hausdorff topological spaces.

Definition 1.1. A multivalued map $F : X \to K(Y)$ ($K(Y)$ denotes the class of nonempty compact subsets of Y) is in the class $\mathcal{A}_m(X,Y)$ if (i). F is continuous, and (ii). for each $x \in X$ the set $F(x)$ consists of one or m acyclic components; here m is a positive integer. We say F is of class $\mathcal{A}_0(X,Y)$ if F is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

Definition 1.2. A decomposition $(F_1, ..., F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \xrightarrow{F_1} X_1 \xrightarrow{F_2} X_2 \xrightarrow{F_3} \; ..... \; \xrightarrow{F_{n-1}} X_{n-1} \xrightarrow{F_n} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ .... \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, ..., F_n)$. The number n is said to be the length of the decomposition $(F_1, ..., F_n)$. We will denote the class of decompositions by $\mathcal{D}(X,Y)$.

Definition 1.3. An upper semicontinuous map $F : X \to K(Y)$ is permissible provided it admits a selector $G : X \to K(Y)$ which is determined by a decomposition $(G_1, ..., G_n) \in \mathcal{D}(X,Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X,Y)$.

Let X be a Hausdorff topological space and let a map Φ be determined by $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X,X)$. Then Φ is said to be a Lefschetz map if the induced homology homomorphism [7, pp 27] $(\Phi_1, ..., \Phi_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [7]) $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\mathbf{\Lambda}(\Phi) = \Lambda((\Phi_1, ..., \Phi_k)_\star).$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{D}$) provided every compact $\Phi : X \to K(X)$ determined by a decomposition $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies Φ has a fixed point.

Example. If X is a metric ANR then X is a Lefschetz space (for the class $\mathcal{D}$) (see [7, pp 42] or [9, Section 50-53]).

A map $\Phi \in \mathcal{P}(X,X)$ is said to be a Lefschetz map provided every selector $G : X \to K(X)$ of Φ which is determined by $(G_1, ..., G_k) \in \mathcal{D}(X,X)$ is such that $(G_1, ..., G_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi \in \mathcal{P}(X,X)$ is a Lefschetz map as described above then we define the Lefschetz set $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\begin{aligned}\mathbf{\Lambda}(\Phi) \;\; = \;\; & \{\Lambda((G_1, ..., G_k)_\star) : \; (G_1, ..., G_k) \in \mathcal{D}(X,X) \\ & \text{and } (G_1, ..., G_k) \text{ determines a selector of } \Phi\}.\end{aligned}$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{P}$) provided every compact $\Phi \in \mathcal{P}(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies

Φ has a fixed point.

Example. If X is a metric ANR then X is a Lefschetz space (for the class $\mathcal{P}$) (see [7, pp 43]).

The following concepts will be needed in Section 3. Let (X, d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x, S) = \inf_{y \in S} d(x, y)$. Also $diam\, S = \sup\{d(x, y) : x, y \in S\}$. We let $B(x, r)$ denote the open ball in X centered at x of radius r and by $B(S, r)$ we denote $\cup_{x \in S} B(x, r)$. For two nonempty subsets S_1 and S_2 of X we define the generalized Hausdorff distance H to be

$$H(S_1, S_2) = \inf\{\epsilon > 0 : \; S_1 \subseteq B(S_2, \epsilon), \; S_2 \subseteq B(S_1, \epsilon)\}.$$

Now suppose $G : S \to 2^X$. Then G is said to be hemicompact if each sequence $\{x_n\}_{n \in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \; \forall \, \alpha, \, \beta \in I, \, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [1, 2] $\cap_{\alpha \in I} E_\alpha$.)

2. Index Theory

In this section we give a slight generalization of the fixed point index for permissible maps. In [15] we defined compact absorbing contractions as follows. Let X be a Hausdorff topological space. A map $F : X \to K(X)$ determined by $(F_1, ..., F_k) \in \mathcal{D}(X, X)$ is said to be a compact absorbing contraction (and we write $F \in CAC(X, X)$ or $F \in CAC(X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$;
(ii) $F|_Y$ (which of course is determined by the restriction of $(F_1, ..., F_k) \in \mathcal{D}(X, X)$ to the subset Y) is a compact map with Y a Lefschetz space;
(iii) for every compact $K \subseteq X$ there is an integer $n = n(K)$ such that $F^n(K) \subseteq Y$.

Remark 2.1. If $Y = U$ is an open subset of X then (iii) could be changed to

(iii)' . for every $x \in X$ there exists an integer $n = n(x)$ such that $F^{n(x)}(x) \subseteq Y = U$.

However for our index below we will need Y in our definition above to be an open ANR. So in this paper X will be a metric space. A map $F : X \to K(X)$ determined by $(F_1, ..., F_k) \in \mathcal{D}(X, X)$ is said to be a compact absorbing contraction (written $F \in CAC(X, X)$ or $F \in CAC(X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$;

(ii) $F|_Y$ (which of course is determined by the restriction of $(F_1, ..., F_k) \in \mathcal{D}(X, X)$ to the subset Y) is a compact map with Y an open ANR;

(iii) for every $x \in X$ there exists an integer $n = n(x)$ such that $F^{n(x)}(x) \subseteq Y$.

We are now in a position to define the index. Let X be a metric space, W open in X, $F \in CAC(X, X)$ (with Y as described above) and $x \notin F\,x$ for $x \in \partial W$ (here ∂W denotes the boundary of W in X). We now define

$$i\,(X, F, W) = ind\,(Y, F|_Y, W \cap Y)$$

where ind is as described in [7, pp 38] (note $W \cap Y$ is an open subset of the ANR Y and note $x \notin Fx$ for $x \in \partial_Y(W \cap Y)$ since $\partial_Y(W \cap Y) = \overline{(W \cap Y)}^Y \backslash (W \cap Y) \subseteq (\overline{W} \cap Y) \backslash (W \cap Y) \subseteq \overline{W} \backslash W = \partial W$). It is worthwhile remarking that if there exists $x \in X$ with $x \in F\,x$ then from (iii) above we have $x \in Y$.

Our definition is independent of our choice of Y. To see this let (i), (ii) and (iii) above hold with Y replaced by Y_1. First note $Y_1 \cap Y$ is an ANR since it is an open subset of the ANR Y. Also note $F(W \cap Y \cap Y_1) \subseteq Y_1 \cap Y$ since $F(Y_1) \subseteq Y_1$ and $F(Y) \subseteq Y$. Now from the contraction property we have

$$\begin{aligned} ind(Y, F, Y_1 \cap Y \cap W) &= ind(Y_1 \cap Y, F, (Y_1 \cap Y) \cap (Y_1 \cap Y \cap W)) \\ &= ind(Y_1 \cap Y, F, Y_1 \cap Y \cap W) \end{aligned}$$

and from the localization property, since $(Y_1 \cap W) \cap Y$ is open in Y and $(Y_1 \cap W) \cap Y \subseteq Y \cap W \subseteq Y$, we have

$$ind(Y, F, Y \cap W) = ind(Y, F, Y_1 \cap Y \cap W).$$

Thus

$$ind(Y, F, Y \cap W) = ind(Y_1 \cap Y, F, Y_1 \cap Y \cap W). \tag{2.1}$$

Similarly from the contraction property we have

$$\begin{aligned} ind(Y_1, F, Y_1 \cap Y \cap W) &= ind(Y_1 \cap Y, F, (Y_1 \cap Y) \cap (Y_1 \cap Y \cap W)) \\ &= ind(Y_1 \cap Y, F, Y_1 \cap Y \cap W) \end{aligned}$$

and from the localization property, since $(Y \cap W) \cap Y_1$ is open in Y_1 and $(Y \cap W) \cap Y_1 \subseteq Y_1 \cap W \subseteq Y_1$, we have

$$ind(Y_1, F, Y_1 \cap W) = ind(Y_1, F, Y_1 \cap Y \cap W).$$

Thus

$$ind(Y_1, F, Y_1 \cap W) = ind(Y_1 \cap Y, F, Y_1 \cap Y \cap W). \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$ind(Y_1, F, Y_1 \cap W) = ind(Y, F, Y \cap W).$$

Now we discuss some properties of the index.

Property I. (Additivity) Let W be an open subset of X, $F \in CAC(X, X)$ and assume $W_1 \subseteq W$, $W_2 \subseteq W$ are disjoint open sets with $Fix\,F|_{\overline{W}} \subseteq W_1 \cap W_2$. Then

$$i(X, F, W) = i(X, F, W_1) + i(X, F, W_2).$$

Proof. Let Y be as described above. Then the additivity property of ind (see [7, pp 39], note $W_1 \cap Y$ and $W_2 \cap Y$ are open in Y and disjoint and $Fix\, F|_{\overline{(W \cap Y)}^Y} \subseteq (W_1 \cap Y) \cup (W_2 \cap Y)$) we have

$$\begin{aligned} i(X, F, W) &= ind(Y, F, W \cap Y) = ind(Y, F, W_1 \cap Y) + ind(Y, F, W_2 \cap Y) \\ &= i(X, F, W_1) + i(X, F, W_2). \end{aligned}$$

□

The following three properties are also immediate.

Property II. (Localization) Let W and W_1 be open subsets of X with $W_1 \subseteq W$ and let $F \in CAC(X, X)$ with $Fix\, F|_{\overline{W}} \subseteq W_1$. Then

$$i(X, F, W) = i(X, F, W_1).$$

Property III. (Contraction) Let W be an open subset of X, $F \in CAC(X, X)$, $x \notin F\, x$ for $x \in \partial W$, with $F(W) \subseteq A$ and $F(A) \subseteq A$. Then

$$i(X, F, W) = i(A, F, A \cap W).$$

Property IV. (Excision) Let W be an open subset of X, $F \in CAC(X, X)$ with $Fix\, F \subseteq W$. Then

$$i(X, F, W) = i(X, F, X).$$

Property V. (Normalization) Let $F \in CAC(X, X)$. Then

$$i(X, F, X) = \mathbf{\Lambda}\,(F)\; (= \Lambda\,((F_1, ..., F_k)_\star));$$

here F is determined by $(F_1, ..., F_k) \in \mathcal{D}(X, X)$.

Proof. Let Y and F be as described above (i.e. F is determined by $(F_1, ..., F_k) \in \mathcal{D}(X, X)$). We consider three homomorphisms

$$(F_1, ..., F_k)_{\star X} : H(X) \to H(X), \quad (F_1, ..., F_k)_{\star Y} : H(Y) \to H(Y)$$

and

$$(F_1, ..., F_k)_\star : H(X, Y) \to H(X, Y).$$

Since Y is an ANR (so in particular a Lefschetz space) then $(F_1, ..., F_k)_{\star Y}$ is a Leray endomorphism. For any compact $K \subseteq X$ there exists an n with $F^n(K) \subseteq Y$ and since we consider Čech homology with compact carriers then $(F_1, ..., F_k)_\star$ is weakly nilpotent. Then [9, pp 53] guarantees that $(F_1, ..., F_k)_\star$ is a Leray endomorphism and $\Lambda\,((F_1, ..., F_k)_\star) = 0$. Also [9, Property 11.5, pp 52] guarantees that $(F_1, ..., F_k)_{\star X}$ is a Leray endomorphism and $\Lambda\,((F_1, ..., F_k)_{\star X}) = \Lambda\,((F_1, ..., F_k)_{\star Y})$. Now [7, pp40] guarantees that

$$ind(Y, F, Y) = \Lambda\,((F_1, ..., F_k)_{\star Y})$$

so

$$\Lambda\,((F_1, ..., F_k)_{\star Y}) = ind(Y, F, Y) = i(X, F, X).$$

□

Other properties, for example the homotopy property, could also be formulated. We leave these to the reader.

Let X be a metric space. A map $F \in \mathcal{P}(X,X)$ is said to be a compact absorbing contraction (written $F \in CACP(X,X)$ or $F \in CACP(X)$) if every selector $G : X \to K(X)$ of F which is determined by $(G_1, ..., G_n) \in \mathcal{D}(X,X)$ is such that $G \in CAC(X,X)$.

Note from the proof of Property V we have that $\Lambda\left((G_1, ..., G_n)_\star\right)$ is defined and as a result we can define the Lefschetz set $\mathbf{\Lambda}\,(F)$ as in Section 1.

Let W be an open subset of X with $Fix\,F \cap \partial W = \emptyset$. We now define

$$\begin{aligned} i(X,F,W) \;\; = \;\; & \{i(X,G,W) : \; G \text{ is a selector of } F \text{ determined by} \\ & (G_1, ..., G_n) \in \mathcal{D}(X,X) \text{ and } G \in CAC(X,X)\}. \end{aligned}$$

The following properties (we just list a few) are immediate.

Property I. Let W be an open subset of X, $F \in CACP(X,X)$ and assume $W_1 \subseteq W$, $W_2 \subseteq W$ are disjoint open sets with $Fix\,F|_{\overline{W}} \subseteq W_1 \cap W_2$. Then

$$i(X,F,W) \subseteq i(X,F,W_1) + i(X,F,W_2).$$

Property II. Let $F \in CACP(X,X)$. Then

$$i(X,F,X) = \mathbf{\Lambda}\,(F).$$

Property III. Let W be an open subset of X, $F \in CACP(X,X)$ and $Fix\,F \cap \partial W = \emptyset$. If $i(X,F,W) \neq \{0\}$ then F has a fixed point in W.

We now briefly discuss the Lefschetz fixed point theorem for pairs of spaces. Let X and A be metric spaces with $A \subseteq X$. Let $F \in CACP(X,X)$ and assume $F(A) \subseteq A$. If a selector G of F is determined by $(G_1, ..., G_k) \in \mathcal{D}(X,X)$ then $G(A) \subseteq A$. The same idea as in the proof of Property V (note $G|_A \in CAC(A,A)$) above guarantees that

$$(G_1, ..., G_k)_{\star X} : H(X) \to H(X) \;\text{ and }\; (G_1, ..., G_k)_{\star A} : H(A) \to H(A)$$

are Leray endomorphisms. Now [9, Property 11.5, pp 52] guarantees that

$$(G_1, ..., G_k)_\star : H(X,A) \to H(X,A)$$

is a Leray endomorphism and

$$\Lambda\left((G_1, ..., G_k)_\star\right) = \Lambda\left((G_1, ..., G_k)_{\star X}\right) - \Lambda\left((G_1, ..., G_k)_{\star A}\right).$$

As a result we can define the relative Lefschetz set $\mathbf{\Lambda}_{X,A}\,(F)$ by

$$\begin{aligned} \mathbf{\Lambda}_{X,A}\,(F) \;\; = \;\; & \{\Lambda((G_1, ..., G_k)_\star) : \; (G_1, ..., G_k)_\star : H(X,A) \to H(X,A) \\ & \text{and } (G_1, ..., G_k) \text{ determines a selector of } F\}. \end{aligned}$$

Theorem 2.2. *Let X and A be metric spaces with $F \in CACP(X,X)$ and $F(A) \subseteq A$. Then $\mathbf{\Lambda}_{X,A}\,(F) \neq \{0\}$ implies F has a fixed point in $\overline{X \backslash A}$.*

Proof. Suppose $(G_1, ..., G_k) \in \mathcal{D}(X,X)$ determines a selector of F with $\Lambda((G_1, ..., G_k)_\star) \neq 0$. As above we have

$$\Lambda\left((G_1, ..., G_k)_\star\right) = \Lambda\left((G_1, ..., G_k)_{\star X}\right) - \Lambda\left((G_1, ..., G_k)_{\star A}\right). \tag{2.3}$$

Suppose F has no fixed points in $\overline{X\backslash A}$. Then $Fix\, G \subseteq X\backslash(\overline{X\backslash A}) = int_X\, A$. Let $W = int_X\, A$ so W is an open subset of X with $W \subseteq A$. Therefore from Property IV and Property V above (note $Fix\, G \subseteq W$) we have

$$(2.4) \qquad i(X, G, W) = i(X, G, X) = \Lambda\,((G_1, ..., G_k)_{\star X}).$$

Similarly since $W = int_X\, A \subseteq A$ is an open subset of A we have

$$(2.5) \qquad i(A, G, W) = i(A, G, A) = \Lambda\,((G_1, ..., G_k)_{\star A}).$$

Also from Property III above (note $G(A) \subseteq F(A) \subseteq A$) we have

$$i(X, G, X) = i(A, G, A \cap W) = i(A, G, W)$$

so from (2.4) and (2.5) we have

$$\Lambda\,((G_1, ..., G_k)_{\star X}) = \Lambda\,((G_1, ..., G_k)_{\star A}).$$

As a result (2.3) implies $\Lambda\,((G_1, ..., G_k)_\star) = 0$, a contradiction. □

3. Fixed point theory in Fréchet spaces

We now present another approach to establishing fixed points based on projective limits (see [1, 2]). Our results improve those in [3, 4]. Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, ...\}$. We assume that the family of seminorms satisfies

$$(3.1) \qquad |x|_1 \le |x|_2 \le |x|_3 \le .... \quad \text{for every} \quad x \in E.$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \le r\, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$(3.2) \qquad x \sim_n y \quad \text{iff} \quad |x - y|_n = 0.$$

We denote by $\mathbf{E}^n = (E\,/\!\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (3.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (3.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m\,/\!\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \le m \le k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \le m$. We now assume the following condition holds:

$$(3.3) \qquad \begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases}$$

Remark 3.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii) In many applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$\left\{ \begin{array}{l} E_1 \supseteq E_2 \supseteq \ldots\ldots\ldots \text{ and for each } n \in N, \\ |j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, x|_n \le |x|_{n+1}\, \forall\, x \in E_{n+1} \end{array} \right. \tag{3.4}$$

(here we use the notation from [1, 2] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [1, 2]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$, $int\, X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x-y|_n \le r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n\, \mu_n\, F\,(x)$ is closed. Let $n \in N$ and $M_n = j_n\, \mu_n(M)$. Since we first consider Volterra type operators we assume (note this assumption is only needed in Theorems 3.2 and 3.6)

$$\text{if } \ x,\, y \in E \ \text{ with } \ |x-y|_n = 0 \ \text{ then } \ H_n(F\,x, F\,y) = 0; \tag{3.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in N, \forall x,\, y \in M$ if $j_n\, \mu_n\, x = j_n\, \mu_n\, y$ then $j_n\, \mu_n\, F\, x = j_n\, \mu_n\, F\, y$ and of course here we do not need to assume that $j_n\, \mu_n\, F\,(x)$ is closed for each $n \in N$ and $x \in M$). Now (3.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\, \mu_n(x)$ and we let

$$F_n\, y = j_n\, \mu_n\, F\, x$$

(we could of course call it $F\,y$ since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\, \mu_n(z)$ then $j_n\, \mu_n\, F\, x = j_n\, \mu_n\, F\, z$ from (3.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Now we present some fixed point theorems in Fréchet spaces which improve results in [4]. Our first two results are motivated by Volterra type operators.

Theorem 3.2. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $X_n \subseteq Z_n$ for each $n \in N$. Also assume for each $n \in N$ and $x \in Z$ that $j_n\, \mu_n\, F\,(x)$ is closed and in* *addition for each* $n \in N$ *that* $F_n : X_n \to 2^{E_n}$ *is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in CACP(X_n, X_n) \tag{3.6}$$

$$\begin{cases} \text{for each } n \in N,\ F_n \text{ has no fixed points in } \partial W_n;\ \text{here} \\ W_n = U_n \cap X_n \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } X_n \end{cases} \tag{3.7}$$

$$\text{for each } n \in N,\ i(X_n, F_n, W_n) \neq \{0\} \tag{3.8}$$

and

$$\begin{cases} \text{for each } n \in \{2,3,....\} \text{ if } y \in W_n \text{ solves } y \in F_n\, y \\ \text{in } E_n \text{ then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in W_k \text{ for } k \in \{1,...,n-1\}. \end{cases} \tag{3.9}$$

In addition assume either

$$\text{for each } n \in N,\ F_n : W_n \to 2^{E_n} \text{ is hemicompact} \tag{3.10}$$

or

$$\text{for each } n \in N,\ F_n : \overline{W_n} \to 2^{E_n} \text{ is hemicompact} \tag{3.11}$$

hold. Then F *has a fixed point in* E*.*

Remark 3.3. Note in Theorem 3.2 if $x \in X_n$ then $x \in Z_n$ so there exists a $y \in Z$ with $x = j_n\, \mu_n\,(y)$ and so $F_n\,(x) = j_n\, \mu_n\, F(y)$.

Proof. Fix $n \in N$. It is well known that $U_n = int\, U_n$. To see this note $U_n \subseteq \overline{U_n} \setminus \partial U_n$ since if $y \in U_n$ then there exists $x \in U$ with $y = j_n \mu_n(x)$ and this together with $U = pseudo - int\, U$ yields $j_n \mu_n(x) \in \overline{U_n} \setminus \partial U_n$ i.e. $y \in \overline{U_n} \setminus \partial U_n$. In addition notice

$$\overline{U_n} \setminus \partial U_n = (int\, U_n \cup \partial U_n) \setminus \partial U_n = int\, U_n \setminus \partial U_n = int\, U_n$$

since $int\, U_n \cap \partial U_n = \emptyset$. Consequently

$$U_n \subseteq \overline{U_n} \setminus \partial U_n = int\, U_n, \quad \text{so } U_n = int\, U_n.$$

Now (3.8) guarantees that there exists $y_n \in W_n = U_n \cap X_n$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in W_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}\,(y_k) \in W_1$ for $k \in N \backslash \{1\}$ from (3.9). Note $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1; to see note for $n \in N$ fixed there exists a $x \in E$ with $y_n = j_n\, \mu_n\,(x)$ so $j_n\, \mu_n\,(x) \in F_n\,(y_n) = j_n\, \mu_n\, F(x)$ on E_n so on E_1 we have

$$\begin{aligned} j_1\, \mu_{1,n}\, j_n^{-1}(y_n) &= j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x) \in j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\, F(x) \\ &= j_1\, \mu_{1,n}\, \mu_n\, F(x) = j_1\, \mu_1\, F(x) = F_1(j_1\, \mu_1\,(x)) \\ &= F_1(j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x)) = F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n)). \end{aligned}$$

As a result $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1, $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \in W_1$ for $n \in N$, together with (3.10) or (3.11) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{W_1}$ (note $z_1 \in W_1$ if (3.10) holds) with $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\, z_1$ since F_1 is upper semicontinuous. Note $z_1 \in W_1$ (this follows from (3.7) if (3.11) holds). Let $N_1 = N_1^\star \setminus \{1\}$. Now $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \in W_2$ for $n \in N_1$ together with (3.10) or (3.11) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{W_2}$ with $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \to z_2$

in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2\, z_2$. Also $z_2 \in W_2$. Note from (3.4) and the uniqueness of limits that $j_1\, \mu_{1,2}\, j_2^{-1}\, z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) = j_1\, \mu_{1,2}\, j_2^{-1}\, j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{W_k}$ with $j_k\, \mu_{k,n}\, j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\, z_k$. Also $z_k \in W_k$. Note $j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k\, z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, j_{k+1}\, \mu_{k+1,k+2}\, j_{k+2}^{-1}\, z_{k+2} \\ &= j_k\, \mu_{k,k+2}\, j_{k+2}^{-1}\, z_{k+2} = ... = j_k\, \mu_{k,m}\, j_m^{-1}\, z_m = \pi_{k,m}\, z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in Z$ since $z_k \in W_k \subseteq X_k \subseteq Z_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k\, \mu_k\,(y) = z_k \in F_k\, z_k = j_k\, \mu_k\, F\, y \quad \text{in} \quad E_k$$

so $y \in F\, y$ in E. □

Remark 3.4. We can replace (3.9) in Theorem 3.2 with

$$\begin{cases} \text{for each } \ n \in \{2, 3, ...\} \ \text{ if } \ y \in W_n \ \text{ solves } \ y \in F_n\, y \\ \text{in } \ E_n \ \text{ then } \ j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in \overline{W_k} \ \text{ for } \ k \in \{1, ..., n-1\} \end{cases}$$

provided (3.11) holds.

Remark 3.5. In Theorem 3.2 it is possible to replace $X_n \subseteq Z_n$ with X_n a subset of the closure of Z_n in E_n provided Z is a closed subset of E so in this case we could have $Z = X$ if X is closed. To see this note from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$ we can conclude that $y \in \overline{Z} = Z$ (note $q \in \overline{Z}$ iff for every $k \in N$ there exists $(x_{k,m}) \in Z$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\, \mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\, \mu_k\,(y) \in Z_k$ and so $j_k\, \mu_k\,(y) \in j_k\, \mu_k\, F\,(y)$ in E_k as before.

Essentially the same reasoning as in Theorem 3.2 yields the following result (in addition here we have the analogue of Remark 3.4 and Remark 3.5).

Theorem 3.6. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $\overline{X_n} \subseteq Z_n$ for each $n \in N$. Also assume for each $n \in N$ and $x \in Z$ that $j_n\, \mu_n\, F\,(x)$ is closed and in addition for each $n \in N$ that $F_n : \overline{X_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } \ n \in N, \ F_n \in CACP(\overline{X_n}, \overline{X_n}) \tag{3.12}$$

$$\begin{cases} \text{for each } \ n \in N, \ F_n \ \text{ has no fixed points in } \ \partial\, W_n; \text{ here} \\ W_n = U_n \cap \overline{X_n} \ \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } \overline{X_n} \end{cases} \tag{3.13}$$

and

$$\text{for each } \ n \in N, \ i(\overline{X_n}, F_n, W_n) \neq \{0\}. \tag{3.14}$$

Also assume (3.9) *and either* (3.10) *or* (3.11) *hold. Then F has a fixed point in E.*

Our next two results are motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (3.20).

Theorem 3.7. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $X_n \subseteq Z_n$ for each $n \in N$. Also for each $n \in N$ assume there exists $F_n : X_n \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in CACP(X_n, X_n) \tag{3.15}$$

$$\begin{cases} \text{for each } n \in N,\ F_n \text{ has no fixed points in } \partial W_n;\ \text{here} \\ W_n = U_n \cap X_n \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } X_n \end{cases} \tag{3.16}$$

$$\text{for each } n \in N,\ i(X_n, F_n, W_n) \neq \{0\} \tag{3.17}$$

$$\begin{cases} \text{for each } n \in \{2, 3, ...\} \text{ if } y \in W_n \text{ solves } y \in F_n\, y \text{ in } E_n \\ \text{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in W_k \text{ for } k \in \{1, ..., n-1\} \end{cases} \tag{3.18}$$

$$\begin{cases} \text{for any sequence } \{y_n\}_{n\in N} \text{ with } y_n \in W_n \\ \text{and } y_n \in F_n\, y_n \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, ...\},\ N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, ...\},\ N_0 = N, \text{ and a } z_k \in \overline{W_k} \text{ with} \\ j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k \end{cases} \tag{3.19}$$

and

$$\begin{cases} \text{if there exists a } w \in Z \text{ and a sequence } \{y_n\}_{n\in N} \\ \text{with } y_n \in W_n \text{ and } y_n \in F_n\, y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, ...\} \text{ of } N \text{ with } j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to j_k\, \mu_k(w) \\ \text{in } E_k \text{ as } n \to \infty \text{ in } S, \text{ then } w \in F\, w \text{ in } E. \end{cases} \tag{3.20}$$

Then F has a fixed point in E.

Remark 3.8. Notice to check (3.19) we need to show that for each $k \in N$ the sequence $\{j_k\, \mu_{k,n}\, j_n^{-1}(y_n)\}_{n\in N_{k-1}} \subseteq \overline{W_k}$ is sequentially compact.

Proof. For each $n \in N$ there exists $y_n \in W_n$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in W_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}(y_k) \in W_1$ for $k \in \{2, 3, ...\}$. Now (3.19) with $k = 1$ guarantees that there exists a subsequence $N_1 \subseteq \{2, 3, ...\}$ and a $z_1 \in \overline{W_1}$ with $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1. Look at $\{y_n\}_{n\in N_1}$. Now $j_2\, \mu_{2,n}\, j_n^{-1}(y_n) \in W_2$ for $k \in N_1$. Now (3.19) with $k = 2$ guarantees that there exists a subsequence $N_2 \subseteq \{3, 4, ...\}$ of N_1 and a $z_2 \in \overline{W_2}$ with $j_2\, \mu_{2,n}\, j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2. Note from (3.4) and the uniqueness of limits that $j_1\, \mu_{1,2}\, j_2^{-1}\, z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) = j_1\, \mu_{1,2}\, j_2^{-1}\, j_2\, \mu_{2,n}\, j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq ......, \quad N_k \subseteq \{k+1, k+2, ...\}$$

and $z_k \in \overline{W_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1,2,...\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = ... = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in Z$ since $z_k \in \overline{W_k} \subseteq Z_k$ for each $k \in N$. Also since $y_n \in F_n\,y_n$ in E_n for $n \in N_k$ and $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k = y$ in E_k as $n \to \infty$ in N_k we have from (3.20) that $y \in F\,y$ in E. □

Remark 3.9. From the proof we see that condition (3.18) can be removed from the statement of Theorem 3.7. We include it only to explain condition (3.19) (see Remark 3.8).

Remark 3.10. Note we could replace $X_n \subseteq Z_n$ above with X_n a subset of the closure of Z_n in E_n if Z is a closed subset of E (so in this case we can take $Z = X$ if X is a closed subset of E).

Remark 3.11. In fact we could replace (3.18) in Theorem 3.7 with

$$\begin{cases} \text{for each } n \in \{2,3,....\} \text{ if } y \in W_n \text{ solves } y \in F_n\,y \text{ in } E_n \\ \text{then } j_k\,\mu_{k,n}\,j_n^{-1}(y) \in \overline{W_k} \text{ for } k \in \{1,...,n-1\} \end{cases}$$

and the result above is again true.

Essentially the same reasoning as in Theorem 3.7 yields the following result (in addition here we have the analogue of Remark 3.10 and Remark 3.11).

Theorem 3.12. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $\overline{X_n} \subseteq Z_n$ for each $n \in N$. Also for each $n \in N$ assume there exists $F_n : \overline{X_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in CACP(\overline{X_n}, \overline{X_n}) \tag{3.21}$$

$$\begin{cases} \text{for each } n \in N,\ F_n \text{ has no fixed points in } \partial\,W_n;\ \text{here} \\ W_n = U_n \cap \overline{X_n} \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } \overline{X_n} \end{cases} \tag{3.22}$$

and

$$\text{for each } n \in N,\ i(\overline{X_n}, F_n, W_n) \neq \{0\}. \tag{3.23}$$

In addition assume (3.18), (3.19) *and* (3.20) *hold. Then F has a fixed point in E.*

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