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ONE-DIMENSIONAL INFINITE HORIZON VARIATIONAL PROBLEMS ARISING IN CONTINUUM MECHANICS

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Abstract. In this paper we discuss the existence and the structure of approximate solutions of autonomous one-dimensional second order variational problems related to a model in thermodynamics. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals.

1. Introduction

In this paper we consider variational problems defined on infinite intervals which arise in the theory of thermodynamical equilibrium for materials [3, 5]. We discuss the existence and the structure of approximate solutions of these variational problems. Given $x \in R^2$ we study the infinite horizon problem of minimizing the expression $\int_0^T f(w(t), w'(t), w''(t))dt$ as T grows to infinity where

$$w \in A_x = \{v \in W^{2,1}_{loc}([0,\infty)) \colon (v(0), v'(0)) = x\}.$$

Here $W^{2,1}_{loc}([0,\infty)) \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative and f belongs to a space of functions to be described below.

The following notion known as the overtaking optimality criterion was introduced in the economic literature [1, 4, 11] and has been used in control theory [2, 17].

A function $u \in A_x$ will be called (f)-overtaking optimal if

$$\limsup_{T\to\infty}[\int_0^T f(u(t), u'(t), u''(t))dt - \int_0^T f(w(t), w'(t), w''(t))dt] \le 0$$

for any $w \in A_x$.

In this paper we employ the following weakened version of this criterion.

A function $u \in A_x$ will be called (f)-weakly optimal if

$$\liminf_{T\to\infty}[\int_0^T f(u(t), u'(t), u''(t))dt - \int_0^T f(w(t), w'(t), w''(t))dt] \le 0$$

for any $w \in A_x$.

Denote by $\mathcal{A}$ the set of all continuous functions $f : R^3 \to R$ such that for each $N > 0$ the function $|f(x, y, z)| \to \infty$ as $|z| \to \infty$ uniformly on the set $\{(x, y) \in R^2 : |x|, |y| \le N\}$. For the set $\mathcal{A}$ we consider the uniformity which is determined by the base

$$E(N, \epsilon, \Gamma) = \{(f, g) \in \mathcal{A} \times \mathcal{A} :$$
$$|f(x_1, x_2, x_3) - g(x_1, x_2, x_3)| \le \epsilon$$

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$$\text{for all } (x_1,x_2,x_3)\in R^3 \text{ such that } |x_i|\le N,\ i=1,2,3,$$
$$(|f(x_1,x_2,x_3)|+1)(|g(x_1,x_2,x_3)|+1)^{-1}\in[\Gamma^{-1},\Gamma]$$
$$\text{for all } (x_1,x_2,x_3)\in R^3 \text{ such that } |x_1|,|x_2|\le N)\},$$

where $N>0$, $\epsilon>0$, $\Gamma>1$. Clearly, the uniform space $\mathcal{A}$ is Hausdorff and has a countable base. Therefore $\mathcal{A}$ is metrizable. It is easy to verify that the uniform space $\mathcal{A}$ is complete.

Let $a=(a_1,a_2,a_3,a_4)\in R^4$, $a_i>0$, $i=1,2,3,4$ and let α, β, γ be positive numbers such that $1\le\beta<\alpha$, $\beta\le\gamma$, $\gamma>1$. Denote by $\mathcal{M}(\alpha,\beta,\gamma,a)$ the set of all functions $f\in\mathcal{A}$ such that:

$$f(w,p,r)\ge a_1|w|^\alpha-a_2|p|^\beta+a_3|r|^\gamma-a_4,\ (w,p,r)\in R^3; \tag{1.1}$$

$$f,\ \partial f/\partial p\in C^2,\ \partial f/\partial r\in C^3,\ \partial^2 f/\partial r^2(w,p,r)>0 \text{ for all } (w,p,r)\in R^3;$$

there is a monotone increasing function $M_f:[0,\infty)\to[0,\infty)$ such that for every $(w,p,r)\in R^3$

$$\max\{f(w,p,r),\ |\partial f/\partial w(w,p,r)|,\ |\partial f/\partial p(w,p,r)|,\ |\partial f/\partial r(w,p,r)|\}\le M_f(|w|+|p|)(1+|r|^\gamma).$$

Denote by $\bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ the closure of $\mathcal{M}(\alpha,\beta,\gamma,a)$ in $\mathcal{A}$ and consider any $f\in\bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f)=\inf\{\liminf_{T\to\infty}T^{-1}\int_0^T f(w(t),w'(t),w''(t))dt : w\in A_x\}. \tag{1.2}$$

It is easy to verify that $\mu(f)$ is well defined and is independent of the initial vector x. A function $w\in W^{2,1}_{loc}([0,\infty))$ is called (f)-good if the function $\phi^f_w : T\to\int_0^T[f(w(t),w'(t),w''(t))-\mu(f)]dt$, $T\in(0,\infty)$ is bounded. For every $w\in W^{2,1}_{loc}([0,\infty))$ the function ϕ^f_w is either bounded or diverges to ∞ as $T\to\infty$ and moreover, if ϕ^f_w is a bounded function, then $\sup\{|(w(t),w'(t))|: t\in[0,\infty)\}<\infty$.

Leizarowitz and Mizel [5] established that for every $f\in\mathcal{M}(\alpha,\beta,\gamma,a)$ satisfying $\mu(f)<\inf\{f(w,0,s):\ (w,s)\in R^2\}$ there exists a periodic (f)-good function. In [12] we generalized their result and proved the following assertion.

Theorem 1.1. *Let $f\in\mathcal{M}(\alpha,\beta,\gamma,a)$. Then there exists an (f)-good function v and a number $T>0$ such that $v(t)=v(t+T)$ for all $t\ge 0$ and*

$$\int_0^T f(v(t),v'(t),v''(t))dt=T\mu(f).$$

In [5] Leizarowitz and Mizel considered for each $T>0$ the function $U^f_T : R^2\times R^2\to R$ which is defined as follows:

$$U^f_T(x,y)=\inf\left\{\int_0^T f(w(t),w'(t),w''(t)dt:\ w\in A^T_{x,y}\right\}, \tag{1.3}$$

where

$$A^T_{x,y}=\{v\in W^{2,1}([0,T]):(v(0),v'(0))=x,\ (v(T),v'(T))=y\},$$

and established the following representation formula

$$U^f_T(x,y)=T\mu(f)+\pi^f(x)-\pi^f(y)+\theta^f_T(x,y),\ x,y\in R^2,\ T>0, \tag{1.4}$$

where $\pi^f : R^2 \to R$ and $(T, x, y) \to \theta_T^f(x, y)$, $x, y \in R^2$, $T > 0$ are continuous functions,

$$(1.5) \quad \pi^f(x) = \inf\{\liminf_{T\to\infty} \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt : \ w \in A_x\}, \ x \in R^2,$$

$\theta_T^f(x, y) \geq 0$ for each $T > 0$, and each $x, y \in R^2$, and for every $T > 0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta_T^f(x, y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathcal{M}(\alpha, \beta, \gamma, a)$, but their result also holds for every $f \in \bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ without change in the proofs.

In [13] we investigated the structure of (f)-good functions and established for a generic $f \in \bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ and for every given $x \in R^2$ the existence of a (f)-weakly optimal solution $v \in A_x$. Most studies which are concerned with the existence of optimal solutions on an infinite horizon assume convex integrands f. One contribution of [13] was in establishing optimal solutions without such convexity assumptions.

2. Existence and structure of good functions

We use the notation and definitions introduced in Section 1 and denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau > 0$ and $v \in W^{2,1}([0, \tau])$ we define $X_v : [0, \tau] \to R^2$ as follows:

$$X_v(t) = (v(t), v'(t)), \ t \in [0, \tau].$$

We also use this definition for $v \in W^{2,1}_{loc}([0, \infty))$.

Fix $a = (a_1, a_2, a_3, a_4) \in R^4$ and positive numbers α, β, γ such that $a_i > 0$, $i = 1, 2, 3, 4$, $1 \leq \beta < \alpha$, $\beta \leq \gamma$, $\gamma > 1$. We denote be $\mathcal{M}_0(\alpha, \beta, \gamma, a)$ the set of continuous functions $f = f(w, p, r) : R^3 \to R$ satisfying (1.1) for any $(w, p, r) \in R^3$.

Denote by $\mathcal{M}_1(\alpha, \beta, \gamma, a)$ the set of all functions $f \in \mathcal{M}_0(\alpha, \beta, \gamma, a)$ such that:
the function $f(w, p, r)$ is convex in r for all $(w, p) \in R^2$;
the function $\partial f/\partial r : R^3 \to R$ is continuous;
there exists a monotone increasing function $M_f : [0, \infty) \to [0, \infty)$ such that

$$f(w, p, r) \leq M_f(|w| + |p|)(1 + |r|^\gamma) \text{ for all } (w, p, r) \in R^3.$$

We consider the topological subspaces $\bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$, $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a) \subset \mathcal{A}$ which have the relative topology. The set $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$ is the closure of $\mathcal{M}_1(\alpha, \beta, \gamma, a)$ in $\mathcal{A}$ and the set $\bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ was defined in Section 1.

We consider functionals of the form

$$(2.1) \qquad I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt$$

where $-\infty < T_1 < T_2 < +\infty$, $w \in W^{2,1}([T_1, T_2])$ and $f \in \mathcal{M}_0(\alpha, \beta, \gamma, a)$.

Let $f \in \bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$. A function $v \in W^{2,1}_{loc}([0, \infty))$ is (f)-weakly optimal if

$$\liminf_{T\to\infty} [I^f(0, T, v) - I^f(0, T, w)] \leq 0$$

for all $w \in W^{2,1}_{loc}([0, \infty))$ satisfying $X_w(0) = X_v(0)$.

Of special interest is the minimal long-run average cost growth rate $\mu(f)$ defined by (1.2). By a result of Leizarowitz and Mizel [5, p. 164] $\mu(f) \in (-\infty, f(0, 0, 0)]$.

A function $v \in W^{2,1}_{loc}([0,\infty))$ is called (f)-good function if $\sup\{|I^f(0,\tau,w) - \tau\mu(f)| : \tau \in (0,\infty)\} < \infty$.

For $f \in \mathcal{M}_0(\alpha,\beta,\gamma,a)$ and $T > 0$ we consider the function $U^f_T : R^2 \times R^2 \to R$ defined by (1.3). In [5] Leizarowitz and Mizel established the representation formula (1.4) where $\pi^f : R^2 \to R$ is a continuous function defined by (1.5) and $(T,x,y) \to \theta^f_T(x,y)$, $T > 0$, $x, y \in R^2$, is a nonnegative continuous function such that for every $T > 0$ and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta^f_T(x,y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathcal{M}(\alpha,\beta,\gamma,a)$, but their result also holds for every $f \in \bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ without changes in the proofs.

In [13] we investigated the structure of (f)-good functions and established for a generic $f \in \bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ and every $x \in R^2$ the existence of a (f)-weakly optimal function $v \in A_x$.

For a function $w \in W^{2,1}_{loc}([0,\infty))$ we denote by $\Omega(w)$ the set of all points $z \in R^2$ such that $X_w(t_j) \to z$ as $j \to \infty$ for some sequence of numbers $t_j \to \infty$.

We denote $d(x,B) = \inf\{|x-y| : y \in B\}$ for $x \in R^n$, $B \subset R^n$ and denote by $\mathrm{dist}(A,B)$ the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$.

A function $w \in W^{2,1}_{loc}(-\infty,+\infty)$ is called almost subperiodic if for every $\epsilon > 0$ there exists a number $T_\epsilon > 0$ such that for each $\tau_1, \tau_2 \in R$ there is $T \in [0,T_\epsilon)$ which satisfies the conditions

$$\begin{aligned} &|X_w(\tau_1+t) - X_w(\tau_2+t+T)| \le \epsilon,\ t \in [0,T_\epsilon - T], \\ &|X_w(\tau_1+t+T_\epsilon - T) - X_w(\tau_2+t)| \le \epsilon,\ t \in [0,T]. \end{aligned} \tag{2.2}$$

A function $w \in W^{2,1}_{loc}([0,\infty))$ is called asymptotically almost subperiodic if for any $\epsilon > 0$ there exist numbers $T_\epsilon > 0$ and $t_\epsilon > 0$ such that for every $\tau_1 \ge t_\epsilon$ and every $\tau_2 \ge t_\epsilon$ there is $T \in [0,T_\epsilon)$ which satisfies (2.2).

In [13] we proved the existence of a set $F \subset \bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ and for which the following theorems are valid.

Theorem 2.1. 1. *Let $f \in F$. Then there is a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for any (f)-good function w. Moreover, for every (f)-good function w and every positive number ϵ there exist $T_\epsilon > 0$ and $t_\epsilon > 0$ such that*

$$dist(\{(w(t),w'(t)) :\ t \in [\tau,\tau+T_\epsilon]\}, H(f)) \le \epsilon \text{ for any } t \ge t_\epsilon.$$

2. *For every $f \in F$ and every $\epsilon > 0$ there exist a number $T > 0$ and a function $v \in C^2([0,\infty))$ such that $v(t+T) = v(t)$ for all $t \ge 0$, and*

$$dist(\{(v(t),v'(t)) : t \in [0,T]\}, H(f)) \le \epsilon.$$

3. *Let $f \in F$. Then every (f)-good function is asymptotically almost subperiodic.*

4. *Let $f \in F$ and let ϵ be a positive number. Then there exist a neighborhood $\mathbf{U}$ of f in $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ and a number $T > 0$ such that for every $g \in \mathbf{U}$ and every (g)-good function w*

$$dist(\{(w(t),w'(t)) : t \in [\tau,\tau+T]\}, H(f)) \le \epsilon \text{ for all large enough } \tau.$$

For $f \in F$ we may consider $H(f)$ as an analog of a turnpike set [6, 9, 10, 17]. Assertion 1 of Theorem 2.1 establishes that for $f \in F$ all (f)-good functions converge to the turnpike set $H(f)$. Assertion 2 shows that for $f \in F$ the set $H(f)$ is approximated by periodic curves in R^2 and Assertion 4 of Theorem 2.1 shows that for every g belonging to a small neighborhood of f and every (g)-good function w, the set $\Omega(w)$ is close enough to $H(f)$ in the Hausdorff metric. If we think of $H(f)$ as an analog of a turnpike set Assertion 4 yields the stability of the turnpike phenomenon.

Theorem 2.2. *Let $f \in F$ and $x \in R^2$. Then there exists $v \in A_x$ such that*

$$I^f(T_1, T_2, v) = (T_2 - T_1)\mu(f) + \pi^f((v(T_1), v'(T_1)) - \pi^f((v(T_2), v'(T_2)) \tag{2.3}$$

for each T_1, T_2 satisfying $0 \le T_1 < T_2$. Moreover, for every $v \in A_x$ which satisfies (2.3) for each $T_1 \ge 0$, $T_2 > T_1$ there exists a sequence of numbers $t_j \to \infty$ as $j \to \infty$ such that:

$$\limsup_{j\to\infty}[I^f(0, t_j, v) - I^f(0, t_j, w)] \le 0 \text{ for all } w \in A_x;$$

$$\text{if } w \in A_x \text{ and } \limsup_{j\to\infty}[I^f(0, t_j, v) - I^f(0, t_j, w)] = 0, \text{ then}$$

$$I^f(T_1, T_2, w) = (T_2 - T_1)\mu(f) + \pi^f((w(T_1), w'(T_1)) - \pi^f((w(T_2), w'(T_2))$$

for each T_1, T_2 satisfying $0 \le T_1 < T_2$.

Theorem 2.2 shows that for every $f \in F$ and every initial value $x \in R^2$ there exists an (f)-weakly optimal solution $v \in A_x$ satisfying (2.2) for each T_1, T_2 such that $0 \le T_1 < T_2$.

Theorem 2.3. *Let $C(R^n)$ be the space of all continuous functions $g : R^n \to R$ with the topology of uniform convergence on bounded subsets of R^n. We define $L : \bar{\mathcal{M}}(\alpha, \beta, \gamma, a) \to R \times C(R^2)$ by*

$$L(f) = (\mu(f), \pi^f), \ f \in \bar{\mathcal{M}}(\alpha, \beta, \gamma, a).$$

Then the set of continuity points of the operator L contains F.

For $f \in \bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ and $x \in R^2$ we set

$$A(f, x) = \{v \in A_x : \ \text{(2.3) holds for each } T_1, T_2 \text{ satisfying } 0 \le T_1 < T_2\}.$$

Let $f \in F$. Theorem 2.4 establishes that for each g belonging to some small neighborhood of f in $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$, each x belonging to some small neighborhood of $H(f)$ and each $v \in A(g, x)$, the point $(v(t), v'(t))$ is contained in a small neighborhood of $H(f)$ for all $t \ge 0$.

Let $f \in F$ and $K > 0$. By Theorem 2.5 for every g belonging to some small neighborhood of f in $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$, every $x \in R^2$ satisfying $|x| \le K$, and every $v \in A(g, x)$, the point $(v(t), v'(t))$ is contained in a small neighborhood of $H(f)$ for all $t \ge Q$, where Q is a constant which depends on K and the neighborhoods, but does not depend on g and x.

Theorem 2.4. *Let $f \in F$, $\epsilon > 0$. Then there exist a neighborhood* $\mathbf{U}$ *of* f *in* $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$ *and numbers* $l > 0$, $\delta \in (0, \epsilon)$ *such that for every* $g \in \mathbf{U}$, *each* $x \in R^2$ *satisfying* $d(x, H(f)) \le \delta$, *every* $v \in A(g, x)$, *and every* $T \ge 0$,

$$dist(\{(v(t+T), v'(t+T)) : t \in [0, l]\}, H(f)) \le \epsilon. \tag{2.4}$$

Theorem 2.5. *Let $f \in F$, $\epsilon > 0$, $K > 0$. Then there exist a neighborhood* $\mathbf{U}$ *of* f *in* $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$ *and numbers* $l > 0$, $Q > 0$ *such that for every* $g \in \mathbf{U}$, *every* $x \in R^2$ *satisfying* $|x| \le K$ *and every* $v \in A(g, x)$, *equation (2.4) holds for all* $T \ge Q$.

Corollary 2.6. *Let* $f \in F$ *and* $v \in W^{2,1}_{loc}(-\infty, +\infty)$. *Assume that (2.3) holds for each* T_1, T_2 *satisfying* $-\infty < T_1 < T_2 < +\infty$ *and* $\liminf_{t \to -\infty} |(v(t), v'(t))| < \infty$. *Then* $(v(t), v'(t)) \in H(f)$ *for all* $t \in R$.

Theorem 2.7. 1. *Let* $f \in F$ *and* $x \in H(f)$. *Then there exists* $v \in W^{2,1}_{loc}(-\infty, +\infty)$ *such that* $(v(t), v'(t)) \in H(f)$ *for all* $t \in R$, $(v(0), v'(0)) = x$ *and (2.3) holds for each* T_1, T_2 *satisfying* $-\infty < T_1 < T_2 < +\infty$.

2. *Let* $f \in F$. *Then each function* $v \in W^{2,1}_{loc}(-\infty, +\infty)$ *such that (2.4) holds for each* T_1, T_2 *satisfying* $-\infty < T_1 < T_2 < +\infty$ *and* $\liminf_{t \to -\infty} |(v(t), v'(t))| < \infty$, *is almost subperiodic.*

Example. Set $a_i = 1$, $i = 1, 2, 3, 4$, $\alpha = 4$, $\beta, \gamma = 2$. Consider the space of functions $\bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ and let F be as assumed in this section. Consider an integrand

$$f(w, p, r) = 8w^2(w-1)^2 + p^2 + r^2 + b, \text{ where } b > 0.$$

It is easy to see that $f \in \mathcal{M}(\alpha, \beta, \gamma, a)$ for all b large enough, $\mu(f) = b$ and the functions $w_1(t) = 0$, $t \in [0, \infty)$ and $w_2(t) = 1$, $t \in [0, \infty)$ are (f)-good functions. Therefore $f \in \bar{\mathcal{M}}(\alpha, \beta, \gamma, a) \setminus F$ for all b large enough.

In [15] for $\alpha = 4, \beta, \gamma = 2$ we constructed a function $\bar{g} \in \mathcal{M}(\alpha, \beta, \gamma, a)$ where $a = (a_1, a_2, a_3, a_4), a_i > 0, i = 1, 2, 3, 4$, which has the following properties:

for each $x \in R^2$ there exists a $(\bar{g})$-weakly optimal function $v \in W^{2,1}_{loc}([0, \infty))$ satisfying $X_v(0) = x$;

there exists a neighborhood $\mathbf{U}$ of $\bar{g}$ in $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a)$ such that for each $f \in \mathbf{U}$ there are no (f)-overtaking optimal functions.

3. The turnpike property

In this section we use the notation and definitions introduced in the previous sections. Fix $a = (a_1, a_2, a_3, a_4) \in R^4$ and positive numbers α, β, γ such that $a_i > 0$, $i = 1, 2, 3, 4$, $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. We consider the topological subspaces $\bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$, $\bar{\mathcal{M}}_1(\alpha, \beta, \gamma, a) \subset \mathcal{A}$ defined in Sections 1 and 2.

For every $f \in \mathcal{M}_0(\alpha, \beta, \gamma, a)$, every $x \in R^2$, and every $T > 0$, we set

$$\sigma(f, x, T) = \inf\{U^f_T(x, y) : y \in R^2\}.$$

For a function $w \in W^{2,1}_{loc}([0, \infty))$ we denote by $\Omega(w)$ the set of all points $z \in R^2$ such that $X_w(t_j) \to z$ as $j \to \infty$ for some sequence of numbers $t_j \to \infty$.

We denote $d(x, B) = \inf\{|x - y| : y \in B\}$ for $x \in R^n$, $B \subset R^n$, and denote by $\mathrm{dist}(A, B)$ the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$.

In [14] we established the existence of a set $F \subset \bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ and for which the following theorems are valid.

Theorem 3.1. *Let $f \in F$. Then there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for any (f)-good function w.*

Theorem 3.1 describes the limit behavior of (f)-good functions for a generic $f \in \bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$. The following results show that for a generic $f \in \bar{\mathcal{M}}(\alpha,\beta,\gamma,a)$ the strong turnpike property holds with the set $H(f)$ being the attractor.

Theorem 3.2. *Let $f \in F$ and $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ and numbers $l_0 > l > 0,\ K_* > K,\ \delta > 0$ such that for each $g \in \mathcal{U}$, each $\tau \geq 2l_0$ and each $v \in W^{2,1}([0,\tau])$ which satisfies*

$$|(v(0),v'(0))|,\ |(v(\tau),v'(\tau))| \leq K \text{ and}$$

$$I^g(0,\tau,v) \leq U^g_\tau((v(0),v'(0)),(v(\tau),v'(\tau))) + \delta,$$

the relation $|(v(t),v'(t))| \leq K_*$ holds for all $t \in [0,\tau]$, and

$$\text{dist}(H(f),\{(v(t),v'(t)) : t \in [T,T+l]\}) \leq \epsilon \tag{3.1}$$

for each $T \in [l_0,\tau - l_0]$.

Theorem 3.3. *Let $f \in F$ and $\epsilon, K > 0$. Then there exist a neighborhood $\mathbf{U}$ of f in $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ and numbers $l_0 > l > 0,\ K_* > K,\ \delta > 0$ such that for each $g \in \mathbf{U}$, each $\tau \geq 2l_0$ and each $v \in W^{2,1}([0,\tau])$ which satisfies*

$$|(v(0),v'(0))| \leq K,\ I^g(0,\tau,v) \leq \sigma(g,(v(0),v'(0)),\tau) + \delta,$$

the relation $|(v(0),v'(0))| \leq K_*$ holds for all $t \in [0,\tau]$ and (3.1) holds for each $T \in [l_0,\tau - l_0]$.

4. Spaces of smooth integrands

In this section we use the notation and definitions introduced in the previous sections. Fix $a = (a_1,a_2,a_3,a_4) \in R^4$ and positive numbers α,β,γ such that $a_i > 0,\ i = 1,2,3,4$ and $1 \leq \beta < \alpha,\ \beta \leq \gamma,\ \gamma > 1$. Let $k \geq 2$ be an integer. Denote by $\mathcal{M}^0_k(\alpha,\beta,\gamma,a)$ the set of all integrands $f = f(w,p,r) \in C^k(R^3)$ such that:

$$f(w,p,r) \geq a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w,p,r) \in R^3;$$

there is an increasing function $M_f : [0,\infty) \to [0,\infty)$ such that for every $(w,p,r) \in R^3$

$$\sup\{f(w,p,r),\ |\partial f/\partial w(w,p,r)|,\ |\partial f/\partial p(w,p,r)|,\ |\partial f/\partial r(w,p,r)|\} \leq$$

$$M_f(|w|+|p|)(1+|r|^\gamma); \quad \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3.$$

For $q = (q_1,q_2,q_3) \in \{0,\dots k\}^3$ such that $q_1+q_2+q_3 \leq k$ and $f \in \mathcal{M}^0_k(\alpha,\beta,\gamma,a)$ we set

$$|q| = q_1+q_2+q_3,\ D^q f = \partial^{|q|} f/\partial w^{q_1}\partial p^{q_2}\partial r^{q_3}.\ (\text{Here } D^0 f = f).$$

For the set $\mathcal{M}_k^0(\alpha,\beta,\gamma,a)$ we consider the uniformity which is determined by the following base

$$E(N,\epsilon,\Gamma) = \{(f,g) \in \mathcal{M}_k^0(\alpha,\beta,\gamma,a) \times \mathcal{M}_k^0(\alpha,\beta,\gamma,a) :$$

$$|D^q f(x_1,x_2,x_3) - D^q g(x_1,x_2,x_3)| \le \epsilon \ (x_i \in R,\ |x_i| \le N,\ i = 1,2,3),$$

$$\text{for each } q \in \{0,\dots k\}^3 \text{ satisfying } |q| \le k, \text{ for each } q = (q_1,q_2,q_3) \in \{0,1,2,3\}^3$$

$$\text{such that } q_1 \ge 1,\ |q| = 3 \text{ and for each } q = (q_1,q_2,q_3) \in$$

$$\{0,1,2,3,4\}^3 \text{ such that } q_3 \ge 1,\ |q| \in \{3,4\},$$

$$(|D^q f(x_1,x_2,x_3)| + 1)(|D^q g(x_1,x_2,x_3)| + 1)^{-1} \in [\Gamma^{-1},\Gamma]$$

$$((x_1,x_2,x_3) \in R^3, |x_1|,|x_2| \le N),\ q \in \{0,1\}^3,\ |q| \le 1\},$$

where $N,\epsilon > 0$, $\Gamma > 1$. Clearly, the uniform space $\mathcal{M}_k^0(\alpha,\beta,\gamma,a)$ is Hausdorff and has a countable base. Therefore $\mathcal{M}_k^0(\alpha,\beta,\gamma,a)$ is metrizable. It is easy to verify that the uniform space $\mathcal{M}_k^0(\alpha,\beta,\gamma,a)$ is complete.

Let $\rho(\cdot,\cdot) : \bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a) \times \bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a) \to R$ be a metric which generates the uniformity for $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$. Set

$$\mathcal{M}_k(\alpha,\beta,\gamma,a) = \{f \in \mathcal{M}_k^0(\alpha,\beta,\gamma,a) :\ \partial^2 f/\partial r^2(w,p,r) > 0 \text{ for all } (w,p,r) \in R^3\}$$

and denote by $\bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a)$ the closure of $\mathcal{M}_k(\alpha,\beta,\gamma,a)$ in $\mathcal{M}_k^0(\alpha,\beta,\gamma,a)$. Clearly

$$\mathcal{M}_k(\alpha,\beta,\gamma,a) = \mathcal{M}_k^0(\alpha,\beta,\gamma,a) \cap \mathcal{M}(\alpha,\beta,\gamma,a),\ \bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a) \subset \bar{\mathcal{M}}(\alpha,\beta,\gamma,a),$$

and $\bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a)$ is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a)$.

In [14] we considered the topological subspace

$$\bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a) \subset \mathcal{M}_k^0(\alpha,\beta,\gamma,a)$$

with the relative topology and established the existence of a set $F_k \subset \mathcal{M}_k(\alpha,\beta,\gamma,a)$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_k(\alpha,\beta,\gamma,a)$ and for which the following theorems are valid.

Theorem 4.1. *Let $f \in F_k$. Then there exist a function $v_f \in C^5(R) \cap C^{k+1}(R)$ and a number $T_f > 0$ such that the following assertions hold:*

1. *$v_f(t+T_f) = v_f(t)$ for all $t \in R$ and $I^f(0,T_f,v_f) = T_f\mu(f)$.*
2. *If $\mu(f) < \inf\{f(z,0,0) : z \in R\}$ then $(v_f(t_1),v_f'(t_1)) \ne (v_f(t_2),v_f'(t_2))$ for each t_1,t_2 satisfying $0 \le t_1 < t_2 < T_f$. Otherwise $v_f(t) = v_f(0)$ for all $t \in R$.*
3. *For every periodic (f)-good function w there exists a number τ such that $w(t) = v_f(t+\tau)$ for all $t \in [0,\infty)$.*
4. *For every $\epsilon > 0$ there exists a neighborhood $\mathbf{U}$ of f in $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ such that for every $g \in \mathbf{U}$, every (g)-good function w and every large enough τ there exists $h \ge 0$ for which*

$$\sup\{|(w(t),w'(t)) - (v_f(t+h),v_f'(t+h))| : t \in [\tau,\tau+T_f]\} \le \epsilon. \tag{4.1}$$

Theorem 4.2. *Let $f \in F_k$, and let $\epsilon, K > 0$. Then there exist a neighborhood* $\mathbf{U}$ *of* f *in* $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ *and numbers* $Q > 0$, $\delta \in (0,\epsilon)$ *such that:*

1. *For every* $g \in \mathbf{U}$, *every* $x \in R^2$ *satisfying* $|x| \le K$, *every* $w \in A(g,x)$ *and every* $\tau \ge Q$, *equation (4.1) holds with some* $h \ge 0$.
2. *For every* $g \in \mathbf{U}$, *every* $x \in R^2$ *satisfying* $d(x, \{(v_f(t), v_f'(t)) : t \in [0,T_f]\}) \le \delta$, *every* $w \in A(g,x)$ *and every* $\tau \ge 0$, *equation (4.1) holds with some* $h \ge 0$.

Corollary 4.3. *Let* $f \in F_k$ *and* $w \in W^{2,1}_{loc}(R)$. *Suppose that*

$$I^f(t_1,t_2,w) = \mu(f)(t_2-t_1) + \pi^f(w(t_1),w'(t_1)) - \pi^f(w(t_2),w'(t_2))$$

for each t_1, t_2 *satisfying* $-\infty < t_1 < t_2 < +\infty$ *and* $\liminf_{t\to-\infty} |(w(t),w'(t))| < \infty$. *Then there exists a number* h *such that* $w(t) = v_f(t+h)$ *for all* $t \in R$.

Theorem 4.4. *Let* $f \in F_k$ *and* $\epsilon, K > 0$. *Then there exist a neighborhood* $\mathbf{U}$ *of* f *in* $\bar{\mathcal{M}}_1(\alpha,\beta,\gamma,a)$ *and numbers* $l > T_f$, $K_* > K$, $\delta > 0$ *such that:*

1. *For each* $g \in \mathbf{U}$, *each* $T \ge 2l$ *and each* $w \in W^{2,1}([0,T])$ *which satisfies*

$$|(w(0),w'(0))|,\ |(w(T),w'(T))| \le K,$$

$$I^g(0,T,w) \le U^g_T((w(0),w'(0)),(w(T),w'(T))) + \delta$$

the relation $|(w(t),w'(t))| \le K_*$ *holds for all* $t \in [0,T]$, *and for each* $\tau \in [l, T-l]$ *equation (4.1) holds with some* $h \ge 0$.

2. *For each* $g \in \mathbf{U}$, *each* $T \ge 2l$ *and each* $w \in W^{2,1}([0,T])$ *which satisfies*

$$|(w(0),w'(0))| \le K,\ I^g(0,T,w) \le \sigma(g,(w(0),w'(0)),T) + \delta,$$

the relation $|(w(t),w'(t))| \le K_*$ *holds for all* $t \in [0,T]$ *and for each* $\tau \in [l, T-l]$ *equations (4.1) holds with some* $h \ge 0$.

Analogs of Theorems 2.2 and 2.3 hold for every $f \in F_k$. By the analog of Theorem 2.2 for every $f \in F_k$ and every $x \in R^2$ there exists an (f)-weakly optimal solution $v \in A_x$ and the analog of Theorem 2.3 establishes that every $f \in F_k$ is a continuity point of the operator

$$g \to (\mu(g), \pi^g),\ g \in \bar{\mathcal{M}}(\alpha,\beta,\gamma,a).$$

5. The structure of periodic good functions

In this section we use the notation and definitions introduced in the previous sections. Fix $a = (a_1,a_2,a_3,a_4) \in R^4$ and positive numbers α,β,γ such that $a_i > 0$, $i = 1,2,3,4$, $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$.

Let $f \in \mathcal{M}(\alpha,\beta,\gamma,a)$.

The following results were established in [7].

Theorem 5.1. *Assume that* $w \in W^{2,1}_{loc}(R^1), T > 0$,

$$w(t+T) = w(t), t \in R^1, I^f(0,T,w) = T\mu(f),$$

and $w'(t) \ne 0$ *for some* $t \in R^1$. *Then there exists* $\tau > 0$ *such that*

$$w(t+\tau) = w(t), t \in R^1, X_w(T_1) \ne X_w(T_2)$$

for each $T_1 \in R^1$ *and each* $T_2 \in (T_1, T_1+\tau)$.

Theorem 5.2. *Assume that* $w \in W^{2,1}_{loc}(R^1), \tau > 0,$

$$w(t+\tau) = w(t), t \in R^1, I^f(0,\tau,w) = \tau\mu(f),$$

$$w(0) = \inf\{w(t) : t \in R^1\}, \text{ and } w'(t) \neq 0 \text{ for some } t \in R^1.$$

Then there exist $\tau_1 > 0, \tau_2 > \tau_1$ *such that the function* w *is strictly increasing in* $[0,\tau_1]$*,* w *is strictly decreasing in* $[\tau_1,\tau_2]$*, and*

$$w(\tau_1) = \sup\{w(t) : t \in R^1\}, \ w(t+\tau_2) = w(t), t \in R^1.$$

6. Asymptotic turnpike property

In this section we use the notation and definitions introduced in the previous sections. The results presented in this section was obtained in [7].

Fix $a = (a_1, a_2, a_3, a_4) \in R^4$ and positive numbers α, β, γ such that $a_i > 0, \ i = 1,2,3,4, \ 1 \le \beta < \alpha, \ \beta \le \gamma, \ \gamma > 1$. Set

$$\mathcal{M} = \mathcal{M}(\alpha,\beta,\gamma,a) \text{ and } \bar{\mathcal{M}} = \bar{\mathcal{M}}(\alpha,\beta,\gamma,a).$$

Let $f \in \bar{\mathcal{M}}$. We will say that w is optimal on compacts, or briefly c-optimal, if $w \in W^{2,1}_{loc}([0,\infty)) \cap W^{1,\infty}([0,\infty))$ and for all $T > 0$,

$$U^f_T((w,w')(0),(w,w')(T)) = I^f(0,T,w).$$

Let $f \in \mathcal{M}$. We say that f has the asymptotic turnpike property, or briefly (ATP), if there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for every (f)-good function w.

Clearly, if f has (ATP) and v is a periodic (f)-good function, then $H(f) = \{(v,v')(t) : 0 \le t < \infty\}$.

The asymptotic turnpike property for optimal control problems was studied in [2, 17]. Here we will consider, besides (ATP), the strong turnpike property, or briefly (STP), which is defined as follows.

Let $f \in \mathcal{M}$ and let w be a periodic (f)-good function with period $T_w > 0$. We say that f has the strong turnpike property if, for every $\epsilon > 0$ and every bounded set $K \subset R^2$, there exists $L > 0$ such that every $v \in W^{2,1}([0,T])$ satisfying

$$X_v(0) = x, \ X_v(T) = y, \ I^f(0,T,v) = U^f_T(x,y),$$

with $x, y \in K$ and $T > T_w + 2L$, satisfies the following:

For every $a \in [L, T-L-T_w]$ there exists $\bar{a} \in [0,T_w)$ such that,

$$|(v,v')(a+t) - (w,w')(\bar{a}+t)| \le \epsilon, \quad \forall t \in [0,T_w].$$

Note that (STP) implies uniqueness up to translation for periodic (f)-good functions.

The following results were obtained in [7].

Theorem 6.1. *Assume that* $g \in \mathcal{M}$ *satisfies (ATP). Let* w *be a periodic* (g)*-good function and let* $T_w > 0$ *be a period of* w*.*

Given ϵ*,* $M > 0$ *there exists a neighbourhood of* g *in* $\bar{\mathcal{M}}$*, say* $\mathcal{U}_g$*, and positive numbers* δ*,* l *such that the following statement holds :*

Let $f \in \mathcal{U}_g$ *and let* $T \ge T_w + 2l$*. Suppose that,*

$$v \in W^{2,1}([0,T]), \ |X_v(0)|, |X_v(T)| \le M, \ I^f(0,T,v) \le U^f_T(X_v(0), X_v(T)) + \delta.$$

Then, for each $s \in [l, T - T_w - l]$ there exists $\xi \in [0, T_w]$ such that,

$$|X_v(s+t) - X_w(\xi + t)| \le \epsilon, \quad \forall t \in [0, T_w]. \tag{6.1}$$

Remark. The conclusion of the theorem can be slightly strengthened as follows:

There exist $\tau_1 \in [0, l]$ and $\tau_2 \in [T - l, T]$ such that, for every $s \in [\tau_1, \tau_2 - T_w]$ there exists $\xi \in [0, T_w]$ such that (6.1) holds. Furthermore, if

$$d(X_v(0), \Omega(w)) \le \delta, \text{ (respectively } d(X_v(T), \Omega(w)) \le \delta),$$

the statement holds with $\tau_1 = 0$, (respectively $\tau_2 = T$).

Theorem 6.2. *Assume that $g \in \mathcal{M}$ has (ATP) and $w \in W^{2,1}_{loc}(R^1)$ is a periodic (g)-good function with a period $T_w > 0$. Then, for every $\epsilon > 0$, there exists a neighborhood $\mathcal{U}$ of g in $\bar{\mathcal{M}}$ such that for each $f \in \mathcal{U}$:*

If v is an (f)-good function then, for every sufficiently large T (depending on v) there exists $\xi \in [0, T_w)$ such that,

$$|X_v(T+t) - X_w(\xi + t)| \le \epsilon, \quad t \in [0, T_w].$$

7. Nonintersecting property

In this section we use the notation and definitions introduced in the previous sections. The results presented in this section was obtained in [8]. Fix $a = (a_1, a_2, a_3, a_4) \in R^4$ and positive numbers α, β, γ such that $a_i > 0,\ i = 1, 2, 3, 4,\ 1 \le \beta < \alpha,\ \beta \le \gamma,\ \gamma > 1$. Set

$$\mathcal{M} = \mathcal{M}(\alpha, \beta, \gamma, a) \text{ and } \bar{\mathcal{M}} = \bar{\mathcal{M}}(\alpha, \beta, \gamma, a).$$

Let $f \in \mathcal{M}$. We continue to study c-optimal functions introduced in Section 6.

Proposition 7.1. *For every point $x = (x_1, x_2) \in R^2$ there exists a c-optimal function w on $[0, \infty)$ such that $(w(0), w'(0)) = x$.*

Another class of minimizers, which plays an important role in our theory, is the class of perfect functions. First we define the concept of a perfect function on an arbitrary interval. The definition requires some additional notation. If $v \in W^{2,1}(D)$, $D = [T_1, T_2]$, put

$$\Gamma^f(D, v) := I^f(T_1, T_2, v) - (T_2 - T_1)\mu(f) + \pi^f(X_v(T_2)) - \pi^f(X_v(T_1)). \tag{1.4}$$

If $\{D_j\}_{j=1}^k$ is a partition of D into disjoint subintervals, then,

$$\Gamma^f(D, v) = \sum_{j=1}^{k} \Gamma^f(D_j, v).$$

We refer to this property of Γ as additivity on intervals.

Proposition 7.2. *For every $x \in R^2$ there exists a perfect function v on $[0, \infty)$ such that $(v(0), v'(0)) = x$.*

We turn now to a description of the main results of the work [8]. The first main result concerns the non-intersecting property of c-optimal functions.

Theorem 7.3. *(a) Let v be a c-optimal function. If there exists $T > 0$ such that*

$$(v, v')(0) = (v, v')(T)$$

then v is periodic with period T.

(b) Let v_1, v_2 be c-optimal functions such that

$$(v_1, v_1')(0) = (v_2, v_2')(0).$$

If there exist $t_1, t_2 \in [0, \infty)$ such that $(t_1, t_2) \neq (0, 0)$ and

$$(v_1, v_1')(t_1) = (v_2, v_2')(t_2),$$

then $v_1(t) = v_2(t)$ for all $t \geq 0$.

The next two results describe the limiting set of c-optimal functions in the phase plane and their asymptotic behavior at infinity. The non-intersecting property plays a crucial role in the derivation of these results. We use the following notation. If $v \in W^{2,1}_{loc}([0, \infty)) \cap W^{1,\infty}([0, \infty))$, then the set of limiting points of (v, v') as $t \to \infty$ is denoted by $\Omega(v)$.

Theorem 7.4. *Let*

$$\text{(N)} \qquad \mu(f) < \inf\{f(x, 0, 0) : \ x \in R^1\}$$

and let v be a c-optimal function. Then there exists a periodic (f)-good function w such that $\Omega(v) = \Omega(w)$ and the following assertion holds:

Let $T > 0$ be a period of w. Then, for every $\epsilon > 0$ there exists $\tau(\epsilon) > 0$ such that for every $\tau \geq \tau(\epsilon)$ there exists $s \in [0, T)$ such that,

$$(7.1) \qquad |(v, v')(t + \tau) - (w, w')(s + t)| \leq \epsilon, \ t \in [0, T].$$

Our next result describes the structure of the limiting set of c-optimal functions, in the absence of assumption (N). In this case the structure of the limiting set is considerably more complicated.

Theorem 7.5. *Suppose that $\mu(f) = \inf\{f(d, 0, 0) : \ d \in R^1\}$ and that the set $\mathcal{D} = \{d \in R^1 : \ f(d, 0, 0) = \mu(f)\}$ is finite. Let v be a c-optimal function. Then $\Omega(v)$ is a compact connected set and the following alternative holds. Either there exists a periodic (f)-good function w such that $\Omega(v) = \Omega(w)$ and (7.1) holds, or $\Omega(v)$ is a finite union of arcs $\cup_{j=1}^{k} \bar{\Xi}_j$ such that each arc Ξ_j is the phase plane image of a perfect function u_j, i.e.,*

$$\Xi_j = \{(u_j, u_j')(t) : \ t \in R^1\}, \quad j = 1, \cdots, k.$$

Furthermore, each function u_j is monotone in neighborhoods of $+\infty$ and $-\infty$ and satisfies,

$$\lim_{t \to \infty} (u_j, u_j')(t) \in \mathcal{D} \times \{0\}, \quad \lim_{t \to -\infty} (u_j, u_j')(t) \in \mathcal{D} \times \{0\}.$$

8. Structure of optimal solutions

In this section we continue to analyze the structure of optimal solutions of the variational problems

$$(P) \qquad \int_0^T f(w(t), w'(t), w''(t))dt \to \min$$

$$w \in W^{2,1}([0,T]),\ (w(0), w'(0)) = x \text{ and } (w(T), w'(T)) = y,$$

where $T > 0$, $x, y \in R^2$, $W^{2,1}([0,T]) \subset C^1$ and f belongs to a space of functions considered in the previous section.

We also consider the following problem on the half line:

$$(P_\infty) \qquad \inf\left\{\liminf_{T\to\infty} T^{-1}\int_0^T f(w(t), w'(t), w''(t))dt : \ w \in W^{2,1}_{loc}([0,\infty))\right\}.$$

Let $a = (a_1, a_2, a_3, a_4) \in R^4$, $a_i > 0$, $i = 1, 2, 3, 4$ and let α, β, γ be positive numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. We consider the spaces $\mathcal{M}(\alpha, \beta, \gamma, a)$ and $\bar{\mathcal{M}}(\alpha, \beta, \gamma, a)$ introduced in Section 2.

Denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau > 0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v : [0,\tau] \to R^2$ as follows:

$$X_v(t) = (v(t), v'(t)),\ t \in [0,\tau].$$

We also use this definition for $v \in W^{2,1}_{loc}([0,\infty))$ and $v \in W^{2,1}_{loc}(R)$.

Put

$$\mathcal{M} = \mathcal{M}(\alpha, \beta, \gamma, a),\ \bar{\mathcal{M}} = \bar{\mathcal{M}}(\alpha, \beta, \gamma, a).$$

We consider functionals of the form

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t), v''(t))dt,$$

$$\Gamma^f(T_1, T_2, v) = I^f(T_1, T_2, v) - (T_2 - T_1)\mu(f) - \pi^f(X_v(T_1)) + \pi^f(X_v(T_2)),$$

where $-\infty < T_1 < T_2 < +\infty$, $v \in W^{2,1}([T_1, T_2])$ and $f \in \bar{\mathcal{M}}$.

We denote by mes(E) the Lebesgue measure of a measurable set $E \subset R$ and by int(D) the interior of a subset D of a metric space.

If $v \in W^{2,1}_{loc}([0,\infty))$ satisfies

$$\sup\{|X_v(t)| : \ t \in [0,\infty)\} < \infty,$$

then the set of limiting points of $X_v(t)$ as $t \to \infty$ is denoted by $\Omega(v)$.

Denote by Card(A) the cardinality of the set A. If $f \in \bar{\mathcal{M}}$, $J = [T_1, T_2]$ with $T_2 > T_1$, $v \in W^{2,1}([T_1, T_2])$, then we set

$$\Gamma^f(J, v) = \Gamma^f(T_1, T_2, v).$$

In [7, 14] we considered certain important subspaces of the space $\mathcal{M}$ equipped with natural uniformities and showed that each of them contains an everywhere dense G_δ subset such that each its element f has the following two properties:

The problem (P_∞) has a unique up to translation periodic minimizer w.

Let $T_w > 0$ be a period of w. For any $\epsilon > 0$ there exists a constant $L > 0$ which depends only on $|x|, |y|$ and ϵ such that for each optimal solution v of problem (P) and each $\tau \in [L, T - L - T_w]$ there exists $s \in [0, T_w)$ such that

$$|(v(\tau+t), v'(\tau+t)) - (w(s+t), w'(s+t))| \le \epsilon \text{ for each } t \in [0, T_w].$$

The results of [7, 14] establish that most integrands (in the sense of Baire's categories) have the turnpike properties. Since the space $\bar{\mathcal{M}}$ and its subspaces considered in [7, 14] contain integrands which do not have the turnpike properties these results cannot be essentially improved. Nevertheless, some questions are still open. It is very important and interesting to obtain some knowledge about the structure of extremals of problem (P) with arbitrary integrand $f \in \mathcal{M}$.

In this section we discuss the results of [16] which show that for each integrand $f \in \mathcal{M}$ the following property holds:

For each pair of positive numbers ϵ, l there exists a constant $L > l$ which depends only on $|x|, |y|, l$ and ϵ such that for each optimal solution v of problem (P) and each closed subinterval $D \in [0, T]$ of length L there exists a closed subinterval $D_1 \subset D$ of length l and a periodic minimizer w of problem (P_∞) such that

$$|(v(t), v'(t)) - (w(t), w'(t))| \le \epsilon \text{ for each } t \in D_1.$$

Let $f \in \mathcal{M}$. Denote by $\sigma(f)$ the set of all $w \in W^{2,1}_{loc}(R)$ which have the following property:

There is $T_w > 0$ such that

$$w(t + T_w) = w(t) \text{ for all } t \in R \text{ and } I^f(0, T_w, w) = \mu(f) T_w.$$

In other words $\sigma(f)$ is the set of all periodic minimizers of (P_∞). By Theorem 4.1 of [12], $\sigma(f) \neq \emptyset$.

The following result established in [7, Lemma 3.1] describes the structure of periodic minimizers of (P_∞).

Proposition 8.1. *Let $f \in \mathcal{M}$. Assume that $w \in \sigma(f)$,*

$$w(0) = \inf\{w(t) :\ t \in R\}$$

and $w'(t) \neq 0$ for some $t \in R$. Then there exist $\tau_1(w) > 0$ and $\tau(w) > \tau_1(w)$ such that the function w is strictly increasing on $[0, \tau_1(w)]$, w is strictly decreasing in $[\tau_1(w), \tau(w)]$,

$$w(\tau_1(w)) = \sup\{w(t) :\ t \in R\} \text{ and } w(t + \tau(w)) = w(t) \text{ for all } t \in R.$$

Corollary 8.2. *Let $f \in \mathcal{M}$, $t_0 \in R$, $w \in \sigma(f)$,*

$$w'(t) \neq 0 \text{ for some } t \in R \text{ and } w(t_0) = \inf\{w(t) :\ t \in R\}.$$

Then there exist $\tau_1(w) > 0$ and $\tau(w) > \tau_1(w)$ such that the function w is strictly increasing in $[t_0, t_0 + \tau_1(w)]$, w is strictly descreasing in $[t_0 + \tau_1(w), t_0 + \tau(w)]$,

$$w(t_0 + \tau_1(w)) = \sup\{w(t) :\ t \in R\} \text{ and } w(t + \tau(w)) = w(t) \text{ for all } t \in R.$$

Let $f \in \mathcal{M}$. By Corollary 8.2, each $w \in \sigma(f)$ which is not a constant has a minimal period which will be denoted by $\tau(w)$. Put

$$\sigma(f, 0) = \{w \in \sigma(f) :\ w \text{ is a constant}\}.$$

For each $T > 0$ set

$$\sigma(f,T) = \sigma(f,0) \cup \{w \in \sigma(f) : w \text{ is not a constant and } \tau(w) \le T\}.$$

The following theorem is the main result of [16].

Theorem 8.3. *Let $f \in \mathcal{M}$ and let l, M_0, M_1, ϵ be positive numbers. Then there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathcal{M}}$ such that for each $g \in \mathcal{U}$, each $T \ge L$ and each $v \in W^{2,1}([0,T])$ which satisfies*

$$|(v(0), v'(0))|,\ |(v(T), v'(T))| \le M_0,$$

$$I^g(0,T,v) \le U_T^g((v(0), v'(0)), (v(T), v'(T))) + M_1$$

the following property holds:

For each $s \in [0, T-L]$ there are $s_1 \in [s, s+L-l]$ and $w \in \sigma(f)$ such that

$$|(v(s_1+t), v'(s_1+t)) - (w(t), w'(t))| \le \epsilon \text{ for all } t \in [0,l]. \tag{8.1}$$

The next theorem describes the structure of good functions.

Theorem 8.4. *Let $f \in \mathcal{M}$ and let l, ϵ be positive numbers. Then there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathcal{M}}$ such that for each $g \in \mathcal{U}$ and each (g)-good function $v \in W^{2,1}_{loc}([0,\infty))$ there exist $T_0 \ge 0$ such that the following property holds:*

For each $s \ge T_0$ there are $s_1 \in [s, s+L-l]$ and $w \in \sigma(f)$ such that inequality (8.1) is valid.

The next two results describe the structure of approximate solutions of problem (P_∞).

Theorem 8.5. *Let $f \in \mathcal{M}$, l, ϵ be positive numbers and let $v \in W^{2,1}_{loc}([0,\infty))$ satisfy*

$$\limsup_{T\to\infty} T^{-1} I^f(0,T,v) = \mu(f)$$

and

$$\sup\{|(v(t), v'(t))| :\ t \in [0,\infty)\} < \infty. \tag{8.2}$$

Then there exists $L_0 > l$ such that the following assertion holds:

For each $\gamma > 0$ there is $T_\gamma > L_0$ such that for each $T \ge T_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_{q_T}$ such that

$$q_T \le \gamma T,$$

$$mes(J_i) \le L_0,\ i = 1, \dots, q_T,$$

$$int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers}$$

$$i, p \in \{1, \dots, q_T\} \text{ such that } i \ne p,$$

and if

$$s \in [0, T-L_0] \text{ and } [s, s+L_0] \cap J_i = \emptyset \text{ for all } i = 1, \dots, q_T,$$

then there are $s_1 \in [s, s+L_0-l]$ and $w \in \sigma(f)$ such that (8.1) is valid.

Theorem 8.6. *Let $f \in \mathcal{M}$, l, ϵ be positive numbers and let $v \in W^{2,1}_{loc}([0,\infty))$ satisfy (8.2). Assume that there exists a strictly increasing sequence of positive numbers $\{T_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} T_i = \infty$ and*

$$\lim_{i\to\infty} T_i^{-1} I^f(0, T_i, v) = \mu(f).$$

Then there exists $L_0 > l$ such that the following assertion holds:

For each $\gamma > 0$ there is a natural number j_γ with $T_{j_\gamma} > L_0$ such that for each integer $j \geq j_\gamma$ the inequality $T_j \geq L_0$ holds and there are a finite number of closed intervals $J_1, \dots, J_{q_j}$ such that

$$q_j \leq \gamma T_j, \ mes(J_i) \leq L_0 \text{ for all } i = 1, \dots, q_j,$$

$$int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers}$$

$$i, p \in \{1, \dots, q_j\} \text{ such that } i \neq p,$$

and if

$$s \in [0, T_j - L_0] \text{ and } [s, s + L_0] \cap J_i = \emptyset \text{ for all } i = 1, \dots, q_j,$$

then there are $s_1 \in [s, s + L_0 - l]$ and $w \in \sigma(f)$ such that (8.1) is valid.

In [7, Lemma 3.2] it was proved the following result.

Proposition 8.7. *Let $f \in \mathcal{M}$ satisfy*

$$\mu(f) < \inf\{f(t, 0, 0) : \ t \in R\}.$$

Then no element of $\sigma(f)$ is a constant and $\sup\{\tau(w) : \ w \in \sigma(f)\} < \infty$.

Let $f \in \mathcal{M}$ satisfy

$$\mu(f) < \inf\{f(t, 0, 0) : \ t \in R\}.$$

We can choose l in Theorems 8.3-8.6 as

$$l = k \sup\{\tau(w) : \ w \in \sigma(f)\}$$

where k is a large natural number. Let $L > l$ be as guaranteed by Theorem 8.3. If an approximate solution v of problem (P) satisfies conditions of Theorem 8.3, then for each closed subinterval $D \in [0, T]$ of length L there exists a closed subinterval $D_1 \subset D$ of length $k \sup\{\tau(w) : \ w \in \sigma(f)\}$ and a periodic minimizer w of problem (P) such that

$$|(v(t), v'(t)) - (w(t), w'(t))| \leq \epsilon \text{ for each } t \in D_1.$$

Clearly, the restriction of v to interval D_1 is a good approximation of the periodic minimizer w.

If $\mu(f) = \inf\{f(t, 0, 0) : \ t \in R\}$, then there is a periodic minimizer $w \in \sigma(f)$ which is a constant and Proposition 8.7 does not hold. Namely, the set $\{\tau(w) : w \in \sigma(f)\}$ can be unbounded. In this case the turnpike property in Theorems 8.3-8.6 (see inequality (8.4)) does not provide sufficient information about the periodic minimizer w if its period is larger than l.

Now we state a result which is a concretization of Theorem 8.3.

Theorem 8.8. *Let $f \in \mathcal{M}$ and let M_0, M_1, ϵ, l_0 be positive numbers. Then there exists $h > l_0$ such that the following assertion holds:*

For each $l > h$ there are $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathcal{M}}$ such that for each $g \in \mathcal{U}$, each $T \geq L$, each $v \in W^{2,1}([0,T])$ satisfying

$$|(v(0), v'(0))|, \ |(v(T), v'(T))| \leq M_0,$$
$$I^g(0, T, v) \leq U_T^g((v(0), v'(0)), (v(T), v'(T))) + M_1$$

and each $s \in [0, T-L]$ there is $s_1 \in [s, s+L-l]$ such that at least one of the following properties holds:

(i) *there exists $w \in \sigma(f, h)$ such that*

$$|(v(s_1+t), v'(s_1+t)) - (w(t), w'(t))| \leq \epsilon \text{ for all } t \in [0, l];$$

(ii) *for each $\tau \in [s_1, s_1 + l - h]$ there are $\tau_1 \in [\tau, \tau + h - l_0]$ and $\xi \in \sigma(f, 0)$ such that*

$$|(v(\tau_1+t), v'(\tau_1+t)) - (\xi(0), 0)| \leq \epsilon \text{ for all } t \in [0, l_0].$$

The next theorem is a concretization of Theorem 8.4.

Theorem 8.9. *Let $f \in \mathcal{M}$ and let l_0, ϵ be positive numbers. Then there exists $h > l_0$ such that the following assertion holds:*

For each $l > h$ there are $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathcal{M}}$ such that for each $g \in \mathcal{U}$, each (g)-good function $v \in W^{2,1}_{loc}([0,\infty))$ and each sufficiently large number s there is $s_1 \in [s, s+L-l]$ such that at least one of the properties (i) and (ii) of Theorem 8.8 holds.

The next two theorems describe the structure of approximate solutions of problem (P_∞).

Theorem 8.10. *Let $f \in \mathcal{M}$ and $v \in W^{2,1}_{loc}([0,\infty))$ satisfy*

$$\sup\{|(v(t), v'(t))| : \ t \in [0,\infty)\} < \infty,$$
$$\limsup_{T \to \infty} T^{-1} I^f(0, T, v) = \mu(f).$$

Assume that ϵ, l_0 are positive numbers. Then there exists $h > l_0$ such that for each $l > h$ there is $L > l$ for which the following assertion holds:

For each $\gamma > 0$ there is $T_\gamma > L$ such that for each $T \geq T_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_q$ such that

$$q \leq \gamma T, \tag{8.3}$$

$$mes(J_i) \leq L, \ i = 1, \dots, q, \tag{8.4}$$

$$int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers}$$
$$i, p \in \{1, \dots, q\} \text{ satisfying } i \neq p \tag{8.5}$$

and if

$$s \in [0, T-L], \ [s, s+L] \cap J_i = \emptyset, \ i = 1, \dots, q, \tag{8.6}$$

then there is $s_1 \in [s, s+L-l]$ such that at least one of the properties (i), (ii) of Theorem 8.8 holds.

Theorem 8.11. *Let $f \in \mathcal{M}$, $v \in W^{2,1}_{loc}([0,\infty))$ satisfy (8.3) and let $\{T_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive numbers such that $\lim_{i\to\infty} T_i = \infty$ and*

$$\lim_{i\to\infty} T_i^{-1} I^f(0, T_i, v) = \mu(f).$$

Assume that $\epsilon > 0$, $l_0 > 0$. Then there exists $h > l_0$ such that for each $l > h$ there is $L > l$ such that the following assertion holds:

For each $\gamma > 0$ there is a natural number i_γ such that $T_{i_\gamma} > L$ and that for each integer $i \geq i_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_q$ such that

$$q \leq \gamma T_i,$$

(8.4), (8.5) hold and that for each number s satisfying (8.6) there is $s_1 \in [s, s+L-l]$ for which at least one of the properties (i), (ii) of Theorem 8.8 holds.

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