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# WEAK AND STRONG CONVERGENCE THEOREMS FOR INEXACT ORBITS OF UNIFORMLY LIPSCHITZIAN MAPPINGS

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Abstract. We study the influence of errors on the convergence of orbits of uniformly Lipschitzian mappings in Banach and metric spaces.

## 1. Introduction

The study of the convergence of iterates of nonexpansive self-mappings is a central topic in Nonlinear Functional Analysis. It began with Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. Banach's celebrated result also yields convergence of iterates to this unique fixed point. There are several generalizations of Banach's fixed point theorem which show that the convergence of iterates holds for larger classes of nonexpansive mappings. For example, Rakotch [9] introduced the class of contractive mappings and showed that their iterates also converged to their unique fixed point.

In view of the above discussion, it is natural to ask if convergence of the iterates of nonexpansive mappings will be preserved in the presence of computational errors. In [2] we provide affirmative answers to this question. Related results can be found, for example, in [3, 5]. More precisely, in [2] we show that if all exact iterates of a given nonexpansive mapping converge (to fixed points), then this convergence continues to hold for inexact orbits with summable errors. In [7] we continued to study the influence of computational errors on the convergence of iterates of nonexpansive mappings in both Banach and metric spaces. We show there that if all the orbits of a nonexpansive self-mapping of a metric space $X$ converge to some closed subset $F$ of $X$, then all inexact orbits with summable errors also converge to the attractor $F$. On the other hand, we also construct examples which show that the convergence of inexact orbits fails if the errors are not summable.

Clearly, each power of a nonexpansive mapping is also nonexpansive. In the present paper we are concerned with a larger class of mappings, namely, the class of uniformly Lipschitzian mappings. (See, for example, [4, p. 34]). Recall that a self-mapping $T$ of a metric space $(X, \rho)$ is said to be uniformly Lipschitzian if there is a constant $C = C(T)$ such that

$$\rho(T^n x, T^n y) \leq C\rho(x, y)$$

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for all $x, y \in X$ and for any $n = 1, 2, \ldots$. This notion is metrically invariant, that is, if a mapping $T$ is uniformly Lipschitzian with respect to a metric $\rho_1$, then it is also uniformly Lipschitzian with respect to any equivalent metric $\rho_2$. Indeed, if $\rho_1(x, y) \le c_1\rho_2(x, y)$ and $\rho_2(x, y) \le c_2\rho_1(x, y)$ for all $x, y \in X$, where $c_1, c_2$ are positive constants, and if $\rho_1(T^n x, T^n y) \le C\rho_1(x, y)$ for all $x, y \in X$ and all natural numbers $n$, then

$$\rho_2(T^n x, T^n y) \le c_2\rho_1(T^n x, T^n y) \le c_2 C\rho_1(x, y) \le c_1 c_2 C\rho_2(x, y).$$

In particular, any nonexpansive mapping becomes uniformly Lipschitzian when we pass to an equivalent metric. Conversely, given a uniformly Lipschitzian mapping $T$, there is an equivalent metric with respect to which $T$ is nonexpansive [4, p. 35].

Our purpose in the present paper is to discuss the convergence of orbits of uniformly Lipschitzian mappings in the presence of computational errors.

## 2. Convergence to fixed points

We begin with the following basic result.

**Theorem 2.1.** *Let $T$ be a uniformly Lipschitzian self-mapping of a complete metric space $(X, \rho)$. Assume that for any $x \in X$, the sequence $\{T^n x\}$ converges in $X$ and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy*

$$\rho(x_{n+1}, Tx_n) \le r_n, \qquad n = 0, 1, \ldots. \tag{2.1}$$

*If $\sum_{n=0}^{\infty} r_n < \infty$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges in $(X, \rho)$ to a fixed point of $T$.*

*Proof.* For arbitrary positive integers $n$ and $k$, we have

$$\rho(T^n x_k, x_{n+k}) \le \sum_{i=1}^{n} \rho(T^i x_{n+k-i}, T^{i-1} x_{n+k-i+1}) \tag{2.2}$$

$$\le C\sum_{i=1}^{n} \rho(Tx_{n+k-i}, x_{n+k-i+1}) \le C\sum_{i=1}^{n} r_{n+k-i} = C\sum_{i=k}^{n+k-1} r_i \le C\sum_{i=k}^{\infty} r_i$$

(here, as usual, $T^0 x = x$ for any $x \in X$).

For an arbitrary given $\epsilon > 0$, we can find a number $k_0$ such that $C\sum_{i=k}^{\infty} r_i < \epsilon/4$ for all $k \ge k_0$. Inequality (2.2) implies that $\rho(T^n x_k, x_{n+k}) < \epsilon/4$ for all $k \ge k_0$ and all $n \ge 1$. Moreover, the assumptions of the theorem ensure the existence of $\lim_{n\to\infty} T^n x_k = y_k$ for any fixed $k \ge k_0$, and thus the existence of a natural number $n_k$ such that $\rho(T^n x_k, y_k) < \epsilon/4$ for any $n \ge n_k$. Consequently,

$$\rho(x_{n+k}, y_k) \le \rho(x_{n+k}, T^n x_k) + \rho(T^n x_k, y_k) < \epsilon/2 \tag{2.3}$$

for all $n \ge n_k$.

Fixing some $k \ge k_0$ and taking arbitrary natural numbers $m, n \ge n_k$, we obtain that

$$\rho(x_{n+k}, x_{m+k}) \le \rho(x_{n+k}, y_k) + \rho(x_{m+k}, y_k) < \epsilon,$$

which means that the sequence $\{x_n\}$ is a Cauchy sequence, and therefore has a limit $\bar{x}$. Passing to the limit in (2.3), we obtain that $\rho(\bar{x}, y_k) \le \epsilon/2$. Since this holds for arbitrary $\epsilon > 0$ and any $k \geq k_0$, we conclude that $\lim_{k\to\infty} y_k = \bar{x}$. But all $y_k$ are fixed points of the mapping $T$, i.e., $Ty_k = y_k$. Hence

$$\rho(T\bar{x}, \bar{x}) \leq \rho(T\bar{x}, Ty_k) + \rho(y_k, \bar{x}) \leq (C+1)\rho(y_k, \bar{x}) \ \longrightarrow\ 0 \ \text{ as } k \to \infty.$$

Thus $T\bar{x} = \bar{x}$, as asserted. □

**Corollary 2.2.** *Let $Y$ be a quasi-normed Abelian group with a generalized triangle inequality:*

$$\|x+y\| \leq M(\|x\| + \|y\|),\ x, y \in Y, \quad M > 1. \tag{2.4}$$

*Let $S$ be a uniformly Lipschitzian (in particular, nonexpansive) self-mapping of the space $Y$, equipped with the quasi-metric $\rho(x,y) = \|x-y\|$. Assume that the sequence of iterates $\{S^n x\}_{n=0}^{\infty}$ converges for any $x \in Y$. Then, for any sequence $\{x_n\}_{n=0}^{\infty} \subset Y$ such that $\rho(x_{n+1}, Sx_n) \leq r_n$ for all nonnegative integers $n$, the condition*

$$\sum_{n=0}^{\infty} r_n^{\alpha} < \infty, \quad \alpha = \frac{\ln 2}{\ln 2 + \ln M}, \tag{2.5}$$

*is sufficient for the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to some fixed point of $S$.*

*Proof.* As shown in [6], any quasi-normed Abelian group can be metrized by some metric $\rho_1(x,y)$ equivalent to $\|x-y\|^{\alpha}$ with the $\alpha$ defined above. In fact, this metric is defined by $\rho_1(x,y) := \|x-y\|_*$, where

$$\|u\|_* = \inf\{\sum_{i=1}^{n} \|u_i\|^{\alpha} :\ u = \sum_{i=1}^{n} u_i,\ n \geq 1\}, \tag{2.6}$$

and the (quasi-)metrics are connected by the inequalities

$$\rho_1(x,y) \leq [\rho(x,y)]^{\alpha} \leq 2\rho_1(x,y). \tag{2.7}$$

Since all powers of the operator $S$ satisfy the inequality $\rho(S^n x, S^n y) \leq C\rho(x,y)$, we have

$$\rho_1(S^n x, S^n y) \leq [\rho(S^n x, S^n y)]^{\alpha} \leq C^{\alpha}[\rho(x,y)]^{\alpha} \leq 2C^{\alpha}\rho_1(x,y),$$

that is, $S$ remains a uniformly Lipschitzian mapping with respect to the metric $\rho_1(x,y)$ as well. Moreover,

$$\rho_1(x_{n+1}, Sx_n) \leq [\rho(x_{n+1}, Sx_n)]^{\alpha} \leq r_n^{\alpha}.$$

Hence condition (2.5) allows us to apply Theorem 2.1 and conclude that the sequence $\{x_n\}_{n=0}^{\infty}$ converges in the metric $\rho_1(x,y)$. By inequalities (2.7), it converges in the quasi-metric $\rho(x,y)$ too. □

## 3. Nonexpansive mapppings

In the previous section we studied the convergence of inexact orbits of uniformly Lipschitzian mappings in the presence of errors. Analogous results for nonexpansive mappings, which are a special case of uniformly Lipschitzian mappings, were obtained in [2, 7, 8]. It should be mentioned that there we investigated not only strong but also weak convergence of iterates.

In this section we recall some of these results.

Let $(X,\rho)$ be a metric space. For each $x \in X$ and each nonempty and closed set $A \subset X$, put

$$\rho(x,A) = \inf\{\rho(x,y) :\ y \in A\}.$$

In [2] we obtain the following result.

**Theorem 3.1.** *Let $(X,\rho)$ be a complete metric space and let $T : X \to X$ satisfy*

$$\rho(Tx,Ty) \le \rho(x,y) \text{ for all } x,y \in X.$$

*Suppose that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges in $(X,\rho)$.*

*Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfies $\sum_{n=0}^{\infty}\gamma_n < \infty$, and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy $\rho(x_{n+1},Tx_n) \le \gamma_n$, $n = 0,1,\dots$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of $T$ in $(X,\rho)$.*

In Theorem 3.1 we obtain convergence of iterates to a point while in the next theorem, established in [7], we deal with convergence to an attracting set.

**Theorem 3.2.** *Let $(X,\rho)$ be a metric space and let $T : X \to X$ satisfy*

$$\rho(Tx,Ty) \le \rho(x,y) \text{ for all } x,y \in X.$$

*Suppose that $F$ is a nonempty and closed subset of $X$ such that for each $x \in X$,*

$$\lim_{i\to\infty} \rho(T^i x, F) = 0.$$

*Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0,\infty)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$, and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy $\rho(x_{n+1},Tx_n) \le \gamma_n$, $n = 0,1,\dots$. Then*

$$\lim_{i\to\infty} \rho(x_i, F) = 0.$$

In the following two theorems, which were also established in [2, 7], respectively, we deal with weak convergence of iterates.

**Theorem 3.3.** *Let $(E,||\cdot||)$ be a Banach space with dual $(E^*,||\cdot||_*)$, $X$ a nonempty and closed subset of $E$, and let $F$ be a nonempty and closed subset of $X$ in the norm topology.*

*Let $T : X \to X$ be such that*

$$||Tx - Ty|| \le ||x - y|| \text{ for all } x,y \in X$$
$$Tx = x \text{ for all } x \in F.$$

*Assume that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges weakly to an element of $F$.*

*Let $\{\gamma_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfy $\sum_{n=0}^{\infty}\gamma_n < \infty$, and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy $||x_{n+1} - Tx_n|| \le \gamma_n$, $n = 0,1,\dots$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ weakly converges to a point of $F$.*

**Theorem 3.4.** *Let $(E,||\cdot||)$ be a reflexive Banach space with dual $(E^*,||\cdot||_*)$, $X$ be a nonempty and closed subset of $E$,*

$$\rho(x,y) = ||x - y||,\ x,y \in E,$$

*and let $F$ be a nonempty and closed subset of $X$ in the norm topology.*

*Let $T : X \to X$ satisfy*

$$||Tx - Ty|| \le ||x - y|| \text{ for all } x,y \in X.$$

*Assume that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to $F$.*

*Let $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfy $\sum_{n=0}^{\infty} \gamma_n < \infty$ and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy $||x_{n+1} - Tx_n|| \leq \gamma_n$, $n = 0, 1, \ldots$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to $F$.*

## 4. Uniformly Lipschitzian mappings

Let $(X, \rho)$ be a metric space and let $T : X \to X$ be a uniformly Lipschitzian mapping, namely,

$$\rho(T^n x, T^n y) \leq c\rho(x, y) \text{ for all natural numbers } n \text{ and all } x, y \in X, \tag{4.1}$$

where $c$ is a positive constant.

In this section we return to the study of the convergence of inexact orbits of such a mapping $T$ in the presence of computational errors, which was begun in Section 2. There we presented a direct proof of Theorem 2.1, which is a generalization of Theorem 3.1. In the next two theorems we show that this generalization of Theorem 3.1, obtained in Section 2, as well as a generalization of Theorem 3.2, can, as a matter of fact, be deduced from Theorems 3.1 and 3.2, respectively. As before, here we also put $T^0 x = x$ for all $x \in X$.

**Theorem 4.1.** *Let the metric space $(X, \rho)$ be complete and let the mapping $T : X \to X$ be uniformly Lipschitzian. Suppose that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges in $(X, \rho)$.*

*Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies $\sum_{n=0}^{\infty} \gamma_n < \infty$, and that a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfies $\rho(x_{n+1}, Tx_n) \leq \gamma_n$, $n = 0, 1, \ldots$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of $T$ in $(X, \rho)$.*

**Theorem 4.2.** *Let $(X, \rho)$ be a metric space and let $T : X \to X$ be a uniformly Lipschitzian mapping. Let $F$ be a nonempty and closed subset of $X$, and suppose that for each $x \in X$,*

$$\lim_{i \to \infty} \rho(T^i x, F) = 0.$$

*Let $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$ and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy $\rho(x_{n+1}, Tx_n) \leq \gamma_n$, $n = 0, 1, \ldots$. Then*

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

*Proofs of Theorems 4.1 and 4.2.* Put

$$\rho_1(x, y) = \sup\{\rho(T^n x, T^n y) : \ n = 0, 1, \ldots\}, \ x, y \in X.$$

By (4.1), $\rho_1$ is a metric on $X$ and for all $x, y \in X$,

$$\rho(x, y) \leq \rho_1(x, y) \leq c\rho(x, y) \tag{4.2}$$

and

$$\rho_1(Tx, Ty) \leq \rho_1(x, y). \tag{4.3}$$

In both Theorems 4.1 and 4.2 we obtain that for all integers $n \geq 0$,

$$\rho_1(x_{n+1}, Tx_n) \leq c\rho(x_{n+1}, Tx_n) \leq c\gamma_n, \ \sum_{n=0}^{\infty} c\gamma_n < \infty.$$

In the case of Theorem 4.1, since $\{T^n x\}_{n=0}^{\infty}$ converges in $(X, \rho_1)$ for each $x \in X$, we can apply Theorem 3.1 with $\rho = \rho_1$ and conclude that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of $T$ in $(X, \rho_1)$ and also in $(X, \rho)$.

In the case of Theorem 4.2, since we have for all $x \in X$,

$$\lim_{i\to\infty} \rho_1(T^i x, F) \le \lim_{i\to\infty} c\rho(T^i x, F) = 0,$$

we can apply Theorem 3.2 with $\rho = \rho_1$ and obtain that

$$\lim_{i\to\infty} \rho(x_i, F) \le \lim_{i\to\infty} \rho_1(x_i, F) = 0.$$

Thus both Theorems 4.1 and 4.2 are proved. □

As in the case of convergence to a fixed point (Theorems 2.1 and 4.1), we can extend Theorem 4.2 to quasi-metric spaces.

**Corollary 4.3.** *Let $Y$ be a quasi-normed Abelian group with a generalized triangle inequality (2.4), equipped with a quasi-metric $\rho(x, y) = \|x-y\|$. Let $S$ be a uniformly Lipschitzian (in particular, nonexpansive) self-mapping of $(Y, \rho)$ such that*

$$\lim_{i\to\infty} \rho(S^i x, F) = 0$$

*for some nonempty and closed set $F \subset Y$ and each $x \in Y$. Then, for any sequence $\{x_n\}_{n=0}^{\infty} \subset Y$ such that $\rho(x_{n+1}, Sx_n) \le \gamma_n$ for all nonnegative integers $n$, the condition*

$$\sum_{n=0}^{\infty} \gamma_n^{\alpha} < \infty, \quad \alpha = \frac{\ln 2}{\ln 2 + \ln M},$$

*is sufficient for the convergence $\lim_{n\to\infty} \rho(x_n, F) = 0$.*

Analogs of Theorems 3.3 and 3.4 need separate proofs. Before stating them we first prove a simple auxiliary result.

**Lemma 4.4.** *Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\{x_n\}_{n=0}^{\infty} \subset X$ and*

$$\rho(x_{n+1}, Tx_n) \le \gamma_n,\ n = 0, 1, \ldots.$$

*Then for each integer $n > 0$ and each integer $k \ge 0$,*

$$\rho(T^n x_k, x_{n+k}) \le c \sum_{i=k}^{\infty} \gamma_i.$$

*Proof.* Let $n > 0$ and $k \ge 0$ be integers. Then by (4.1),

$$\rho(T^n x_k, x_{n+k}) \le \sum_{i=1}^{n} \rho(T^i x_{n+k-i}, T^{i-1} x_{n+k-i+1})$$

$$\le c \sum_{i=1}^{n} \rho(T x_{n+k-i}, x_{n+k-i+1}) \le c \sum_{i=1}^{n} \gamma_{n+k-i} = c \sum_{i=k}^{n+k-1} \gamma_i \le c \sum_{i=k}^{\infty} \gamma_i.$$

Lemma 4.4 is proved. □

In the next two theorems we assume that $(E, ||\cdot||)$ is a Banach space with dual $(E^*, ||\cdot||_*)$, $X$ is a nonempty and closed subset of $E$, $F$ is a nonempty and closed subset of $X$ in the norm topology,

$$\rho(x,y) = ||x-y||,\ x,y \in X,$$

and that $T : X \to X$ satisfies (4.1) with $c > 0$. The following theorem is a generalization of Theorem 3.3.

**Theorem 4.5.** *Let $Tx = x$ for all $x \in F$. Assume that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^\infty$ converges weakly to an element of $F$.*

*Let $\{\gamma_n\}_{n=0}^\infty \subset (0,\infty)$ satisfy $\sum_{n=0}^\infty \gamma_n < \infty$ and let a sequence $\{x_n\}_{n=0}^\infty \subset X$ satisfy*

$$||x_{n+1} - Tx_n|| \le \gamma_n,\ n = 0,1,\dots. \tag{4.4}$$

*Then the sequence $\{x_n\}_{n=1}^\infty$ weakly converges to a point of $F$.*

*Proof.* Let $k \ge 0$ be an integer and define a sequence $\{y_i^{(k)}\}_{i=k}^\infty$ by

$$y_k^{(k)} = x_k,\ y_{i+1}^{(k)} = Ty_i^{(k)},\ i = k, k+1, \dots. \tag{4.5}$$

For each integer $k \ge 0$, there exists $y_*^{(k)} \in F$ which is a weak limit of $\{y_i^{(k)}\}_{i=k}^\infty$. By (4.5), Lemma 4.4 and (4.4), for each integer $n \ge 0$ and each integer $k \ge 0$,

$$||y_{n+k}^{(k)} - x_{n+k}|| = ||T^n x_k - x_{n+k}|| \le c\sum_{i=k}^\infty \gamma_i. \tag{4.6}$$

By (4.6), for each pair of integers $k_1, k_2 \ge 0$ and each integer $p \ge k_1, k_2$,

$$||y_p^{(k_1)} - y_p^{(k_2)}|| \le 2c \sum_{i=\min\{k_1,k_2\}}^\infty \gamma_i. \tag{4.7}$$

We show that the sequence $\{y_*^{(k)}\}_{k=0}^\infty$ is a Cauchy sequence. Let $\epsilon > 0$. Choose a natural number $\bar{k}$ such that

$$2(c+1)\sum_{i=\bar{k}}^\infty \gamma_i < \epsilon/2. \tag{4.8}$$

Let $k_1, k_2 \ge \bar{k}$ be integers. We claim that

$$||y_*^{(k_1)} - y_*^{(k_2)}|| \le 2(c+1)\sum_{i=\bar{k}}^\infty \gamma_i. \tag{4.9}$$

To prove this, it is sufficient to show that for each $f \in E^*$ satisfying $||f||_* \le 1$, we have

$$|f(y_*^{(k_1)} - y_*^{(k_2)})| \le 2(c+1)\sum_{i=\bar{k}}^\infty \gamma_i. \tag{4.10}$$

Assume that $f \in E^*$ and $||f||_* \le 1$. Then by (4.7),

$$|f(y_*^{(k_1)} - y_*^{(k_2)})| = |\lim_{i\to\infty} f(y_i^{(k_1)} - y_i^{(k_2)})|$$

$$\leq \limsup_{i\to\infty} ||y_i^{(k_1)} - y_i^{(k_2)}|| \leq 2c \sum_{i=\bar{k}}^{\infty} \gamma_i$$

and (4.10) holds. Therefore (4.9) is true for each pair of integers $k_1, k_2 \geq \bar{k}$, as claimed. When combined with (4.8), this implies that

$$||y_*^{(k_1)} - y_*^{(k_2)}|| < \epsilon/2 \text{ for all integers } k_1, k_2 \geq \bar{k}. \tag{4.11}$$

Since $\epsilon$ is an arbitrary positive number, we conclude that $\{y_*^{(k)}\}_{k=0}^{\infty}$ is a Cauchy sequence which converges in norm to $\bar{y} \in F$. In view of (4.11), we have

$$||y_*^{(k)} - \bar{y}|| \leq \epsilon/2 \text{ for all integers } k \geq \bar{k}. \tag{4.12}$$

Let

$$f \in E^*, \ ||f||_* \leq 1. \tag{4.13}$$

By (4.13), (4.6), (4.12) and (4.8), for each integer $m > \bar{k}$,

$$\begin{aligned} |f(x_m - \bar{y})| &\leq |f(x_m - y_m^{(\bar{k})})| + |f(y_m^{(\bar{k})} - y_*^{(\bar{k})})| + |f(y_*^{(\bar{k})} - \bar{y})| \\ &\leq ||x_m - y_m^{(\bar{k})}|| + |f(y_m^{(\bar{k})} - y_*^{(\bar{k})})| + ||y_*^{(\bar{k})} - \bar{y}|| \\ \leq c \sum_{i=\bar{k}}^{\infty} \gamma_i &+ |f(y_m^{(\bar{k})} - y_*^{(\bar{k})})| + \epsilon/2 \leq |f(y_m^{(\bar{k})} - y_*^{(\bar{k})})| + (3/4)\epsilon. \end{aligned}$$

Since $y_m^{(\bar{k})}$ converges weakly for $y_*^{(\bar{k})}$ as $m \to \infty$, we conclude that for all sufficiently large natural numbers $m$, $|f(x_m - \bar{y})| < \epsilon$. Thus $\{x_i\}_{i=0}^{\infty}$ does indeed converge weakly to $\bar{y}$ and Theorem 4.5 is proved. □

Our last theorem is a generalization of Theorem 3.4.

**Theorem 4.6.** *Let the Banach space $(E, ||\cdot||)$ be reflexive. Assume that for each $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to $F$.*

*Let $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfy $\sum_{n=0}^{\infty} \gamma_n < \infty$ and let a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy*

$$||x_{n+1} - Tx_n|| \leq \gamma_n, \ n = 0, 1, \ldots. \tag{4.14}$$

*Then the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to $F$.*

*Proof.* Let $k \geq 0$ be an integer and define a sequence $\{y_i^{(k)}\}_{i=k}^{\infty}$ by (4.5). By (4.5), (4.14) and Lemma 4.1, inequality (4.6) holds. By (4.6), the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Let $y_*$ be a weak limit point of $\{x_n\}_{n=0}^{\infty}$. In order to prove the theorem it is sufficient to show that $y_* \in F$. To this end, consider a subsequence $\{x_{i_p}\}_{p=1}^{\infty}$ which converges to $y_*$ in the weak topology.

Let $\epsilon > 0$ be given. There is a natural number $k$ such that

$$2(c+1) \sum_{i=k}^{\infty} \gamma_i < \epsilon/2. \tag{4.15}$$

By (4.15) and (4.6), for all integers $j > k$, we have

$$||y_j^{(k)} - x_j|| \leq c \sum_{i=k}^{\infty} \gamma_i < \epsilon/4. \tag{4.16}$$

Extracting a subsequence and re-indexing if necessary, we may assume without loss of generality that the subsequence $\{y_{i_p}^{(k)}\}_{p=1}^{\infty}$ weakly converges to $y \in F$.

Assume that

$$f \in E^*, \ ||f||_* \leq 1. \tag{4.17}$$

By (4.17) and (4.16),

$$|f(y_* - y)| = \lim_{p\to\infty} |f(x_{i_p} - y_{i_p}^{(k)})| \leq \limsup_{p\to\infty} ||x_{i_p} - y_{i_p}^{(k)}|| \leq \epsilon/4.$$

Since $f$ is an arbitrary linear functional satisfying (4.17), we conclude that $||y_* - y|| \leq \epsilon/4$. Since $\epsilon$ is an arbitrary positive number and $y \in F$, we see that $y_* \in F$ too. Theorem 4.6 is proved. □

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