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SUFFICIENT CONDITIONS FOR TURNPIKE PROPERTIES OF EXTREMALS OF AUTONOMOUS VARIATIONAL PROBLEMS WITH VECTOR-VALUED FUNCTIONS

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Abstract. In this work we study the structure of approximate solutions of autonomous variational problems with vector-valued functions. We are interested in turnpike properties of these solutions which are independent of the length of the interval, for all sufficiently large intervals.

1. Introduction

In this paper we analyze the structure of extremals of the variational problems

$$(P) \qquad \int_0^T f(z(t), z'(t))dt \to \min,\ z(0) = x,\ z(T) = y,$$

$$z:\ [0,T] \to R^n \text{ is an absolutely continuous (a. c.) function,}$$

where $T > 0$, $x, y \in R^n$ and $f : R^n \times R^n \to R^1$ is an integrand. We are interested in a turnpike property of the extremals which is independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions.

Turnpike properties are well known in mathematical economics [6, 13]. The term was first coined by Samuelson in 1948 (see [7]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics. See, for example, [3, 6] and the references mentioned therein. For variational problems the turnpike properties were studied in [4, 5, 8, 9, 11, 12]. Many turnpike results can be found in [13].

Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$. Denote by $\mathcal{A}$ the set of all continuous functions $f :\ R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1$ is convex;

A(ii) $f(x,u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x,u) \in R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \leq \epsilon \max\{f(x_1,u_1), f(x_2,u_2)\}$$

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for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1,2,\ |u_i| \ge \Gamma,\ i = 1,2, \quad |x_1 - x_2|,\ |u_1 - u_2| \le \delta.$$

It is easy to show that an integrand $f = f(x,u) \in C^1(R^{2n})$ belongs to $\mathcal{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 : [0,\infty) \to [0,\infty)$ such that

$$\max\{|\partial f/\partial x(x,u)|,\ |\partial f/\partial u(x,u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $x, u \in R^n$.

For the set $\mathcal{A}$ we consider the uniformity which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{A} \times \mathcal{A} :\ |f(x,u) - g(x,u)| \le \epsilon \text{ for all } u, x \in R^n \text{ satisfying } |x|, |u| \le N\}$$
$$\cap \{(f,g) \in \mathcal{A} \times \mathcal{A} :\ (|f(x,u)| + 1)(|g(x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \text{ for all } x, u \in R^n \text{ satisfying } |x| \le N\},$$

where $N, \epsilon > 0$ and $\lambda > 1$. It was shown in [9] that the uniform space $\mathcal{A}$ is metrizable and complete.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{1.1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$ and $x :\ [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y, z \in R^n$ and real numbers T_1, T_2 satisfying $T_1 < T_2$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) :\ x :\ [T_1, T_2] \to R^n \text{ is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}. \tag{1.2}$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $f \in \mathcal{A}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $-\infty < T_1 < T_2 < \infty$.

Let $f \in \mathcal{A}$. For any a.c. function $x : [0,\infty) \to R^n$ we set

$$J(x) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x). \tag{1.3}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(x) :\ x : [0,\infty) \to R^n \text{ is an a.c. function}\}. \tag{1.4}$$

Clearly $-\infty < \mu(f) < \infty$. By a simple modification of the proof of Proposition 4.4 in [2] (see [9, Theorems 8.1, 8.2]) we obtained the representation formula

$$U^f(0, T, x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x,y), \quad x, y \in R^n,\ T \in (0,\infty), \tag{1.5}$$

where $\pi^f : R^n \to R^1$ is a continuous function and $(T, x, y) \to \theta_T^f(x,y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x, y \in R^n$,

$$\pi^f(x) = \inf\{\liminf_{T \to \infty} [I^f(0, T, v) - \mu(f)T] :\ v : [0,\infty) \to R^n \text{ is an a.c. function satisfying } v(0) = x\},\ x \in R^n \tag{1.6}$$

and for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta_T^f(x,y) = 0$.

An a. c. function $x : [0, \infty) \to R^n$ is called (f)-good if the function $T \to I^f(0, T, x) - \mu(f)T$, $T \in (0, \infty)$ is bounded. In [9] we showed that for each $f \in \mathcal{A}$ and each $z \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ satisfying $v(0) = z$.

Propositions 1.1 and 3.2 of [9] imply the following result.

Proposition 1.1. *For any a.c. function $x : [0, \infty) \to R^n$ either $I^f(0, T, x) - T\mu(f) \to \infty$ as $T \to \infty$ or*

$$\sup\{|I^f(0, T, x) - T\mu(f)| : T \in (0, \infty)\} < \infty.$$

Moreover any (f)-good function $x : [0, \infty) \to R^n$ is bounded.

We denote $d(x, B) = \inf\{|x - y| : y \in B\}$ for $x \in R^n$ and $B \subset R^n$ and denote by $\mathrm{dist}(A, B)$ the distance in the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$. For every bounded a.c. function $x : [0, \infty) \to R^n$ define

$$\Omega(x) = \{y \in R^n : \text{ there exists a sequence } \{t_i\}_{i=1}^{\infty} \subset (0, \infty) \text{ for which } t_i \to \infty,\ x(t_i) \to y \text{ as } i \to \infty\}. \tag{1.7}$$

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly (ATP), if $\Omega(v_2) = \Omega(v_1)$ for all (f)-good functions $v_i : [0, \infty) \to R^n$, $i = 1, 2$ [4, 9].

In [9, Theorem 2.1] we established the following result.

Theorem 1.2. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ possesses (ATP).*

Therefore most elements of $\mathcal{A}$ (in the sense of Baire category) possess (ATP). In other words, (ATP) holds for a generic (typical) element of $\mathcal{A}$ [1, 13].

By Proposition 1.1 for each integrand $f \in \mathcal{A}$ which possesses (ATP) there exists a compact set $H(f) \subset R^n$ such that $\Omega(v) = H(f)$ for each (f)-good function $v : [0, \infty) \to R^n$.

Let $f \in \mathcal{A}$. We say that the integrand f has the strong turnpike property, or briefly (STP), with a turnpike $D \subset R^n$, where D is a nonempty compact subset of R^n, if for each $\epsilon, K > 0$ there exist real numbers $\delta > 0$ and $l_0 > l > 0$ such that the following assertion holds:

For each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq K,\ I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + \delta$$

the inequality

$$\mathrm{dist}(D, \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon \tag{1.8}$$

for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), D) \leq \delta$, then (1.8) holds for all $\tau \in [0, T - l_0]$ and if $d(v(T), D) \leq \delta$, then (1.8) holds for each $\tau \in [l_0, T - l]$.

Denote by $\mathcal{N}$ the set of all functions $f \in C^1(R^{2n})$ which satisfy the following assumptions:

$$\partial f/\partial u_i \in C^1(R^{2n}) \text{ for } i = 1, \dots, n;$$

the matrix $(\partial^2 f/\partial u_i \partial u_j)(x, u)$, $i, j = 1, \dots, n$ is positive definite for all $(x, u) \in R^{2n}$;

$$f(x, u) \geq \max\{\psi(|x|),\ \psi(|u|)|u|\} - a \text{ for all } (x, u) \in R^n \times R^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i : [0, \infty) \to [0, \infty)$, $i = 0, 1, 2$ such that

$$\phi_0(t)/t \to \infty \text{ as } t \to \infty,$$
$$f(x,u) \geq \phi_0(c_0|u|) - \phi_1(|x|),\ x, u \in R^n,$$
$$\max\{|\partial f/\partial x_i(x,u)|,\ |\partial f/\partial u_i(x,u)|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)),\ x, u \in R^n,\ i = 1, \ldots, n.$$

It is easy to see that $\mathcal{N} \subset \mathcal{A}$.

In [11, Theorem 1.2] we established the following result.

Theorem 1.3. *Assume that an integrand $f \in \mathcal{N}$ has (ATP). Then f possesses (STP) with the set $H(f)$ being the turnpike.*

The set $\mathcal{N}$ contains many integrands but actually it is a small subset of the space $\mathcal{A}$. In this paper we enlarge the set of integrands $f \in \mathcal{A}$ which possess (STP). We introduce the following notation.

Let $f \in \mathcal{A}$. For each pair of real numbers $T_2 > T_1$ and each a.c. function $x : [T_1, T_2] \to R^n$ set

$$\sigma^f(T_1, T_2, x) = I^f(T_1, T_2, x) - (T_2 - T_1)\mu(f) - \pi^f(x(T_1)) + \pi^f(x(T_2)). \tag{1.9}$$

By (1.2), (1,5) and (1.9),

$$\sigma^f(T_1, T_2, v) \geq 0$$

(1.10) for each $T_1 \in R^1$, each $T_2 > T_1$ and each a. c. function $v : [T_1, T_2] \to R^n$.

We have the following proposition.

Proposition 1.4 ([9, Theorem 8.3])**.** *For every $x \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ such that $v(0) = x$ and $\sigma^f(T_1, T_2, v) = 0$ for each $T_1 \geq 0$ and each $T_2 > T_1$.*

The proof of Theorem 1.2 in [11] is based on the following auxiliary result.

Proposition 1.5 ([11, Lemma 4.4])**.** *Let $f \in \mathcal{N}$ possess (ATP) and $\epsilon > 0$. Then there exists $q > 0$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $v : [0, q] \to R^n$ which satisfies $v(0) = h_1$, $v(q) = h_2$ and $\sigma^f(0, q, v) \leq \epsilon$.*

The following theorem is our main result.

Theorem 1.6. *Let $f \in \mathcal{A}$ possess (ATP) and let the following property hold:*

(P) For each $\epsilon > 0$ there exists $q > 0$ such that for each $x, y \in H(f)$ there exists an a.c. function $v : [0, q] \to R^n$ such that

$$v(0) = x,\ v(q) = y \text{ and } \sigma(0, q, v) \leq \epsilon.$$

Then f has (STP) with the turnpike $H(f)$.

Recall that $H(f)$ is a nonempty compact subset of R^n such that $\Omega(v) = H(f)$ for all (f)-good functions $v : [0, \infty) \in R^n$.

We will show that Theorem 1.6 implies the following result.

Theorem 1.7. *Let $f \in \mathcal{A}$ possess (ATP) and $H(f)$ be a singleton. Then f has (STP) with the turnpike $H(f)$.*

It was shown in [14] that if $f \in \mathcal{A}$ is strictly convex, then f has (ATP) and $H(f)$ is a singleton and therefore in view of Theorem 1.7 f has (STP).

Denote by $\mathcal{M}$ the set of all $f \in \mathcal{A}$ which possess (ATP) and the property (P) and denote by $\bar{\mathcal{M}}$ the closure of $\mathcal{M}$ in $\mathcal{A}$. We suppose that the set $\mathcal{M}$ is nonempty, consider the topological subspace $\bar{\mathcal{M}} \subset \mathcal{A}$ with the relative topology and prove the following result.

Theorem 1.8. *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}$ such that each $f \in \mathcal{F}$ has (STP) and the property (P).*

2. Auxiliary results for Theorem 1.6

In this paper we use the following auxiliary results.

Lemma 2.1 ([11, Proposition 2.10]). *Assume that $f \in \mathcal{A}$ has (ATP) and $\epsilon \in (0,1)$. Then there exist $L > 0$, $\delta > 0$ such that for each $T \in [L, \infty)$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$d(v(0), H(f)) \le \delta,\ d(v(T), H(f)) \le \delta,\ \sigma^f(0,T,v) \le \delta$$

the inequality $dist(H(f), \{v(t) :\ t \in [\tau, \tau + L]\}) \le \epsilon$ holds for all $\tau \in [0, T - L]$.

Proposition 2.2 ([10, Theorem 1.2]). *For each $f \in \mathcal{A}$ there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and a number $M > 0$ such that for each $g \in \mathcal{U}$ and each (g)-good function $x : [0,\infty) \to R^n$ the relation $\limsup_{t\to\infty} |x(t)| < M$ holds.*

Proposition 2.3 ([10, Proposition 2.5]). *Assume that $f \in \mathcal{A}$, $M_1 > 0$, $0 \le T_1 < T_2$ and that $x_i : [T_1, T_2] \to R^n$, $i = 1, 2, \dots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \le M_1$, $i = 1, 2 \dots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $x : [T_1, T_2] \to R^n$ such that*

$$I^f(T_1, T_2, x) \le M_1,$$

$x_{i_k}(t) \to x(t)$ as $k \to \infty$ uniformly on $[T_1, T_2]$ and $x'_{i_k} \to x'$ as $k \to \infty$ weakly in $L^1(R^n; (T_1, T_2))$.

Proposition 2.4 ([10, Theorem 1.3]). *Let $f \in \mathcal{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, each $T_2 \in [T_1 + c, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ satisfying*

$$|v(T_i)| \le M_1,\ i = 1, 2,\ I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + M_2$$

the following inequality holds:

$$|v(t)| \le S,\ t \in [T_1, T_2].$$

Proposition 2.5 ([9, Lemma 10.2]). *Let $f \in \mathcal{A}$ possess (ATP), $\epsilon_0 \in (0,1)$, $K_0, M_0 > 0$ and let l be a positive integer such that for each (f)-good function $x : [0,\infty) \to R^n$ the inequality*

$$dist(H(f), \{x(t) :\ t \in [T, T + l]\}) \le 8^{-1}\epsilon_0$$

holds for all large T (the existence of l follows from Theorem 5.1 of [9]). Then there exist an integer $N \geq 10$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$, each $S \in [0, \infty)$ and each a.c. function $x : [S, S+Nl] \to R^n$ satisfying

$$|x(S)|, |x(S+Nl)| \leq K_0, \ I^g(S, S+Nl, x)$$
$$\leq U^g(S, S+Nl, x(S), x(S+Nl)) + M_0$$

there exists an integer $i_0 \in [0, N-8]$ such that

$$dist(H(f), \{x(t) : \ t \in [T, T+l]\}) \leq \epsilon_0$$

for all $T \in [S + i_0 l, S + (i_0 + 7)l]$.

Lemma 2.6. *Let $f \in \mathcal{A}$ have (ATP) and $z \in H(f)$. Then there exists an a. c. function $v : R^1 \to H(f)$ such that $v(0) = z$ and $\sigma^f(-T, T, v) = 0$ for all $T > 0$.*

Proof. Consider any (f)-good function $u : [0, \infty) \to R^n$. Then $\Omega(u) = H(f)$. By Proposition 2.2 the function u is bounded. Together with Proposition 1.1 this implies that

$$\sup\{\sigma^f(0, T, u) : \ T > 0\} < \infty.$$

Combined with (1.10) this implies the following property:

(P1) For each $\epsilon > 0$ there exists $T_\epsilon > 0$ such that for each $T_2 > T_1 \geq T_\epsilon$ the inequality $\sigma^f(T_1, T_2, u) \leq \epsilon$ holds.

There exists a sequence of numbers $\{T_p\}_{p=1}^\infty \subset (0, \infty)$ such that

$$T_{p+1} \geq T_p + 4, \ p = 1, 2, \ldots, \ u(T_p) \to z \text{ as } p \to \infty. \tag{2.1}$$

For every integer $p \geq 1$ we set

$$u_p(t) = u(t + T_p), \ t \in [-T_p, \infty). \tag{2.2}$$

In view of the boundedness of u, the continuity of π^f, (P1) and (2.2), for each natural number q the sequence $I^f(-q, q, u_p)$ (where p is a natural number satisfying $T_p \geq q$) is bounded. Together with Proposition 2.3 this implies that there exist a subsequence $\{u_{p_i}\}_{i=1}^\infty$ and an a. c. function $v : R^1 \to R^n$ such that for each natural number q

$$u_{i_p}(t) \to v(t) \text{ as } p \to \infty \text{ uniformly on } [-q, q], \tag{2.3}$$

$$I^f(-q, q, v) \leq \liminf_{i \to \infty} I^f(-q, q, u_{i_p}). \tag{2.4}$$

Relations (2.2) and (2.3) imply that

$$v(R^1) \subset \Omega(u) = H(f). \tag{2.5}$$

It follows from (2.1), (2.2) and (2.3) that

$$v(0) = z. \tag{2.6}$$

Let q be a natural number. By (2.1)-(2.4), the continuity of π^f and (P1),

$$\sigma^f(-q, q, v) = I^f(-q, q, v) - 2q\mu(f) - \pi^f(v(-q)) + \pi^f(v(q))$$
$$\leq \liminf_{i \to \infty} [I^f(-q, q, u_{i_p}) - 2q\mu(f) - \pi^f(u_{i_p}(-q)) + \pi^f(u_{i_p}(q))]$$
$$= \liminf_{i \to \infty} [I^f(-q + T_{i_p}, q + T_{i_p}, u) - 2q\mu(f) - \pi^f(u(T_{i_p} - q)) + \pi^f(u(q + T_{i_p}))]$$

$$= \liminf_{i\to\infty} \sigma^f(T_{i_p} - q, T_{i_p} + q, u) = 0.$$

Lemma 2.6 is proved. □

Lemma 2.7. *Let $f \in \mathcal{A}$ have (ATP) and the property (P) and let $\epsilon > 0$. Then there are $q > 0$, $\delta > 0$ such that for each $x, y \in R^n$ satisfying*

$$d(x, H(f)) \le \delta,\ d(y, H(f)) \le \delta$$

and each $T \ge q$ there is an a. c. function $v : [0, T] \to R^n$ such that

$$v(0) = x,\ v(q) = y,\ \sigma^f(0, T, v) \le \epsilon.$$

Proof. By the property (P) there is $q_0 > 0$ such that for each $x, y \in H(f)$ there exists an a. c. function $v : [0, q_0] \to R^n$ such that

$$v(0) = x,\ v(q_0) = y,\ \sigma^f(0, q_0, v) \le \epsilon/8. \tag{2.7}$$

Set

$$q = q_0 + 2. \tag{2.8}$$

By the continuity of π^f and $U^f(0, 1, \cdot, \cdot)$ there is $\delta \in (0, 1)$ such that for each $x_1, x_2, y_1, y_2 \in R^n$ satisfying

$$|x_i|, |y_i| \le \sup\{|z| :\ z \in H(f)\} + 8,\ i = 1, 2, \tag{2.9}$$
$$|x_1 - x_2|,\ |y_1 - y_2| \le 2\delta$$

the following inequalities hold:

$$|\pi^f(x_1) - \pi^f(x_2)| \le \epsilon/16,\ |\pi^f(y_1) - \pi^f(y_2)| \le \epsilon/16, \tag{2.10}$$
$$|U^f(0, 1, x_1, y_1) - U^f(0, 1, x_2, y_2)| \le \epsilon/16.$$

Assume that $T \ge q$ and that $x, y \in R^n$ satisfy

$$\mathrm{dist}(x, H(f)) \le \delta,\ \mathrm{dist}(y, H(f)) \le \delta. \tag{2.11}$$

There exist $x_1, y_1 \in R^n$ such that

$$x_1, y_1 \in H(f),\ |x - x_1|, |y - y_1| \le \delta. \tag{2.12}$$

By Lemma 2.6 and (2.12) there exist a.c. functions $u_1, u_2 : R^1 \to H(f)$ such that for each $i = 1, 2$ and each $T > 0$

$$\sigma^f(-T, T, u_i) = 0 \tag{2.14}$$

and

$$u_1(0) = x_1,\ u_2(0) = y_1. \tag{2.15}$$

By the choice of q_0 (see (2.7)) there exists an a.c. function $u_3 : [1, q_0 + 1] \to R^n$ such that

$$u_3(1) = u_1(1) \in H(f),\ u_3(q_0 + 1) = u_2(q_0 + 1 - T) \in H(f), \tag{2.16}$$
$$\sigma^f(1, q_0 + 1, u_3) \le \epsilon/8.$$

There exists an a. c. function $v : [0, T] \to R^n$ such that

$$v(0) = x,\ v(t) = u_3(t),\ t \in [1, q_0 + 1],$$

$$I^f(0, 1, v) = U^f(0, 1, x, u_3(1)), \tag{2.17}$$

$$v(t) = u_2(t - T),\ t \in [q_0 + 1, T - 1],$$

$$v(T) = y,\ I^f(T - 1, T, v) = U^f(0, 1, u_2(-1), y).$$

By (2.17) in order to complete the proof of the lemma it is sufficient to show that $\sigma^f(0, T, v) \le \epsilon$. It follows from (2.17), (2.16) and (2.14) that

$$\sigma^f(0, T, v) = \sigma^f(0, 1, v) + \sigma^f(1, q_0 + 1, v) + \sigma^f(q_0 + 1, T - 1, v) + \sigma^f(T - 1, T, v)$$

$$= \sigma^f(0, 1, v) + \sigma^f(T - 1, T, v) + \sigma^f(1, q_0 + 1, u_3) + \sigma^f(q_0 + 1 - T, -1, u_2)$$

$$\le \sigma^f(0, 1, v) + \sigma^f(T - 1, T, v) + \epsilon/8. \tag{2.18}$$

By the choice of δ (see (2.9), (2.10)), (2.13), (2.17), (2.15) and (2.12),

$$|U^f(0, 1, x, u_1(1)) - U^f(0, 1, u_1(0), u_1(1))| \le \epsilon/16,$$

$$|\pi^f(x) - \pi^f(u_1(0))| \le \epsilon/16,$$

$$|U^f(0, 1, u_2(-1), y) - U^f(0, 1, u_2(-1), u_2(0))| \le \epsilon/16,$$

$$|\pi^f(y) - \pi^f(u_2(0))| \le \epsilon/16.$$

Together with (2.17), (2.14) and (2.16) these inequalities imply that

$$\sigma^f(0, 1, v) = I^f(0, 1, v) - \mu(f) - \pi^f(v(0)) + \pi^f(v(1))$$

$$= U^f(0, 1, x, u_1(1)) - \mu(f) - \pi^f(x) + \pi^f(u_1(1))$$

$$\le U^f(0, 1, u_1(0), u_1(1)) + \epsilon/16 - \mu(f) - \pi^f(u_1(0)) + \epsilon/16 + \pi^f(u_1(1))$$

$$\le \sigma^f(0, 1, u_1) + \epsilon/8 = \epsilon/8,$$

$$\sigma^f(T - 1, T, v) = I^f(T - 1, T, v) - \mu(f) - \pi^f(v(T - 1)) + \pi^f(v(T))$$

$$= U^f(0, 1, u_2(-1), y) - \mu(f) - \pi^f(u_2(-1)) + \pi^f(y)$$

$$\le U^f(0, 1, u_2(-1), u_2(0)) + \epsilon/16 - \mu(f) - \pi^f(u_2(-1)) + \pi^f(u_2(0)) + \epsilon/16$$

$$\le \sigma^f(-1, 0, u_2) + \epsilon/8 = \epsilon/8.$$

In view of these inequalities and (2.18)

$$\sigma^f(0, T, v) \le \epsilon/8 + \epsilon/8 + \epsilon/8 < \epsilon.$$

Lemma 2.7 is proved.

□

3. Proof of Theorem 1.6

Let $\epsilon \in (0,1)$ and

$$K > \sup\{|z| : z \in H(f)\} + 1. \tag{3.1}$$

By Lemma 2.1 there exist $l > 0$, $\delta_0 \in (0,1)$ such that for each $T \in [l,\infty)$ and each a. c. function $v : [0,T] \to R^n$ which satisfies

$$d(v(0), H(f)) \le \delta_0,\ d(v(T), H(f)) \le \delta_0, \tag{3.2}$$
$$\sigma^f(0,T,v) \le \delta_0$$

the inequality

$$\mathrm{dist}(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \le \epsilon \tag{3.3}$$

holds for all $\tau \in [0, T-l]$. By Lemma 2.7 there are $\delta_1 \in (0,1)$, $q > 0$ such that for each $x, y \in R^n$ satisfying

$$d(x, H(f)) \le \delta_1,\ d(y, H(f)) \le \delta_1 \tag{3.4}$$

and each $T \ge q$ there is an a. c. function $v : [0,T] \to R^n$ such that

$$v(0) = x,\ v(q) = y,\ \sigma^f(0,T,v) \le \delta_0/16. \tag{3.5}$$

Set

$$\delta = \min\{\delta_0, \delta_1\}/4. \tag{3.6}$$

By Proposition 2.4 there is $K_1 > K$ such that for each $T \ge 1$ and each a. c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \le K,\ I^f(0,T,v) \le U^f(0,T,v(0),v(T)) + 1 \tag{3.7}$$

the following inequality holds:

$$|v(t)| \le K_1,\ t \in [0,T]. \tag{3.8}$$

By Proposition 2.5 there is $N > 0$ such that for each a. c. function $v : [0,N] \to R^n$ satisfying

$$|v(0)|, |v(N)| \le K_1,\ I^f(0,N,v) \le U^f(0,N,v(0),v(N)) + 1$$

there is $\tau \in [0,N]$ for which

$$d(v(\tau), H(f)) \le \delta. \tag{3.9}$$

Set

$$l_0 = 8(N + l + q + 2). \tag{3.10}$$

Assume that $T \ge 2l_0$ and $v : [0,T] \to R^n$ is an a. c. function such that

$$|v(0)|,\ |v(T)| \le K,\ I^f(0,T,v) \le U^f(0,T,v(0),v(T)) + \delta. \tag{3.11}$$

By (3.11) and the definition of K_1 the inequality (3.8) is true. In view of (3.8), (3.10), (3.11) and the definition of N there are

$$\tau_1 \in [0,N],\ \tau_2 \in [T-N,T] \tag{3.12}$$

such that

$$d(v(\tau_i), H(f)) \le \delta,\ i = 1,2. \tag{3.13}$$

If $d(v(0), H(f)) \leq \delta$, then we choose $\tau_1 = 0$. If $d(v(T), H(f)) \leq \delta$, then we choose $\tau_2 = T$.

By (3.12), (3.13), (3.10) and the definition of δ_1, q there is an a.c. function $u : [\tau_1, \tau_2] \to R^n$ such that

$$u(\tau_i) = v(\tau_i),\ i = 1, 2,\ \sigma^f(\tau_1, \tau_2, u) \leq \delta_0/16. \tag{3.14}$$

Relations (3.14) and (3.11) imply that

$$\sigma^f(\tau_1, \tau_2, v) - \sigma^f(\tau_1, \tau_2, u) = I^f(\tau_1, \tau_2, v) - I^f(\tau_1, \tau_2, u)$$
$$\leq I^f(\tau_1, \tau_2, v) - U^f(0, \tau_2 - \tau_1, v(\tau_1), v(\tau_2)) \leq \delta.$$

Combined with (3.14) and (3.6) this implies that

$$\sigma^f(\tau_1, \tau_2, v) \leq \delta + \sigma^f(\tau_1, \tau_2, u) \leq \delta + \delta_0/16 \leq \delta_0/2. \tag{3.15}$$

By (3.12), and (3.10),

$$\tau_2 - \tau_1 \geq T - 2N \geq l_0 - 2N \geq 8l. \tag{3.16}$$

It follows from (3.15), (3.16), (3.14) and (3.13) and the definition of l, δ_0 that (3.3) is valid for all $\tau \in [\tau_1, \tau_2 - l]$. Theorem 1.6 is proved. □

4. Proof of Theorem 1.7

By Theorem 1.6 it is sufficient to show that the property (P) holds. Let $H(f) = \{z\}$ with $z \in R^n$. Set $v(t) = z$ for all $t \in R^1$. In view of Lemma 2.6 $\sigma^f(-T, T, v) = 0$ for all $T > 0$. Thus (P) holds. Theorem 1.7 is proved. □

5. Proof of Theorem 1.8

We need the following auxiliary results. □

Proposition 5.1 ([13, Chapter 4, Theorem 4.1.1]). *Let $f \in \mathcal{A}$ have (ATP) and $\epsilon, K > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$ and each $x \in R^n$ satisfying $|x| \leq K$, $|\pi^f(x) - \pi^g(x)| \leq \epsilon$.*

Proposition 5.2 ([13, Chapter 4, Theorem 4.1.1]). *Let $f \in \mathcal{A}$ have (ATP). Then the functional $g \to \mu(g)$, $g \in \mathcal{A}$ is continuous at the point f.*

Proposition 5.3 ([9, Theorem 2.3]). *Let $f \in \mathcal{A}$ have (ATP) and let $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$ and each (g)-good function $v : [0, \infty) \to R^n$, $dist(\Omega(v), H(f)) \leq \epsilon$.*

Proposition 5.4 ([10, Proposition 2.8]). *Let $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$ and let $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a. c. function $x : [T_1, T_2] \to R^n$ satisfying*

$$\min\{I^g(T_1, T_2, x),\ I^f(T_1, T_2, x)\} \leq D$$

the inequality $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq \epsilon$ holds.

For each $g \in \mathcal{A}$ put

$$\Omega_g = \cup\{\Omega(v) :\ v : [0, \infty) \to R^n \text{ is an } (g) - \text{good function}\}. \tag{5.1}$$

Lemma 5.5. *Let $f \in \mathcal{A}$ have (ATP) and the property (P) and let $\epsilon \in (0,1)$. Then there exist $q > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$ and each $x, y \in \Omega_g$ there is an a. c. function $v : [0, q] \to R^n$ such that $v(0) = x$, $v(q) = y$ and $\sigma^g(0, q, v) \le \epsilon$.*

Proof. Since f has the property (P) it follows from Lemma 2.7 that there are $q > 0$ and $\delta \in (0, \epsilon)$ such that for each $x, y \in R^n$ satisfying

$$d(x, H(f)) \le \delta,\ d(y, H(f)) \le \delta \tag{5.2}$$

there is an a. c. function $v : [0, q] \to R^n$ such that

$$v(0) = x,\ v(q) = y,\ \sigma^f(0, q, v) \le \epsilon/16. \tag{5.3}$$

By Proposition 5.3 there is a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}_1$

$$\mathrm{dist}(\Omega_g, H(f)) \le \delta. \tag{5.4}$$

By Propositions 5.1 and 5.2 there is a neighborhood $\mathcal{U}_2$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}_2$

$$|\mu(f) - \mu(g)| \le (\epsilon/16)(q + 1)^{-1} \tag{5.5}$$

and that for each $x \in R^n$ satisfying $d(x, H(f)) \le 4$

$$|\pi^f(x) - \pi^g(x)| \le \epsilon/16. \tag{5.6}$$

By Proposition 5.4 there is a neighborhood $\mathcal{U}_3$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}_3$ and each a.c. function $v : [0, q] \to R^n$ satisfying

$$\min\{I^f(0, q, v),\ I^g(0, q, v)\}$$

$$\le (q + 1)|\mu(f)| + 2 \sup\{|\pi^f(z)| :\ z \in R^n,\ d(z, H(f)) \le 4\} + 8 \tag{5.7}$$

the following inequality holds:

$$|I^f(0, q, v) - I^g(0, q, u)| \le \epsilon/16. \tag{5.8}$$

Set

$$\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3. \tag{5.9}$$

Assume that

$$g \in \mathcal{U},\ x, y \in \Omega_g. \tag{5.10}$$

By (5.10), (5.9) and the definition of $\mathcal{U}_1$ (see (5.4)),

$$d(x, H(f)),\ d(y, H(f)) \le \delta. \tag{5.11}$$

By (5.11), the definition of δ, q (see (5.2) and (5.3)) there is an a. c. function $v : [0, q] \to R^n$ which satisfies (5.3). Relation (5.3) implies that

$$\begin{aligned} I^f(0, q, v) &= \mu(f)q + \pi^f(x) - \pi^f(y) + \sigma^f(0, q, v) \\ &\le \mu(f)q + \pi^f(x) - \pi^f(y) + 1. \end{aligned}$$

Together with (5.9), (5.11), (5.10) and the definition of $\mathcal{U}_3$ (see (5.7), (5.8)) this inequality implies that

$$|I^f(0, q, v) - I^g(0, q, v)| \le \epsilon/16. \tag{5.12}$$

In view of (5.9), (5.10) and the definition of $\mathcal{U}_2$, the relation (5.5) is true. By (5.9), the definition of $\mathcal{U}_2$ (see (5.6)) and (5.11),

$$|\pi^f(x) - \pi^g(x)|,\ |\pi^f(y) - \pi^g(y)| \le \epsilon/16. \tag{5.13}$$

It follows from (5.3), (5.12), (5.5) and (5.13) that

$$\sigma^g(0,q,v) = I^g(0,q,v) - \mu(g)q - \pi^g(x) + \pi^g(y)$$

$$\le I^f(0,q,v) + \epsilon/16 - \mu(f)q + \epsilon/16 - \pi^f(x) + \pi^f(y) + \epsilon/8$$

$$= \sigma^f(0,q,v) + \epsilon/4 \le \epsilon/4 + \epsilon/16.$$

Lemma 5.5 is proved.

□

Proof of Theorem 1.8 Let $f \in \mathcal{M}$ and let $n \ge 1$ be an integer. By Proposition 5.3 there is an open neighborhood $\mathcal{U}(f,n)$ of f in $\bar{\mathcal{M}}$ and $q(f,n) > 0$ such that for each $g \in \mathcal{U}(f,n)$ the following properties hold:

(1) For each (g)-good function $v : [0,\infty) \to R^1$, $\mathrm{dist}(\Omega(v), H(f)) \le 1/n$.

(2) For each $x, y \in \Omega_g$ there is an a. c. function $v : [0, q(f,n)] \to R^n$ such that

$$v(0) = x,\ v(q(f,n)) = y,\ \sigma^g(0, q(f,n), v) \le 1/n.$$

Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f,n) :\ f \in \mathcal{M}\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}$.

Let $g \in \mathcal{F}$ and $\epsilon > 0$. Choose a integer $n \ge 1$ such that $8/n < \epsilon$. There is $f \in \mathcal{M}$ such that

$$g \in \mathcal{U}(f,n). \tag{5.14}$$

Let $v_1, v_2 : [0,\infty) \to R^n$ be (g)-good functions. By (5.14) and the property (1),

$$\mathrm{dist}(\Omega(v_i), H(f)) \le 1/n,\ i = 1, 2,$$

$$\mathrm{dist}(\Omega(v_1), \Omega(v_2)) \le 2/n < \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that $\Omega(v_1) = \Omega(v_2)$, g has (ATP) and

$$\mathrm{dist}(H(g), H(f)) \le 1/n.$$

Now let $x, y \in H(g) = \Omega_g$. By (5.14) and the property (2) there is an a. c. function $v : [0, q(f,n)] \to R^n$ such that

$$v(0) = x,\ v(q(f,n)) = y,\ \sigma^g(0, q(f,n), v) \le 1/n < \epsilon.$$

Since ϵ is any positive number we conclude that f has (P).

Theorem 1.8 is proved. □

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