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ON ASYMPTOTICALLY OPTIMAL INVESTMENT WITH THE RANK DEPENDENT EXPECTED UTILITY CRITERION

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Abstract. This paper concerns the well known paradox of inconsistency of the maximum-expected-utility (MEU) and the maximum-expected-log (MEL) criteria in investment dynamic models for large horizons. The goal of the paper is to consider this phenomenon at the level of premises, and to suggest a generalized criterion, namely the rank dependent expected utility (RDEU) approach which allows to "bridge the gap" between the MEU and MEL criteria. The preference order in the RDEU approach is preserved by the functional

$$U(F) = \int_0^\infty u(x)d\Psi(F(x)),$$

where F is a probability distribution, u is a utility function, and Ψ is a transforming or weighting function: the subject "transforms" the real distribution function $F(x)$ into another one, $\Psi(F(x))$, assigning different weights to different probabilities.

One of main goals of the paper is to establish conditions on the tail of Ψ, and on the utility function u, under which the *asymptotically optimal investment in the long run* corresponds to the MEL policy.

The result of the paper is relevant also to the questions of the survival of economic agents in the market and the accuracy of their predictions or beliefs.

1. Introduction and an example

1.1. Background and motivations.

1.1.1. *The MEU and MEL criteria.* This paper considers optimal investment in time, and concerns the long-known fact that the maximum-expected-utility (MEU) and the maximum-expected-log (MEL) criteria prove to be inconsistent even for large time horizons. In a certain sense, this is a paradox since both criteria have reasonable justifications based on assumptions which, though maybe are restrictive (as, say, the independence axiom), but are natural at least as the first approximation. Below, we explain the inconsistency mentioned at the level of premises, and give a possible solution to the problem proceeding from two issues: some relatively recent achievements of the modern utility theory, and a technique of determining asymptotically optimal controls in the long run.

More specifically, we make use of the rank dependent expected utility (RDEU) approach which, as will be seen, allows to "bridge the gap" between the MEU and MEL criteria, and in a sense to "reconcile" the results based on the two approaches.

In Section 1.1.3, we will also connect this problem with the questions of the survival of economic agents in the market and the accuracy of their predictions or beliefs.

The history of the question will be considered later in Section 1.1.4 after we state the problem.

Let the initial wealth of an investor $W_0 = 1$, and the wealth after T periods of time

$$W_T = W_{T\boldsymbol{\pi}} = \prod_{t=1}^{T}(1 + R_t(\boldsymbol{\pi})),$$

where $R_t(\boldsymbol{\pi})$ is a (random) return at time t corresponding to an investment policy $\boldsymbol{\pi}$. More formally, the map $R_t : \Pi \to \mathcal{K}$, where Π is a space of a general nature whose elements $\boldsymbol{\pi}$ are viewed as investment policies, and $\mathcal{K}$ is a space of random variables with values in $\mathbb{R}$.

In the standard framework, $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)$, a portfolio-vector, where n is the number of securities in the market, and π_k is the share of the capital invested in the kth security. In this case, $\Pi = \{\boldsymbol{\pi} = (\pi_1, ..., \pi_n) : \sum_{k=1}^{n} \pi_i = 1\}$, the return $R_t(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \mathbf{R}_t$, where $\mathbf{R}_t$ is the random vector of the returns of the securities, and $\cdot$ stand for dot product. Below, we do not need such a specification and may view $\boldsymbol{\pi}$ as a policy of a general nature.

Suppose that for each $\boldsymbol{\pi}$, the random returns $\{R_i(\boldsymbol{\pi})\}$ are independent and identically distributed (i.i.d.), and at each period t, the investor chooses the same policy not depending on the previous history. It is worth noting that in the asymptotic analysis, the last assumption does not, in essence, restrict generality; we discuss this issue in more detail in Section 1.1.3. Nevertheless strategies $\boldsymbol{\pi}$ may depend on the horizon T.

Let $m(\boldsymbol{\pi}) := E\{\ln(1 + R_1(\boldsymbol{\pi}))\}$, and there exist a policy

$$\hat{\boldsymbol{\pi}} = Arg \max_{\boldsymbol{\pi}} m(\boldsymbol{\pi})$$

which we call a maximum-expected-log (MEL) policy. Formally, we do not assume the uniqueness of such a policy. Note also that for results below we do not need to impose particular conditions on the map R_t for the existence of $\hat{\boldsymbol{\pi}}$: we just suppose it exists. In Section 2.3, we introduce some additional conditions on $\hat{\boldsymbol{\pi}}$ and R_t itself.

If for a $\boldsymbol{\pi}$ and some $c > 0$, it is true that $m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}) \geq c$, then by the strong law of large numbers (see also references in Section 1.1.4)

$$(W_{T\hat{\boldsymbol{\pi}}}/W_{T\boldsymbol{\pi}}) \to \infty, \quad \text{as } T \to \infty, \text{ almost surely.} \tag{1.1.1}$$

In view of (1.1.1), it might seem that in the long run any "reasonable" policy $\boldsymbol{\pi}$ should be close, in a sense, to $\hat{\boldsymbol{\pi}}$. However, as is well known, it is not the case for the MEU criterion

$$E\{u(W_{T\boldsymbol{\pi}})\} = \int_0^\infty u(x)dF_{T\boldsymbol{\pi}}(x), \tag{1.1.2}$$

where $u(x)$ is the *utility function* of the investor, and $F_{T\boldsymbol{\pi}}$ is the (cumulative) distribution function of the random variable (r.v.) $W_{T\boldsymbol{\pi}}$. Say, if $u(x) = x^\alpha$, then

$$E\{u(W_{T\boldsymbol{\pi}})\} = (E\{(1 + R_1(\boldsymbol{\pi})^\alpha\})^T,$$

and the maximum is attained under a policy $\boldsymbol{\pi}'$ which maximizes $E\{(1+R_1(\boldsymbol{\pi}))^\alpha\}$. Clearly, $\boldsymbol{\pi}'$ is not close to $\hat{\boldsymbol{\pi}}$ in general, and cannot be close asymptotically since $\boldsymbol{\pi}'$ does not depend on T at all. For more complicated u, the analysis is more complex, but the conclusion is the same; see, e.g., Merton and Samuelson (1974), Markowitz (1976), references therein and in Section 1.1.4.

There has been a great deal of discussion on which criterion, MEU or MEL, is more preferable or realistic; for references see again Section 1.1.4. In this paper, we are not concerned about which approach is better, but rather what makes them different, and the answer to this question is simple. Any integral criterion of the type (1.1.2) takes into account possibilities of "very large" values occurring with "very small" probabilities, while the property (1.1.1) has to do only with probabilities, and in a certain sense eliminates events with negligible probabilities.

If one deals with fixed, not growing, variables as, say, in a one-time fixed investment, the difference between the two criteria can be not significant. However, once we deal with growing variables (as $W_{T\boldsymbol{\pi}}$ in dynamics; for example, in a long period investment into a retirement fund), the difference mentioned may be dramatic.

The question is whether it is possible, maintaining at least some features of the classical MEU criterion, to make the decision process more flexible with respect to large deviations. One of possible answers is in making use of the *Rank Dependent Expected Utility (RDEU)* approach.

1.1.2. *The RDEU criterion.* On a space of probability distributions F on $[0, \infty)$ consider a preference order preserved by the functional

$$U(F) = \int_0^\infty u(x) d\Psi(F(x)), \tag{1.1.3}$$

where u is a *utility* function, and the function Ψ is assumed to be non-decreasing, $\Psi(0) = 0$, $\Psi(1) = 1$.

The "*transformation*" function Ψ reflects the attitude of the subject to different probabilities: the subject "transforms" the real distribution function $F(x)$ into another one, $\Psi(F(x))$, assigning different weights to different probabilities.

Historical comments on the RDEU approach and a rich bibliography may be found in monographs Wakker (1989), Quiggin (1993), and Luce (2000). Some remarks and references are also given in Section 1.1.4.

There are several axiomatic justifications of the criterion (1.1.3), for references see also Section 1.1.4. A key axiom is either the trade-off consistency requirement or the different, though in a certain sense similar, ordinal dependence axiom. The latter axiom requires that, if two probability distribution functions (d.f.'s) coincide on an interval, then their values on that interval do not affect the preference order between these distributions. This is very much similar to the Savage sure-thing principle. Certainly, such an axiom is considerably weaker than the independence axiom.

A simple example is $\Psi(p) = p^\beta$. If $\beta = 1$, the subject perceives F as it is, and hence deals with the "usual" expected utility (1.1.2). If $\beta < 1$, the investor overestimates the probability for the wealth to be less that a fixed value: the investor is "security-minded". In the case $\beta > 1$, the investor underestimates the probability mentioned, being "potentially-minded".

A limiting example is a truncation: if for a fixed $q \in [0,1]$

$$\Psi(p) = \begin{cases} p & \text{if } p < 1-q, \\ 1 & \text{if } p \geq 1-q, \end{cases}$$

then

$$U(F) = qu(\gamma_q(F)) + \int_0^{\gamma_q(F)} u(x)dF(x), \tag{1.1.4}$$

where $\gamma_q(F)$ is the $(1-q)$-quantile of F. In this case, the investor does not distinguish values greater than $\gamma_q(F)$ (viewed as too large) and occurring with a probability of q (viewed as too small). One may view it as the existence of a perception threshold. The functional (1.1.4) is not linear and should be distinguished from the naive criterion where truncation is carried out at a fixed, perhaps, big value not depending on F.

If F is a distribution of a r.v. taking only two values, say, a and $b > a$, with probabilities p and $1-p$, respectively, then

$$U(F) = u(a)\Psi(p) + u(b)[1 - \Psi(p)], \tag{1.1.5}$$

and $\Psi(p)$ "transforms" the probability p.

Consider a simple example. Let an investor having, say, the utility function $u(x) = \sqrt{x}$, choose from two future retirement plans: either the annual pension will be equal to $X = \$100,000$, or to $Y = \$50,000$ or $\$200,000$ with equal probabilities. (We do not consider here annuities in dynamics.) For the numbers above, the expected utility criterion leads to a slight preference for the latter plan ($E\{u(X)\} \approx 316$ and $E\{u(Y)\} \approx 335$), which does not look realistic. (At least the author would choose X.) On the other hand, under the criterion (1.1.5), as is easy to calculate, the investor would prefer X if $\Psi(1/2) > 0.59 > 1/2$, which means that such an investor would slightly overestimate the probability of the unlucky event to get $\$50,000$. So, one can expect $\Psi(p)$ to be concave for large p's. Certainly, the above primitive example is given merely for illustration.

1.1.3. *The goal of the paper.* In this paper, we establish conditions on Ψ, under which the optimal policy converges to the MEL-policy as $T \to \infty$. Roughly, these conditions require the tail $1-\Psi(p)$ as $p \to 1$, or/and $\Psi(p)$ as $p \to 0$, to vanish sufficiently fast.

The former condition means that the investor underestimates the probabilities of "too large" values of the income, viewing them as "very small". One may say that the investor *practically does not count on too large values,* considering them "too non-plausible (though very good) for taking them into account".

The smallness of $\Psi(p)$ for "very small" p means that the investor underestimates the probabilities of very small values of the income. So to speak, the investor views "very bad" events as too rare for taking them into account when determining a routine investment strategy.

The particular conditions we state below are non-necessary for the optimal policy to be an asymptotically MEL-policy. However, as will be seen in Section 1.2, in a certain sense, these conditions are close to minimal. The asymptotic optimality of the MEL-policy is understood as follows.

Let $\{\boldsymbol{\pi}_T\}$ denote a sequence of policies where the integer $T \to \infty$. Suppose that for such a sequence, and some $c > 0$

$$m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}_T) \geq c$$

at least for large T, that is, $\boldsymbol{\pi}_T$ is not optimal with respect to (w.r.t.) the MEL-criterion at least for large T. Then, under the conditions established in the next sections, there exists T_0 such that for all $T > T_0$

$$U(F_{T\boldsymbol{\pi}_T}) < U(F_{T\hat{\boldsymbol{\pi}}}). \tag{1.1.6}$$

If the space of policies under consideration is endowed by a metric $\|\cdot\|$ - in the standard framework it may be Euclidean, and if the policy $\hat{\boldsymbol{\pi}}$ is, in a certain sense, unique with respect to this metric (see Section 2.3 for detail), then for the policy $\tilde{\boldsymbol{\pi}}_T$ maximizing $U(F_{T\boldsymbol{\pi}})$, (1.1.6) would imply that $\|\tilde{\boldsymbol{\pi}}_T - \hat{\boldsymbol{\pi}}\| \to 0$, as $T \to \infty$.

It is worth noting that such a type of results is connected with the question of the "survival" of investors with not accurate predictions of future states of the market. The distortion of probabilities under the rank dependent criterion leads to a discrepancy between the investor's beliefs and real probabilities. Nevertheless we see that in the long run and under some conditions, the policy which the investor views as optimal proceeding from her/his beliefs, proves to be optimal with respect to the *real* probability measure also, if optimality is understood in the sense of the maximization of the expected logarithm of the return.

This fact, though being correct asymptotically, in the long run, is relevant to survival since, as was proved for example in Blume and Easley (1992), under some conditions on saving rules, the agents who maximize the expected logarithm occur eventually "most prosperous". See also results in the same direction in De Long, Shleifer, Summers and Waldman (1991), Palomino (1996), and references therein.

This paper does not aim to consider the survival issue in detail; it would require the consideration of a special model with several agents, and comparison of different restrictions on savings, investment decisions, and beliefs. Say, Sandroni (2000) considered the situation when agents with incorrect beliefs are driven out of the market by agents with correct beliefs. Sandroni's paper contains also an interesting discussion and examples. The goal of the remarks above was just to point to a connection between the results below and the survival issue.

The last but important remark concerns the fact that we deal here only with policies not depending on the previous history. This is certainly a restriction but not as serious as it might look: under the i.i.d. assumption and some rather mild additional conditions optimal strategies are asymptotically, for large time horizons, close to stationary strategies of the type mentioned. However, the rigorous proof of this fact (especially when we apply the RDEU criterion) would take long calculations and make the framework and the result much less explicit. For this reason, to make the exposition clearer, we start with stationary strategies from the very beginning: in the asymptotic analysis it does not, in essence, restrict generality.

In this connection, it worthwhile to note that in practically all papers where the difference between the MEL and MEU criteria has been discussed (for example, in basic papers by Merton and Samuelson (1974), and Markowitz (1976)) the choice of policies under consideration was the same and apparently for the same reason.

The goal of this paper is in the introduction and analysis of a new criterion. So, it makes sense probably, at least in the first stage, to do this in the framework of the same model as has been considered before.

1.1.4. *Further historical remarks.* The MEL criterion itself was considered, for example, in Markowitz (1959), Latane (1959) and Breiman (1961). For properties of the MEL-portfolio as applied to bounded utilities see also Goldman (1974). A rather general model was investigated later in Algoet and Cover (1988), see also references therein.

It is worth noting that the maximization of the expected logarithm appears also in the analysis of stochastic analogues of the von Neumann-Gale model. In particular, the only natural stochastic analogue of the von Neumann ray is the balanced path that maximizes the expected logarithm of the growth rate. See, e.g., Arnold, Evstigneev, Gundalach (1994), Evstigneev and Taksar (2001), and references therein.

Different aspects of the application of the MEU criterion to portfolio optimization were considered in a great many of papers; see, e.g., Samuelson (1969), Hakansson (1971), Merton (1973), Breeden (1979); these papers also contain substantial reference lists.

The comparison of the two criteria, and a deep sophisticated discussion may be found in Samuelson (1969, 1971), Goldman (1974), Merton and Samuelson (1974), Markowitz (1976), Ophir (1978, 1979), Latane (1979), Samuelson (1979), and also in Markowitz's remarks following Samuelson (1988).

One can find in the literature some remarks on the relevancy of large deviations to the inconsistency of the MEU and MEL criteria (see, e.g., Latane (1959), Ophir (1978, 1979); Samuelson (1979)), though all these remarks are implicit. To my knowledge, the only paper where the inconsistency of the MEL and MEU criteria has been explicitly connected with large deviations, is the working paper by Kim, Omberg and Russell (1993) [20].

In [20] the authors suggest to divide the whole space of elementary events in two groups: "non-extreme events" which correspond to "moderate" values of the wealth, and extreme events "to be remaining events farther out in the tails". The authors suggest to consider the expected utility only over the "non-extreme outcomes", that is, to truncate integration in (1.1.2) by a "large" number depending on T (the same for all F). Some examples in [20] show that it may lead to the MEL policy.

Next remarks concern the RDEU approach. For binary gambles the RDEU criterion was suggested by Kahneman and Tversky (1979) in their Prospect Theory, the full model was considered in Quiggin (1982), though some earlier Quigin's papers contained some relevant ideas; see references in Quiggin (1993). A special case of RDEU was independently considered in the "dual model" of Yaari (1987) developed further by Roell (1987). To my knowledge, an axiomatic system for the most general case including continuous distributions was considered in Wakker (1993). As was told already, a rather full history of the question and a rich bibliography may be found in monographs Wakker (1989), Quiggin (1993), and Luce (2000), and brief survey of facts concerning the weighting function Ψ - in Rotar (2002). Some terminology and interpretations may be found in Lopes (1987, 1990).

It is worth noting that the general modern RDEU model is more flexible than (1.1.3), and deals not only with probability distributions but with the corresponding events structures as well. The description of this model which coincides with that of (1.1.3) in the case of so called coalescing, may be found in Luce (2000); one of the most recent axiomatic systems - in Marley and Luce (1992). In the present paper we restrict ourselves to (1.1.3), and hence to orders on spaces of probability distributions.

As was told, there are several axiomatic justifications of the criterion (1.1.3); all of them can be found in books Wakker (1989), Quiggin (1993), and Luce (2000). A key axiom is either the trade-off consistency requirement [Wakker (1993)] or the ordinal dependence axiom [Green and Jullien (1988), Quiggin (1989), Segal (1989)].

To the author's knowledge, there are few papers where the RDEU approach are applied to portfolio optimization (for example, Chew, Karni and Safra (1987), Dentcheva and Ruszczynski (2005, 2006); though in general the idea of using RDEU in Economics is not new [see, e.g., Simonsen & Sérgio (1991), Dow and Sérgio (1992), Epstein and Tan Wang (1994), Mukerji & Tallon (1998), Tallon (1998), which does not exhaust the whole possible references.

The rest of the paper is organized as follows. In Section 1.2 we consider a particular example with the power transforming function in the standard geometrical Brownian motion framework. This example shows to what extent we should narrow the class of Ψ's.

The rest results concern the discrete time model as more difficult for analysis. The reader will easily see that similar results are true for the continuous time model too. General results are given in Section 2. Section 3 concerns a truncation criterion; see also comments in the end of Section 2. Proofs are given in Section 4.

1.2. **An example with power transformation functions for a simple continuous time model.** Next we consider a simple example when Ψ is a power function. This is the case of the so called first order reduction of compound gambles; see for a definition and comments, e.g., Luce (2000, p.84). In the context of the modern utility theory, this case is viewed as too simple to be "realistic", but it can serve as a good preliminary illustration of what one can expect in the RDEU framework.

To make an example simpler, we consider here just a power utility function and the standard continuous time scheme with a risk free and one risky securities governed, respectively, by the equations

$$dB_t = rB_t dt, \quad \text{and} \quad dS_t = S_t(mdt + \sigma dZ_t),$$

where Z_t is a Wiener process.

Let W_t be the total wealth at time t, $W_0 = 1$, and θ be the share of the wealth invested into the risky security. We assume θ to be a constant perhaps depending on the time horizon T. As is well known, in this case

$$W_t = \exp\{\mu(\theta)t + \theta\sigma Z_t\}, \tag{1.2.1}$$

where $\mu(\theta) = r + (m-r)\theta - \theta^2\sigma^2/2$. From (1.2.1), one gets the well known optimal θ under the MEL-criterion: $\theta_{\text{MEL}} = (m-r)/\sigma^2$.

Next, we apply the RDEU criterion. We will see that the result should depend on the type of the utility function. Let first $u(x) = x^a,\ 0 < \alpha < 1$.

In this case, large values of W_t matter, so the main property of $\Psi(p)$ should concern its behavior for p close to one. Set $\Psi(p) = 1 - (1-p)^\beta,\ \ \beta \geq 1$. Then for the distribution F_T of W_T

$$U(F_T) = -\int_{-\infty}^{\infty} \exp\left\{\alpha\left(\mu(\theta)T + \theta\sigma\sqrt{T}z\right)\right\} d[1-\Phi(z)]^\beta, \tag{1.2.2}$$

where Φ is the standard normal distribution function. Calculations show that the maximizer of (1.2.2) is

$$\theta = \theta(T) = \frac{m-r}{(1-\alpha/\beta)\sigma^2}\left(1 + (\beta-1)O\left(\frac{1}{\sqrt{T}}\right)\right). \tag{1.2.3}$$

For $\beta = 1$, we naturally get the optimal policy in the Merton's MEU model: $\theta_{\text{MEU}} = (m-r)\,/[(1-\alpha)\sigma^2]$ (see, e.g., Duffie [11], Merton [32]). For $\beta > 1$ and large T, the value θ shifts to θ_{MEL}, and

$$\bar{\theta} := \lim_{T\to\infty} \theta(T) = \frac{m-r}{(1-\alpha/\beta)\sigma^2}.$$

The greater the value of β, the closer $\bar{\theta}$ to θ_{MEL}, and farther from θ_{MEU}. In the limiting case $\beta \to \infty$, one has $\bar{\theta} \to \theta_{\text{MEL}}$. [If $T \to \infty$, and $\beta \to \infty$ simultaneously, the picture is more complicated, depending on which characteristic grows faster.]

Consider now $u(x) = -x^{-\alpha},\ \alpha > 0$. Unlike in the previous case, here small values of W_t matter. Hence now the asymptotics of $\Psi(p)$ for $p \to 0$ is important. Set $\Psi(p) = p^\beta,\ \beta \geq 1$. It is not difficult to calculate that in this case the maximizer of $U(F_T)$ is

$$\theta(T) = \frac{m-r}{(1+\alpha/\beta)\sigma^2}\left(1 + (\beta-1)O\left(\frac{1}{\sqrt{T}}\right)\right), \tag{1.2.4}$$

to which similar comments apply.

Thus, though the power transforming function causes a shift towards the MEL policy, for the optimal policy to converge to the MEL strategy, as $T \to \infty$, the tail of $\Psi(p)$ should vanish faster than a power function. It is lucky that this is the case that - for other reasons - attracted the attention of a number of researchers last years. We use one of these results in the next section.

2. A discrete time model: general results

2.1. **Conditions on Ψ.** Note first that the criterion (1.1.3) is often specified in the literature in terms of the function $w(p) = 1 - \Psi(1-p)$ which is referred to as a *weighting function.* If $u(0) = 0$ (which does not restricts generality if $u(x)$ is bounded from below), integration by parts leads to

$$U(F) = \int_0^\infty w(1-F(x))du(x).$$

In some calculations, the last representation proves to be more convenient.

We make here use of the well known result by Prelec (1998), who established axioms under which the weighting function admits the explicit representation

$$w(p) = w_{\beta\eta}(p) = \exp\left\{-\left[-\beta\ln(p)\right]^{\eta}\right\}, \tag{2.1.1}$$

where β, η are positive parameters. The corresponding transformation function $\Psi_{\beta\eta}(p) = 1 - w_{\beta\eta}(1-p)$.

Luce (2000, 2001) suggested another, and in the author's opinion, simpler system of axioms leading to (2.1.1); see also a discussion and generalizations in Luce (2000).

A detailed discussion of properties of the representation mentioned may be found, e.g., in Luce (2000). Here we just note that if $\eta = 1$, then (2.1.1) directly leads to the power function p^{β}, while for $\eta > 1$ the function $w_{\beta\eta}$ is S-shaped, and $w_{\beta\eta}(p) \to 0$ as $p \to 0$ (and hence $\Psi_{\beta\eta}(p) \to 1$ as $p \to 1$) faster than any power function. In this case, the subject (investor) is "strongly security-minded", and in a rather small degree takes into account possibilities of "lucky" large deviations.

Below we use Prelec's representation. However, since we are concerned only with large deviations, we do *not need to specify the precise shape of* $\Psi(p)$ *but rather its asymptotic behavior* for $p \to 1$*, and/or* $p \to 0$. Namely, we impose

Condition (Ψ): *For some* $\beta > 0$ *and* $\eta > 1$

$$\Psi(p) \le w_{\beta\eta}(p) \qquad \text{for } p \le 1/2, \tag{2.1.2}$$

$$\Psi(p) \ge 1 - w_{\beta\eta}(1-p) \qquad \text{for } p > 1/2. \tag{2.1.3}$$

If, as a matter of fact, (2.1.2) and (2.1.3) hold separately for different parameters β and η, we can choose the "worst", that is, the smallest, β and η. Since, when taking a smaller positive β, we still have a property (Ψ), we can choose β small enough for $w_{\beta\eta}(1/2) \ge 1/2$, which makes (2.1.2) and (2.1.3) consistent.

As will be seen, if the utility function u is bounded from below, (2.1.2) may be replaced by the weaker condition

$$\Psi(p) \le Mp^{a} \qquad \text{for } p \le 1/2, \tag{2.1.4}$$

and some positive M, a. Clearly, for (2.1.4) and (2.1.3) to be consistent, one should have $M2^{-a} \ge 1 - w_{\beta\eta}(1/2)$. Note also that, as is easy to check, Prelec's function $\Psi_{\beta\eta}$ itself satisfies (2.1.4).

If u is bounded from above, (2.1.3) may be replaced by the condition

$$\Psi(p) \ge 1 - M(1-p)^{a} \qquad \text{for } p > 1/2. \tag{2.1.5}$$

and some $M, a > 0$. The similar remarks on consistency apply to this case too.

2.2. **Conditions on** u**.** The goal here is to make these conditions rather weak. This leads to a bit complicated formulations, so it is worth emphasizing that *conditions below are very mild* in the context of the utility maximization, and *exclude just "very bad" utility functions.* In particular, these conditions hold for "classical" functions $u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1,$ or $u(x) = \ln x$, or more generally for functions

$$u(x) = \frac{h(x)x^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1, \tag{2.2.1}$$

where h is a slowly varying function (s.v.f.)[1], or for s.v.f.'s themselves from a very large class of those. More detailed comments are given below.

We could avoid some conditions, if we had restricted ourselves to regularly varying functions but it would have narrowed the setup of the problem too much, and not only from a mathematical point of view: the behavior of human beings is not always regular.

It proves also to be convenient to consider the case $u(\infty) < \infty$ separately, which we do in Section 2.4. Nevertheless the scheme of this section includes some bounded from above functions too.

Clearly, we can consider only non-decreasing and taking at least two values functions u. Then without loss of generality we can assume that $u(x) > 0$ for sufficiently large x. (As usual, here and below, when saying that something is true for large x, we mean that it is true for all x greater than a fixed x_0.)

Condition $(u1)$. *There exist non-negative constants s and C_1 such that*
(a) *for all $y \geq 1$ and sufficiently large x*

$$u(xy) \leq C_1 u(x) y^s; \tag{2.2.2}$$

(b) *for all $y \leq 1$ and sufficiently small* (that is, close to zero) $x > 0$

$$|u(xy)| \leq C_1 |u(x)| / y^s . \tag{2.2.3}$$

If (2.2.2) and (2.2.3) hold separately with different C_1 and s, we can choose the "worst" from these parameters. Next, we discuss (2.2.2) and (2.2.3).

1. If $u(\infty) < \infty$, (2.2.2) holds automatically. The same concerns (2.2.3) if $u(0) > -\infty$.
2. If $u(x) = h(x)x^{\alpha}$ where $\alpha \geq 0$, and h is s.v. as $x \to \infty$, then (2.2.2) holds. For (2.2.3) to hold, it suffices that $u(x) = \bar{h}(x)x^{-\alpha}$ where $\alpha \geq 0$, and $\bar{h}$ is s.v. as $x \to 0$.
3. If $u(0) = 0$ and $u(x)$ is concave, then (2.2.2) is true for all x with $C = s = 1$, while (2.2.3) holds automatically.
4. It is easy to see that, if (2.2.2) holds, $u(x) \leq Cx^s$ for some C and large x, that is, $u(x)$ grows, as $x \to \infty$, not faster than a power function. For instance, the exponential utility function e^x does not satisfy (2.2.2). Similarly, if (2.2.3) holds, $u(x) \geq -Cx^{-s}$ for sufficiently small x.

Conditions (2.2.2) and (2.2.3) require u to grow not too fast and not too irregularly. The next condition requires u to grow not too slow.

Condition $(u2)$. *For any $d > 1$ and any $l > 0$, there exists a constant $C(d,l) > 0$ such that for large x,*

$$\frac{u(x)}{u(x^d)} \leq 1 - \frac{C(d,l)}{x^l}. \tag{2.2.4}$$

[1]A function h is s.v. as $x \to \infty$ if $[h(kx)/h(x)] \to 1$, as $x \to \infty$, for any $k > 0$. Say, $\ln x$, $(\ln x)^s$ for any s, $\ln\ln x$, etc., are s.v.f.'s. Certainly, if $h(x) \to$ *const*, it is s.v. If h is a s.v.f., then $[h(x)/x^{\varepsilon}] \to 0$, as $x \to \infty$ for any $\varepsilon > 0$. Similarly one defines a s.v.f. as $x \to 0$. For details, see, e.g., Seneta (1976) [50], also Ibragimov and Linnik (1971), Rotar (1998).

This condition covers a large class of utility functions but also requires some remarks.

1. Clearly (2.2.4) is true if

$$\limsup_{x\to\infty}\left[u(x)\Big/u(x^d)\right] < 1 \tag{2.2.5}$$

   As to (2.2.5), it obviously holds, say, for all $u(x) = h(x)x^\alpha$ where $\alpha > 0$ and h is s.v., or if, for example, $u(x) \sim \ln^\alpha(1+x)$, $\alpha > 0$, as $x \to \infty$. On the other hand, (2.2.5) excludes u growing "too slowly", say, as $\ln\ln(e+x)$. Since for such functions the theorem below, as a matter of fact, is true, we impose the weaker condition (2.2.4) that covers such functions.
2. Condition (2.2.4) holds for some bounded functions also, as, for example, $u(x) = 1 - 1/\ln(e+x)$, but for instance, for $u(x) = 1 - 1/(1+x)^\alpha$, $\alpha > 0$, (2.2.4) is not true for all l. We consider such functions in Section 2.4 in a bit different terms.
3. Functions that grow as power ones at least for sequences of x's, but do not satisfy (2.2.4), certainly exist but look exotic. Let, say, $u(x_k) = \sqrt{x_k}$ for $x_k = \exp\{4^k\}$, $k = 1, 2, \dots$, and $u(x)$ is constant on $[x_k, x_{k+1})$. Then $\left[u(x_k)\,/u(x_k^2)\right] = 1$ for all k, that is, (2.2.4) is not true for $d = 2$.
4. It is not true however that (2.2.4) holds for any s.v.f. tending to infinity, but examples are rather exotic and we skip them here.

2.3. **The main theorem.** First, we need some conditions on r.v.'s $R_t(\boldsymbol{\pi})$.

Conditions (R). (1) *There exists a policy* $\hat{\boldsymbol{\pi}}$ *maximizing* $m(\boldsymbol{\pi})$, *and* $\hat{m} := m(\hat{\boldsymbol{\pi}}) > 0$.

(2) For $\sigma^2(\boldsymbol{\pi}) := Var\{\ln(1 + R_1(\boldsymbol{\pi}))\}$, it is true that

$$\sup_{\boldsymbol{\pi}} \sigma(\boldsymbol{\pi}) < \infty. \tag{2.3.1}$$

(3) *For a positive* c_0, *and all* $\boldsymbol{\pi}$

$$|\ln(1 + R_1(\boldsymbol{\pi})) - m(\pi)| \le c_0\sigma(\boldsymbol{\pi}), \tag{2.3.2}$$

that is, the normalized r.v. $[\ln(1 + R_1(\boldsymbol{\pi})) - m(\pi)]/\sigma(\boldsymbol{\pi})$ is bounded uniformly in π.

The third requirement is imposed to avoid complicated conditions on the tails of the distributions of $R_t(\boldsymbol{\pi})$. It means, in particular, that the r.v. $\ln(1 + R_1(\boldsymbol{\pi}))$ is bounded, and it excludes policies π for which the variance $\sigma^2(\boldsymbol{\pi})$ is "very small" while the r.v. $\ln(1 + R_1(\boldsymbol{\pi}))$ itself may take "not small" values with "very small" probabilities.

We turn to the first result. When comparing the policy $\hat{\boldsymbol{\pi}}$ with a policy $\boldsymbol{\pi}$, we presuppose that $\boldsymbol{\pi} = \boldsymbol{\pi}_T$, that is, perhaps depends on T. This is the case, for example, if we consider a policy maximizing $U(F_{T\boldsymbol{\pi}})$. When considering a sequence of policies $\{\boldsymbol{\pi}_T\}$ we assume that the integer parameter $T \to \infty$ but perhaps takes on not all sequential natural values, that is, $\{\boldsymbol{\pi}_T\} = \{\boldsymbol{\pi}_{T_1}, \boldsymbol{\pi}_{T_2}, ...\}$ where $\{T_k\}$ is an increasing sequence of integers, and $T_k \to \infty$, as $k \to \infty$.

Theorem 1. *Assume that conditions* (Ψ), (u1), (u2) *and* (R) *hold. Suppose that for a sequence* $\{\boldsymbol{\pi}_T\}$ *and a fixed* $c > 0$ *it is true that* $m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}_T) \geq c$ *at least for large* T*. Then there exists* T_0 *perhaps depending on the sequence* $\{\boldsymbol{\pi}_T\}$ *and such that for all* $T > T_0$

$$U(F_{T\boldsymbol{\pi}_T}) < U(F_{T\hat{\boldsymbol{\pi}}}). \tag{2.3.3}$$

If $u(0) > -\infty$*, condition (2.2.3) in* (u1) *holds automatically, and instead of (2.1.2) in* (Ψ) *it suffices to assume (2.1.4) to be true for some positive* M *and* a.

Let now the space of all possible policies be endowed by a metric $\|\cdot\|$, and $\hat{\boldsymbol{\pi}}$ is unique w.r.t. this metric in the following usual sense.

Condition ($\hat{\boldsymbol{\pi}}$)*: For each* $\delta > 0$ *there exists a positive* ε *depending only on* δ *and such that, if* $\|\hat{\boldsymbol{\pi}} - \pi\| \geq \delta$*, then* $m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}) \geq \varepsilon$.

Clearly, Theorem 1 implies

Corollary 2. *Suppose that all conditions of Theorem 1 plus condition* ($\hat{\boldsymbol{\pi}}$) *hold, and for each* T *there exists a policy* $\tilde{\boldsymbol{\pi}}_T$ *maximizing* $U(F_{T\boldsymbol{\pi}})$*. Then* $\|\tilde{\boldsymbol{\pi}}_T - \hat{\boldsymbol{\pi}}\| \to 0$*, as* $T \to \infty$.

2.4. **The case of** u **bounded from above.** In this case, we slightly change conditions. First, without loss of generality, we assume $u(\infty) = 0$. Next, instead of both conditions ($u1$), we consider

Condition ($u1+$). *There exist non-negative* s *and* C_1 *such that (2.2.3) hold for all* x *and all* $y \leq 1$.

The sense of *(2.2.3)* for small x's has been already discussed. Regarding large x's, note the following.

1. Condition (2.2.3) is true for all x and $y \leq 1$ for strictly negative $u(x) = h(x)/x^{\alpha}$, where $\alpha \geq 0$, and h is s.v.f. for $x \to 0$, and $x \to \infty$, as well.
2. As is easy to see, (2.2.3) implies $u(x) \geq -C/x^s$ for a constant C $(= u(1)/C_1)$, and $x > 1$, that is, $u(x) \to 0$, as $x \to \infty$, not faster than a power function.

The last remark means that the theorem below does not cover bounded functions $u(x)$ converging to $u(\infty)$ too fast; say, the exponential utility function $-e^{-\alpha x}, \alpha > 0$. An analysis of proofs below allows to conjecture that this reflects the essence of the matter, and the corresponding result cannot be proved in the rank-dependent-utility framework.

Next we formulate the following counterpart of (2.2.4).

Condition ($u2+$). *For any* $d > 1$ *and any* l *there exists a constant* $C(d,l)$ *such that for large* x

$$\left|\frac{u(x^d)}{u(x)}\right| \leq 1 - \frac{C(d,l)}{x^l}. \tag{2.4.1}$$

Note that in (2.4.1) we deal with $u(x) \to 0$, as $x \to \infty$. The remarks similar to those following (2.2.4) may apply to this case too.

Theorem 3. *Assume that conditions (u1+), (u2+) and (R) hold. Regarding the function* Ψ *suppose that (2.1.2) for some* $\beta > 0$ *and* $\eta > 1$*, and (2.1.5) for some positive* M *and* a*, are true. Let* $m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}_T) \geq c$ *for a sequence* $\{\boldsymbol{\pi}_T\}$ *and a fixed* $c > 0$ *at least for large* T*. Then there exists* T_0 *such that* $U(F_{T\boldsymbol{\pi}_T}) < U(F_{T\hat{\boldsymbol{\pi}}})$ *for all* $T > T_0$*. If in addition condition* ($\hat{\boldsymbol{\pi}}$) *holds, and there exists a policy* $\tilde{\boldsymbol{\pi}}_T$ *maximizing* $U(F_{T\boldsymbol{\pi}})$*, then* $\|\tilde{\boldsymbol{\pi}}_T - \hat{\boldsymbol{\pi}}\| \to 0$*, as* $T \to \infty$*.*

The above general results should be viewed rather as qualitative. In concrete situations, even if we accept the hypothesis that the investor assigns different weights to different probabilities, it could be still a problem to figure out the particular weighting function of the investor. One can hope that in future some models for a "typical" investor will be elaborated - and Prelec's representation is a step in this direction, but for now this topic is not so developed. In the light of this, the simple but explicit truncation criterion could occur to be useful. In the next section we consider it in detail. It will allow to remove or essentially weaken some conditions, and to consider a quantitative estimate for the time T_0.

3. On the truncation criterion

To include into consideration a larger class of utility function we consider truncation from both sides. More specifically, we fix $q \in [0,1]$, and consider the criterion

$$U(F) = q[u(\gamma_{q+}(F)) + u(\gamma_{q-}(F))] + \int_{\gamma_{q-}(F)}^{\gamma_{q+}(F)} u(x)dF(x), \tag{3.1}$$

where $\gamma_{q-}(F)$ and $\gamma_{q+}(F)$ are q- and $(1-q)$-quantiles of F, respectively. As was noted in Section 1.1.2, this is a limiting case of the rank dependent utility: the investor does not distinguish "too large" values occurring with a small probability of q, as well as "too small values" occurring with the same probability. Truncation in the areas of large and small values should be interpreted differently.

If an investor does not distinguish too large values of gains, she/he exhibits a threshold of perception in the zone of "lucky" events, demonstrating a cautious behavior. On the other hand, if the investor does not distinguish too small values of the wealth, that is, too large values of losses, she has a threshold of perception in the area of ruin. It reflects the well known phenomenon when people exhibit different types of behavior depending on whether they deal with gains or losses (see, e.g., Luce (2000)).

By the significance of q, it is small, so we can assume $q \leq 1/4$.

By virtue of (2.3.1),

$$W_{T\boldsymbol{\pi}} = \exp\left\{m(\boldsymbol{\pi})T + \sigma(\boldsymbol{\pi})\sqrt{T} \cdot \xi_{T\boldsymbol{\pi}}\right\}, \tag{3.2}$$

where

$$\xi_{T\boldsymbol{\pi}} = \frac{\ln W_{T\boldsymbol{\pi}} - m(\boldsymbol{\pi})T}{\sigma(\boldsymbol{\pi})\sqrt{T}} = \frac{\sum_{t=1}^{T}[\ln(1+R_t(\boldsymbol{\pi})) - m(\boldsymbol{\pi})]}{\sigma(\boldsymbol{\pi})\sqrt{T}}.$$

The last r.v. is asymptotically normal for each $\boldsymbol{\pi}$ such that $\sigma(\boldsymbol{\pi}) \neq 0$. If $\sigma(\boldsymbol{\pi}) = 0$, we define $\xi_{T\boldsymbol{\pi}}$ as a standard normal r.v.; it will not cause a misunderstanding below.

For each pair of distributions F and G, set $\|F-G\|_\infty = \sup_x |F(x)-G(x)|$. Let, as before, $F_{T\boldsymbol{\pi}}(x) = P(W_{T\boldsymbol{\pi}} \le x)$, $F^*_{T\boldsymbol{\pi}}(x) = P(\xi_T \le x)$ and $\Delta_T(\boldsymbol{\pi}) = \|F^*_{T\boldsymbol{\pi}} - \Phi\|_\infty$. Clearly, $\Delta_T(\boldsymbol{\pi}) \to 0$, as $T \to \infty$, for each $\boldsymbol{\pi}$.

Condition (*UNA: Uniform Normal Approximation*):

$$\sup_{\boldsymbol{\pi}} \Delta_T(\boldsymbol{\pi}) \to 0, \quad \text{as} \quad T \to \infty. \tag{3.3}$$

It is a rather weak condition. For example, by the Berry-Esseen theorem (see, e.g., [14]) if $\sigma(\boldsymbol{\pi}) \neq 0$, then

$$\Delta_T(\boldsymbol{\pi}) \le \frac{\beta(\boldsymbol{\pi})}{\sigma^3(\boldsymbol{\pi})\sqrt{T}}, \tag{3.4}$$

where $\beta(\boldsymbol{\pi}) = E\{|\ln R_1(\boldsymbol{\pi}) - m(\boldsymbol{\pi})|^3$. Hence, condition (3.3) holds, say, if the supremum of the Lyapunov ratios,

$$L = \sup_{\boldsymbol{\pi}:\sigma(\boldsymbol{\pi})\neq 0} (\beta(\boldsymbol{\pi})/\sigma^3(\boldsymbol{\pi})) < \infty. \tag{3.5}$$

Theorem 4. *Let the policy $\hat{\boldsymbol{\pi}}$ exist, and $m(\hat{\boldsymbol{\pi}}) > 0$. Let $m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}_T) \ge c$ for a sequence $\{\boldsymbol{\pi}_T\}$ and some fixed $c > 0$. Then, in the case of the criterion (3.1), $U(F_{T\boldsymbol{\pi}_T}) < U(F_{T\hat{\boldsymbol{\pi}}})$ for T greater than some T_0 if the following conditions hold: (2.3.1), UNA, and either (2.2.4) if $u(\infty) = \infty$, or (2.4.1) if $u(\infty) = 0$.*

If we deal just with (3.1), we can estimate the above threshold time T_0. To make the expressions below simpler, instead of (3.3), we consider condition (3.5) - the latter implies the former, and instead of (2.2.4) or (2.4.1) - a simpler though slightly stronger

Condition ($u <$). *There exist a positive k_u such that for $k > k_u$*

$$\tilde{u}(k) := \sup_{x\ge 1} \frac{u(x)}{u(kx)} < \frac{1}{4}, \tag{3.6}$$

and

$$\tilde{u}(k) := \sup_{x\ge 1} \left|\frac{u(kx)}{u(x)}\right| < \frac{1}{4}, \quad \text{if } u(\infty) = 0. \tag{3.7}$$

We write $1/4$ just for simplicity; any number less than $1/2$ can be chosen. Clearly, any function (2.2.1) satisfies (3.7).

Let $\varphi(x)$ be the standard normal density, and

$$C_q = 1/\varphi(\gamma_q(\Phi) + 1) = \sqrt{2\pi}e^{(\gamma_q(\Phi)+1)^2/2}. \tag{3.8}$$

Theorem 5. *Consider the criterion (3.1), and assume conditions (σ), (3.5), and ($u <$) to hold. Then $U(F_{T\boldsymbol{\pi}}) \le U(F_{T\hat{\boldsymbol{\pi}}})$ for all*

$$T > T_0 = \Gamma_1 + \frac{\Gamma_2}{m(\hat{\boldsymbol{\pi}})} + \frac{\Gamma_3}{m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi})} + \frac{\Gamma_4}{(m(\hat{\boldsymbol{\pi}}) - m(\boldsymbol{\pi}))^2}, \tag{3.9}$$

where

$$\Gamma_1 = 4L^2(C_q^2 + 4), \ \Gamma_2 = \ln k_u, \ \Gamma_3 = 2\ln k_u,$$

$$\Gamma_4 = 4(\gamma_q(\Phi) + 1)^2\bar{\sigma}^2, \ \textit{and } \bar{\sigma} = \sup_{\boldsymbol{\pi}} \sigma(\boldsymbol{\pi}).$$

The above values of Γ's are just rough estimates. More accurate estimation may lead to much more precise though more cumbersome expressions for Γ's.

4. Proofs

4.1. **Proof of Theorem 1.** Below, we will consider only sequences of policies $\{\boldsymbol{\pi}_T\}$ for which $\inf_T m(\boldsymbol{\pi}_T) > 0$ and $\inf_T \sigma(\boldsymbol{\pi}_T) > 0$; otherwise calculations are much simpler.

We make use of the exponential bounds for large deviations (see, e.g., [14], [23], [43]). For the r.v. $\xi_{T\boldsymbol{\pi}}$ under consideration it may be formulated as follows: for all $\boldsymbol{\pi}$

$$P(\xi_{T\boldsymbol{\pi}} > x) \le \exp\left\{-\frac{x^2}{2}\left(1 - \frac{xc_0}{2\sqrt{T}}\right)\right\} \quad \text{for} \quad 0 \le x \le \sqrt{T}/c_0,$$

$$P(\xi_{T\pi} > x) \le \exp\left\{-\frac{x\sqrt{T}}{4c_0}\right\} \quad \text{for} \quad x \ge \sqrt{T}/c_0,$$

where c_0 is the constant from (2.3.2). The same bounds are true for $P(\xi_T < -x),\ \ x > 0$.

We simplify this as

$$P(\xi_{T\pi} > x) \le g_T(x), \qquad P(\xi_{T\pi} < -x) \le g_T(x), \tag{4.1.1}$$

where x is arbitrary, and

$$g_T(x) = \begin{cases} 1 & \text{for } x < 0, \\ \exp\{-x^2/4\} & \text{for } 0 \le x \le \sqrt{T}/c_0, \\ \exp\{-x\sqrt{T}/4c_0\} & \text{for } x \ge \sqrt{T}/c_0. \end{cases} \tag{4.1.2}$$

Set, as before, $F_{T\boldsymbol{\pi}}(x) = P(W_{T\boldsymbol{\pi}} \le x)$, and $F^*_{T\boldsymbol{\pi}}(x) = P(\xi_{T\boldsymbol{\pi}} \le x)$. If it cannot cause a misunderstanding, we omit sometimes the index T in $\boldsymbol{\pi}_T$, and the index $\boldsymbol{\pi}$ in $\xi_{T\boldsymbol{\pi}}$, $F_{T\boldsymbol{\pi}}$ and $F^*_{T\boldsymbol{\pi}}$, and write m and σ instead of $m(\boldsymbol{\pi})$ and $\sigma(\boldsymbol{\pi})$.

As usual, saying below that something, for example, an inequality, is true for large T we mean that it is true for all T greater or equal than some fixed T_1. Once it has been said, for the rest of the proof we consider only $T \ge T_1$. It means, in particular, that if, say, another inequality is true for $T \ge$ some T_2, than we consider $T \ge \max\{T_1, T_2\}$. We will not repeat it each time. The numbers T_1, T_2, etc., mentioned depend perhaps on parameters of the problem: c_0, C_1, M, a, etc.

Let c be a continuity point of $\Psi(F^*_T(x))$. Then,

$$\begin{aligned} U(F_T) &= \int_0^\infty u(x)d\Psi(F_T(x)) \\ &= \int_{-\infty}^\infty u(e^{mT}e^{\sigma\sqrt{T}y})d\Psi(F^*_T(y)) \\ &= \int_{-\infty}^{c-0} u(e^{mT}e^{\sigma\sqrt{T}y})d\Psi(F^*_{T\boldsymbol{\pi}}(y)) - \int_c^\infty u(e^{mT}e^{\sigma\sqrt{T}y})d[1 - \Psi(F^*_T(y))] \end{aligned}$$

$$= \int_{-\infty}^{c} u(e^{mT}e^{\sigma\sqrt{T}y})d\Psi(F^*_{T\boldsymbol{\pi}}(y)) - \int_{c}^{\infty} u(e^{mT}e^{\sigma\sqrt{T}y})d[1-\Psi(F^*_T(y))]$$

$$= u(e^{mT}e^{\sigma\sqrt{T}c}) + \int_{c}^{\infty}[1-\Psi(F^*_T(y))]du(e^{mT}e^{\sigma\sqrt{T}y}) \tag{4.1.3}$$

$$- \int_{-\infty}^{c}\Psi(F^*_T(y))du(e^{mT}e^{\sigma\sqrt{T}y}).$$

Hence, by (4.1.1) and since u is non-decreasing, for $c \geq 0$,

$$U(F_T) \leq u(e^{mT}e^{\sigma\sqrt{T}c}) + \int_{c}^{\infty}[1-\Psi(1-g_T(y))]du(e^{mT}e^{\sigma\sqrt{T}y}). \tag{4.1.4}$$

Let $\tilde{\Psi}(p)$ be any non-decreasing and continuous at zero function such that $\tilde{\Psi}(0) = 0, \tilde{\Psi}(1) = 1$, and $\tilde{\Psi}(p) \leq \Psi(p)$ for all p. Then (4.1.4) implies that

$$\begin{aligned} U(F_T) &\leq u(e^{mT}e^{\sigma\sqrt{T}c}) + \int_{c}^{\infty}[1-\tilde{\Psi}(1-g_T(y))]du(e^{mT}e^{\sigma\sqrt{T}y}) \\ &= u(e^{mT}e^{\sigma\sqrt{T}c})\tilde{\Psi}(1-g_T(c)) + \int_{c}^{\infty} u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(1-g_T(y)). \end{aligned}$$

Since c can be chosen arbitrary close to zero and $\tilde{\Psi}(p)$ is continuous at zero, eventually

$$U(F_T) \leq \int_{0}^{\infty} u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(1-g_T(y)). \tag{4.1.5}$$

Recall that in the case of Theorem 1 we assume that $u(x) > 0$ for sufficiently large x. Consequently, since we consider only the case $\inf_T m(\pi_T) > 0$, for sufficiently large T all values of $u(e^{mT}e^{\sigma\sqrt{T}y})$ in the r.-h.s. of (4.1.5) are positive for all y. Set

$$k_1 = k_1(T) = 2\beta^{-1/2}T^{1/2\eta}. \tag{4.1.6}$$

Since $\eta > 1$, there exists $T_1 = T_1(\beta,\eta)$ such that for $T \geq T_1$ one has $k_1(T) \leq \sqrt{T}/c_0$. So, for $T \geq \max\{T_1, (\beta \ln 2)^\eta\}$

$$g_T(k_1) = \exp\{-k_1^2/4\} = \exp\{-T^{1/\eta}/\beta\} \leq 1/2. \tag{4.1.7}$$

Then we can use (2.1.3) and set $\tilde{\Psi}(p) = 1 - w_{\beta\eta}(1-p)$ for $p \geq 1/2$, and $\tilde{\Psi}(p) = \min\{\Psi(p), 1 - w_{\beta\eta}(1-p)\}$ for $p < 1/2$. The $\tilde{\Psi}$ so chosen is continuous at zero. By (4.1.5)

$$U(F_T) \leq \int_{0}^{\infty} u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(1-g_T(y)) = \int_{0}^{k_1} + \int_{k_1}^{\infty}$$

$$= J_1 + J_2. \tag{4.1.8}$$

For J_1 it suffices to write

$$J_1 \leq u(\exp\{mT + k_1\sigma\sqrt{T}\}) = u(\exp\{mT + 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\}), \tag{4.1.9}$$

where the r.-h.s. is positive for large T.

For large T we have also $1 - g_T(k_1(T)) > 1/2$, and hence $\tilde{\Psi}(1 - g_T(y)) = 1 - w_{\beta\eta}(g_T(y))$ for $y \geq k_1(T)$. Next we use (2.2.2) which is true for $x >$ some x_1. Since $\exp\{mT + k_1(T)\sigma\sqrt{T}\} > x_1$ for large T, we can use (2.2.2) when estimating J_2. Consequently, for sufficiently large T

$$\begin{aligned} J_2 &= \int_{k_1}^{\infty} u(e^{mT} e^{\sigma\sqrt{T}y}) d[-w_{\beta\eta}(g_T(y))] \\ &\leq C_1 u(e^{mT}) I, \end{aligned} \tag{4.1.10}$$

where

$$I = I(T) = \int_{k_1(T)}^{\infty} \exp\{s\sigma\sqrt{T}y\} d[-w_{\beta\eta}(g_T(y))]. \tag{4.1.11}$$

As was told already, $k_1(T) \leq \sqrt{T}/c_0$ for large T, and hence we can write

$$I = \int_{k_1}^{\infty} = \int_{k_1}^{\sqrt{T}/c_0} + \int_{\sqrt{T}/c_0}^{\infty} = I_1 + I_2.$$

Furthermore,

$$\begin{aligned} I_1 &= I_1(T) = \int_{k_1(T)}^{\sqrt{T}/c_0} \exp\{s\sigma\sqrt{T}y\} d[-w_{\beta\eta}(g_T(y))] \\ &= \int_{k_1(T)}^{\sqrt{T}/c_0} \exp\left\{s\sigma\sqrt{T}y\right\} d\left[-\exp\left\{-\left[\beta y^2/4\right]^{\eta}\right\}\right] \\ &= \int_{\sqrt{\beta}k_1(T)/2}^{\sqrt{\beta}\sqrt{T}/2c_0} \exp\left\{2\beta^{-1/2}s\sigma\sqrt{T}y\right\} d\left[-\exp\left\{-y^{2\eta}\right\}\right] \\ &\leq \int_{\sqrt{\beta}k_1(T)/2}^{\infty} \exp\left\{2\beta^{-1/2}s\sigma\sqrt{T}y\right\} d\left[-\exp\left\{-y^{2\eta}\right\}\right]. \end{aligned} \tag{4.1.12}$$

Thus, we need an estimate for integrals of the type

$$\begin{aligned}\int_R^\infty x^l \exp\{tx - x^{2\eta}\}dx &\le \int_R^\infty x^l \exp\{tx - x^2 R^{2\eta-2}\}dx \\ &= \frac{1}{R^{(\eta-1)(l+1)}} \int_{R^\eta}^\infty y^l \exp\{(t/R^{\eta-1})y - y^2\}dx \\ &\le \frac{C(l)}{R^{(\eta-1)(l+1)}} \int_{R^\eta}^\infty \exp\{(t/R^{\eta-1})y - y^2/2\}dx \\ &= \frac{C(l)}{R^{(\eta-1)(l+1)}} \exp\left\{\frac{1}{2}\left(t/R^{\eta-1}\right)^2\right\} \int_{R^\eta}^\infty \exp\{-(y-(t/R^{\eta-1}))^2/2\}dx \\ &\le \frac{C(l)}{R^{(\eta-1)(l+1)}} \exp\left\{\frac{1}{2}\left(t/R^{\eta-1}\right)^2\right\} [1-\Phi(R^\eta - (t/R^{\eta-1}))].\end{aligned}$$

where $C(\cdot)$ as usual, denotes a constant depending only on the argument in $(\cdot)$, and which may be *different in different formulas*. Applying the last inequality to (4.1.12), we have

$$\begin{aligned} I_1(T) &\le \frac{C(\eta)(2\beta^{-1/2})^{(\eta-1)2\eta}}{k_1^{(\eta-1)2\eta}(T)} \exp\left\{\frac{1}{2}\left((2/\sqrt{\beta})^\eta s\sigma\sqrt{T}/k_1^{\eta-1}\right)^2\right\} \\ &\quad \times \left[1-\Phi\left((\sqrt{\beta}k_1/2)^\eta - (2/\sqrt{\beta})^\eta s\sigma\sqrt{T}/k_1^{\eta-1}\right)\right] \\ &\le \frac{C(\eta)}{T^{(\eta-1)}} \exp\left\{2\beta^{-1}s^2\sigma^2 T^{1/\eta}\right\}\left[1-\Phi\left(\sqrt{T} - 2\beta^{-1/2}s\sigma T^{1/2\eta}\right)\right]. \end{aligned} \tag{4.1.13}$$

It suffices now to use that

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) \le 1 - \Phi(x) = \Phi(-x) \le \frac{1}{x}\varphi(x), \quad \text{for} \quad x > 0, \tag{4.1.14}$$

where $\varphi(x) = \Phi'(x)$ (see, e.g., [14], [23], [43]). Set $\bar{\sigma} = \sup_{\boldsymbol{\pi}} \sigma(\boldsymbol{\pi}) < \infty$ in view of (2.3.1). It is straightforward to derive from (4.1.13) and (4.1.14) that, if $\eta > 1$, then there exists $T_2 = T_2(\bar{\sigma}, C_1, \beta, \eta, s)$ such that for $T \ge T_2$

$$I_1(T) \le e^{-T/4}. \tag{4.1.15}$$

Furthermore,

$$\begin{aligned} I_2(T) &= \int_{\sqrt{T}/c_0}^\infty \exp\{s\sigma\sqrt{T}y\} d[-w_{\beta\eta}(\exp\{-y\sqrt{T}/4c_0\})] \\ &= \int_{\sqrt{T}/c_0}^\infty \exp\left\{s\sigma\sqrt{T}y\right\} d\left[-\exp\left\{-\left[\frac{\beta y\sqrt{T}}{4c_0}\right]^\eta\right\}\right] \end{aligned}$$

$$(4.1.16) \qquad = \int_{\beta T/4c_0^2}^{\infty} \exp\left\{(4c_0\beta^{-1}s\sigma)z\right\} d\left[-\exp\left\{-z^{\eta}\right\}\right].$$

We see that there exists $T_3 = T_3(c_0, C_1, \beta, \eta, s)$ such that for all $T \geq T_3$

$$(4.1.17) \qquad I_2 \leq e^{-T}.$$

From (4.1.15) and (4.1.17) it follows that for large T

$$(4.1.18) \qquad I(T) \leq e^{-T/4} + e^{-T} \leq 2e^{-T/4},$$

and hence

$$(4.1.19) \qquad J_2 \leq C_1 u(e^{mT})(e^{-T/4} + e^{-T}) \leq u(e^{mT})e^{-T/5}$$

for sufficiently large T.

Collecting (4.1.8), (4.1.9), and (4.1.19), we obtain that for sufficiently large T

$$\begin{aligned} U(F_T) &\leq u(\exp\{mT + 2\bar{\sigma}\beta^{-1/2}T^{(1+\eta)/2\eta}\}) + u(e^{mT})e^{-T/5} \\ (4.1.20) \qquad &\leq u(\exp\{mT + 2\bar{\sigma}\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1 + e^{-T/5}], \end{aligned}$$

where the r.-h.s. is positive.

We turn to lower bounds for $U(F_T)$. Let c be a continuity point of $\Psi(F_T^*(y))$, and $c \leq 0$. Then, by (4.1.4),

$$\begin{aligned} U(F_T) &\geq u(e^{mT}e^{\sigma\sqrt{T}c}) - \int_{-\infty}^{c} \Psi(F_T^*(y))du(e^{mT}e^{\sigma\sqrt{T}y}) \\ (4.1.21) \qquad &\geq u(e^{mT}e^{\sigma\sqrt{T}c}) - \int_{-\infty}^{c} \Psi(g_T(-y))du(e^{mT}e^{\sigma\sqrt{T}y}). \end{aligned}$$

Let now $\tilde{\Psi}(p)$ be any non-decreasing and continuous at $p = 1$ function such that $\tilde{\Psi}(0) = 0, \tilde{\Psi}(1) = 1$, and $\tilde{\Psi}(p) \geq \Psi(p)$ for all p. Then it follows from (4.1.21) that

$$\begin{aligned} U(F_T) &\geq u(e^{mT}e^{\sigma\sqrt{T}c}) - \int_{-\infty}^{c} \tilde{\Psi}(g_T(-y))du(e^{mT}e^{\sigma\sqrt{T}y}) \\ &= u(e^{mT}e^{\sigma\sqrt{T}c})(1 - \tilde{\Psi}(g_T(-c))) + \int_{-\infty}^{c} u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(g_T(-y)) \\ &= u(e^{mT}e^{\sigma\sqrt{T}c})(1 - \tilde{\Psi}(g_T(-c))) + \int_{|c|}^{\infty} u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))]. \end{aligned}$$

Since c can be chosen arbitrary close to zero and $\tilde{\Psi}(p)$ is continuous at one,

$$(4.1.22) \qquad U(F_T) \geq \int_{0}^{\infty} u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))]$$

Let now x_0 be the number such that $u(x) \leq 0$ for $x < x_0$, and $u(x) > 0$ for $x > x_0$. By condition $(u1)$ there exists $x_2 \leq x_0$ such that (2.2.3) holds for $x \leq x_2$. Let $y_0 = y_0(T) = m\sqrt{T}\sigma^{-1} - (\ln x_0)(\sigma\sqrt{T})^{-1}$, $y_2 = y_2(T) = m\sqrt{T}\sigma^{-1} - (\ln x_2)(\sigma\sqrt{T})^{-1}$. Clearly, $k_1(T) < y_0(T) \leq y_2(T)$ for large T.

Set $\tilde{\Psi}(p) = w_{\beta\eta}(p)$ for $p \leq 1/2$, and $\tilde{\Psi}(p) = \max\{w_{\beta\eta}(p), \Psi(p)\}$ otherwise. Note that $\tilde{\Psi}(p)$ is continuous at $p = 1$. Furthermore

$$\begin{aligned} U(F_T) &\geq \int_0^\infty u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))] = \int_0^{k_1} + \int_{k_1}^{y_0} + \int_{y_0}^{y_2} + \int_{y_2}^\infty \\ &= J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{4.1.23}$$

Considering T large enough for $g_T(k_1(T)) \leq 1/2$, we get that

$$\begin{aligned} J_3 &\geq u(\exp\{mT - k_1(T)\sigma\sqrt{T}\})[1 - \tilde{\Psi}(g_T(k_1(T)))] \\ &= u(\exp\{mT - k_2(T)\sigma\sqrt{T}\})[1 - w_{\beta\eta}(g_T(k_1(T)))] \\ &= u(\exp\{mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1 - e^{-T}], \end{aligned} \tag{4.1.24}$$

where the r.-h.s. is positive for large T.

It suffices to write $J_4 \geq 0$. The next integral

$$\begin{aligned} J_5 &\geq -|u(x_2)|\tilde{\Psi}(g_T(y_0)) \geq -|u(x_2)|w_{\beta\eta}(g_T(y_0)) \\ &\geq -|u(x_2)|w_{\beta\eta}(g_T(k_1)) = -|u(x_2)|e^{-T}. \end{aligned} \tag{4.1.25}$$

Furthermore, for $y \geq y_2$ we can use (2.2.3) in the following way

$$\begin{aligned} J_6 &= \int_{y_2}^\infty u(e^{mT}e^{-\sigma\sqrt{T}y})d[-w_{\beta\eta}(g_T(y))] \\ &= \int_{y_2}^\infty u(x_2\exp\{-\sigma\sqrt{T}(y - y_2)\})d[-w_{\beta\eta}(g_T(y))] \\ &\geq \int_{y_2}^\infty u(x_2\exp\{-\sigma\sqrt{T}y\})d[-w_{\beta\eta}(g_T(y))] \\ &\geq C_1u(x_2)\int_{y_2}^\infty \exp\{s\sigma\sqrt{T}y\})d[-w_{\beta\eta}(g_T(y))] \\ &\geq -C_1|u(x_2)|\int_{k_1}^\infty \exp\{s\sigma\sqrt{T}y\})d[-w_{\beta\eta}(g_T(y))] \\ &\geq -2C_1|u(x_2)|e^{-T/4} \end{aligned} \tag{4.1.26}$$

for large T in view of (4.1.18).

Since there exists a constant $C_2 > 0$, perhaps depending on parameters of the problem, such that $u(mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}) \geq C_2$ for large T, (4.1.23), (4.1.24), (4.1.25), and (4.1.26) imply for large T that

$$\begin{aligned} U(F_T) &\geq u(\exp\{mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1 - e^{-T} - C_2^{-1}|u(x_2)|(e^{-T} + 2C_1e^{-T/4})] \\ &\geq u(\exp\{mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1 - e^{-T/5}], \end{aligned} \tag{4.1.27}$$

where the r.-h.s. is positive.

Turning to the last step of the proof, we fix a policy $\boldsymbol{\pi}$ and write F_T, m, σ for $F_{T\boldsymbol{\pi}}, m(\boldsymbol{\pi}), \sigma(\pi)$, respectively. Symbols $\hat{F}_T, \hat{m}, \hat{\sigma}$ will correspond to the policy $\hat{\boldsymbol{\pi}}$.

Let $\partial m := \hat{m} - m \geq c > 0$. Combining (4.1.20) and (4.1.27), taking into account that $\eta > 1$, and making use of (2.2.4), we have for large T

$$\begin{aligned} \frac{U(F_T)}{U(\hat{F}_T)} &\leq \frac{u(\exp\{mT + 2\bar{\sigma}\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1+e^{-T/5}]}{u(\exp\{\hat{m}T - 2\hat{\sigma}\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1-e^{-T/5}]} \\ &\leq \frac{u(\exp\{(m+\partial m/4)T)\})[1+e^{-T/5}]}{u(\exp\{(\hat{m}-\partial m/4)T)\})[1-e^{-T/5}]} \\ &\leq [1 - C(l,d)\exp\{-l(m+\partial m/4)T\}] \cdot \left[1+3e^{-T/5}\right], \end{aligned} \tag{4.1.28}$$

where $d = \frac{\hat{m}-\partial m/4}{m+\partial m/4} \geq \frac{4\hat{m}-c}{4\hat{m}-3c} > 1$. (The denominator is positive since $\hat{m} > m \geq 0$.) Since the positive l can be arbitrary small, we can set, say, $l = \frac{1}{6(m+\partial m/4)} \geq \frac{1}{6\hat{m}} > 0$. Note also that without loss of generality we can assume $C(l,d)$ increasing in both arguments. Similarly, larger l and/or d, larger the set of x's for which (2.2.4) is true. Hence, as is now easy to see, the r.-h. side of (4.1.28) is less than one for large T.

It remains to consider the case when $u(0) > -\infty$, which means that we can set $u(0) = 0$. In this case we keep the bound (4.1.20) as it is, and for a lower bound we can appeal to condition (2.1.4). Set $k_2 = k_2(T) = 2\sqrt{(\nu/a)T}$, where a is a parameter from (2.1.4), and a positive fixed $\nu \leq a/4c_0^2$ will be specified later. Set $\tilde{\Psi}(p) = Mp^a$ for $p \leq 1/2$, and $\tilde{\Psi}(p) = 1$ otherwise. (Note that in (2.1.4) we can consider $M2^{-a} \leq 1$.)

Then

$$\begin{aligned} U(F_T) &\geq \int_0^\infty u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))] \geq \int_0^{k_2} u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))] \\ &\geq u(\exp\{mT - 2(\nu/a)^{1/2}\sigma T\})[1 - \tilde{\Psi}(g_T(k_2))]. \end{aligned}$$

Since $k_2(T) \leq \sqrt{T}/c_0$, and $g_T(k_2(T)) \leq 1/2$ for large T, by (2.1.4) and (4.1.2) $\tilde{\Psi}(g_T(k_2(T))) = M\exp\{-\nu T\}$. So, for large T

$$U(F_T) \geq u(\exp\{(m - 2(\nu/a)^{1/2}\sigma)T\})[1 - M\exp\{-\nu T\}], \tag{4.1.29}$$

where the r.-h.s. is positive. Thus, (4.1.20) and (4.1.29) imply that

$$\frac{U(F_T)}{U(\hat{F}_T)} \leq \frac{u(\exp\{mT + 2\bar{\sigma}\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1+e^{-T/5}]}{u(\exp\{(\hat{m} - 2(\nu/a)^{1/2}\hat{\sigma})T\})[1 - M\exp\{-\nu T\}]}. \tag{4.1.30}$$

Let $\nu = \min\{(a/4c_0^2), a(\partial m)^2/64\hat{\sigma}^2\}$. Then, since $\eta > 1$, for large T

$$\frac{U(F_T)}{U(\hat{F}_T)} \leq \frac{u(\exp\{(m+\partial m/4)T\})}{u(\exp\{(\hat{m}-\partial m/4)T\})} \cdot \frac{1+e^{-T/5}}{1 - M\exp\{-\nu T\}}. \tag{4.1.31}$$

Proceeding similar to (4.1.28) and applying (2.2.4) it is easy to show now that the r.-h.s. of (4.1.31) is less than one for large T.

The proof is complete. □

4.2. **Proof of Theorem 3.** The proof repeats the previous proof with the following exceptions. First, we set $u(\infty) = 0$. Second, the upper bound for $U(F_T)$ can be provided in the following way. Let $k_2 = k_2(T)$ be defined as above, and $\tilde{\Psi} = 1 - M(1-p)^a$ for $p > 1/2$, and $\Psi(p) = 0$ otherwise. Then, since $u(x) \le 0$,

$$\begin{aligned} U(F_T) &\le \int_0^\infty u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(1-g_T(y)) \le \int_0^{k_2} u(e^{mT}e^{\sigma\sqrt{T}y})d\tilde{\Psi}(1-g_T(y)) \\ &\le u(\exp\{mT + 2(\nu/a)^{1/2}\sigma T\})\tilde{\Psi}(1-g_T(k_2(T))). \end{aligned}$$

Similar to what we did before we get that $\tilde{\Psi}(1-g_T(k_2(T))) = 1 - M\exp\{-\nu T\}$ for large T and

$$U(F_T) \le u(\exp\{(m + 2(\nu/a)^{1/2}\sigma)T\})[1 - M\exp\{-\nu T\}], \tag{4.2.1}$$

where the r.-h.s. is negative.

Consider a lower bound. In this case we set $\tilde{\Psi}(p) = w_{\beta\eta}(p)$ for $p \le 1/2$, and $\tilde{\Psi}(p) = \max\{w_{\beta\eta}(p), \Psi(p)\}$. For the same $k_1 = k_1(T)$ as above

$$U(F_T) \ge \int_0^\infty u(e^{mT}e^{-\sigma\sqrt{T}y})d[-\tilde{\Psi}(g_T(y))] = \int_0^{k_1} + \int_{k_1}^{\infty} = J_7 + J_8. \tag{4.2.2}$$

First,

$$J_7 \ge u(\exp\{mT - k_1(T)\sigma\sqrt{T}\}) = u(\exp\{mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\}), \tag{4.2.3}$$

where the l.-h.s. is negative.

For J_8 we have to use a condition on $u(x)$ of the type (2.2.3) for all x. Taking into account (4.1.7), we have for large T

$$\begin{aligned} J_8 &= \int_{k_1}^\infty u(e^{mT}e^{s\sigma\sqrt{T}y})d[-w_{\beta\eta}(g_T(y))] \\ &\ge -C_1|u(e^{mT})| \int_{k_1}^\infty e^{s\sigma\sqrt{T}y}d[-w_{\beta\eta}(g_T(y))] \\ &= -C_1|u(e^{mT})|I, \end{aligned}$$

where I is the integral (4.1.11). So, in virtue of (4.1.18), for sufficiently large T

$$U(F_T) \ge u(\exp\{mT - 2\sigma\beta^{-1/2}T^{(1+\eta)/2\eta}\})[1 + e^{-T/5}], \tag{4.2.4}$$

where the r.-h.s. is negative.

The rest of the proof is similar to what has been done before. We should only take into account that $U(F_T)$ is now negative. □

Because of the lack of room we skip here simpler proofs of Theorems 4 and 5.

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