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INDETERMINACY IN AGGREGATE MODELS WITH SMALL EXTERNALITIES: AN INTERPLAY BETWEEN PREFERENCES AND TECHNOLOGY

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Abstract. In this paper we consider a Ramsey-type aggregate model with general preferences and technology, endogenous labor and factor-specific productive external effects arising from average capital and labor. First, we show that indeterminacy cannot arise when there are only capital externalities but that it does when there are only labor external effects. Second, we prove that only the additively-separable and linear homogeneous specifications for the utility function allow to get local indeterminacy under small externalities and plausible restrictions on the main parameters. Third, we show that the existence of sunspot fluctuations is intimately related to the occurrence of periodic cycles through a Hopf bifurcation.

1. Introduction

Since the seminal contribution of Benhabib and Farmer [2], the existence of local indeterminacy and sunspot fluctuations based on self-fulfilling expectations has been widely studied within aggregate Ramsey-type models with externalities. The success of this literature is mainly based on the fact that by focussing on business cycle fluctuations derived from shocks on expectations, it provides an alternative explanation of macroeconomic volatility and instability to the standard real business cycle approach which is based on the consideration of real shocks on the fundamentals.

Over the last 10 years, the main effort has been devoted to finding plausible conditions on the main parameters to get local indeterminacy. In particular, following the empirical evidences provided by Basu and Fernald [1], decreasing the amount of externalities needed to produce sunspot fluctuations has been a driving force for most of the recent papers. Although a number of important contributions has been produced, the main message of this literature remains unclear as many different specifications for preferences and technologies have been considered within discrete-time or continuous-time frameworks, and produce quite different and sometimes contradictory conclusions.

The objective of this paper is to provide a unified analysis of local indeterminacy within an aggregate model with small externalities and to produce a clear picture of the main sufficient conditions. We consider a continuous-time formulation which is simpler to deal with. The production side is defined on the basis of a general

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technology which admits as a particular case only the Cobb-Douglas formulation with a unitary elasticity of capital-labor substitution. The preferences side is defined on the basis of four standard specifications for the utility function which are widely used in the growth literature: i) an additively separable formulation which has been initially used by Benhabib and Farmer [2], ii) a linearly homogeneous formulation for which the marginal rate of substitution between consumption and leisure depends on their ratio, iii) a King, Plosser and Rebello [13] (KPR) formulation which is compatible with both balanced growth and stationary worked hours, and finally iv) a Greenwood, Hercovitz and Huffman [9] (GHH) formulation for which the marginal rate of substitution between consumption and leisure depends on the latter only. We assume that the factor-specific externalities are small enough to be compatible with aggregate demand functions for capital and labor which are decreasing respectively with respect to the rental rate and the wage rate.

We show that the existence of multiple equilibrium paths results from a complex interplay between preferences and technology. First we prove that even with minimal assumptions on the fundamentals, local indeterminacy cannot arise when there are only capital externalities but fundamentally requires labor external effects. Second we show that among the four specifications for preferences, only two are compatible with sunspot fluctuations under plausible restrictions on the main parameters: i) The additively separable formulation provided the elasticity of capital-labor substitution is larger than one. ii) The linear homogeneous formulation provided the elasticity of capital labor substitution is larger than a critical bound which is smaller than one. In this case, and contrary to the first one, local indeterminacy becomes compatible with a Cobb-Douglas technology. The common conditions for these two configurations require a large enough elasticity of intertemporal substitution in consumption and a large enough elasticity of the labor supply. We also show that in both cases the occurrence of local indeterminacy is intimately linked with the existence of a Hopf bifurcation and periodic cycles.

Third we prove that uniqueness of the equilibrium is a robust result for the two other preferences' specifications, namely the KPR and GHH utility functions. In the KPR case, the necessary conditions for the existence of local indeterminacy are highly unplausible with respect to standard empirical evidences, while in the GHH case, local indeterminacy is completely ruled out with small externalities.

This paper is organized as follows: The next section sets up the basic model, defines the intertemporal equilibrium and proves the existence of a normalized steady state. In Section 3 we provide our main results: we start by the benchmark formulation with a Cobb-Douglas technology and then we study the general case with a non-unitary elasticity of capital-labor substitution. Section 4 contains concluding comments and all the proofs are gathered into a final Appendix.

2. The model

2.1. **The production structure.** Consider a perfectly competitive economy in which the final output is produced using capital K and labor L. Although production takes place under constant returns to scale, we assume that each of the many firms benefits from positive externalities due to the contributions of the average levels of capital and labor, respectively $\bar{K}$ and $\bar{L}$. Capital external effects are usually interpreted as coming from learning by doing while labor externalities are associated with thick market effects. The production function of a representative firm is thus $AF(K,L)e(\bar{K},\bar{L})$, with $F(K,L)$ homogeneous of degree one, $e(\bar{K},\bar{L})$ increasing in each argument and $A>0$ a scaling parameter. Denoting, for any $L\neq 0$, $x=K/L$ the capital stock per labor unit, we define the production function in intensive form as $Af(x)e(\bar{K},\bar{L})$.

Assumption 2.1. $f(x)$ is $\mathbf{C}^r$ over $\mathbb{R}_+$ for r large enough, increasing ($f'(x)>0$) and concave ($f''(x)<0$) over $\mathbb{R}_{++}$.

The rental rate of capital $r(t)$ and the wage rate $w(t)$ then satisfy:

$$r(t) = Af'(x(t))e(\bar{K}(t),\bar{L}(t)) \tag{2.1}$$

$$w(t) = A[f(x(t))-x(t)f'(x(t))]e(\bar{K}(t),\bar{L}(t)) \tag{2.2}$$

We can also compute the share of capital in total income:

$$s(x)=\tfrac{xf'(x)}{f(x)}\in(0,1) \tag{2.3}$$

the elasticity of capital-labor substitution:

$$\sigma(x)=-\tfrac{(1-s(x))f'(x)}{xf''(x)}>0 \tag{2.4}$$

and the elasticities of $e(\bar{K}_t,\bar{L}_t)$ with respect to capital and labor:

$$\varepsilon_{eK}(\bar{K},\bar{L})=\tfrac{e_1(\bar{K},\bar{L})\bar{K}}{e(\bar{K},\bar{L})},\quad \varepsilon_{eL}(\bar{K},\bar{L})=\tfrac{e_2(\bar{K},\bar{L})\bar{L}}{e(\bar{K},\bar{L})} \tag{2.5}$$

with $e_1(\bar{K},\bar{L})$ and $e_2(\bar{K},\bar{L})$ the partial derivatives of $e(\bar{K},\bar{L})$ with respect to $\bar{K}$ and $\bar{L}$. We consider positive externalities:

Assumption 2.2. For any given $\bar{K},\bar{L}>0$, $\varepsilon_{eK}(\bar{K},\bar{L})\geq 0$ and $\varepsilon_{eL}(\bar{K},\bar{L})\geq 0$.

Considering the aggregate consumption $C(t)$, the capital accumulation equation is then

$$\dot{K}(t)=L(t)Af(x(t))e(\bar{K}(t),\bar{L}(t))-\delta K(t)-C(t) \tag{2.6}$$

with $\delta\geq 0$ the depreciation rate of capital and $K(0)$ given.

2.2. **Preferences and intertemporal equilibrium.** We consider an economy populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one, i.e. $N=1$. At each period a representative agent supplies elastically an amount of labor $l\in[0,\ell]$, with $\ell>0$ his endowment of labor. He then derives utility from consumption c and leisure $\mathcal{L}=\ell-l$ according to a function $U(c,\mathcal{L})$ which satisfies:

Assumption 2.3. $U(c,\mathcal{L})$ is $\mathbf{C}^r$ over $\mathbb{R}_+\times[0,\ell]$ for r large enough, increasing with respect to each argument and concave.

We also introduce a standard normality assumption between consumption and leisure which ensures that the demands for these two goods are increasing functions of the agent's total income

Assumption 2.4. Consumption c and leisure $\mathcal{L}$ are normal goods.

Actually, within these general properties for the utility function, we will consider four different particular specifications which are widely used in the literature.

i) An additively separable utility function $U(c,\mathcal{L}) = u(c) + v(\mathcal{L}/B)$, with $B > 0$ a normalization constant.[1] Additive separability implies that Assumption 2.4 holds and beside Assumption 2.3, $U(c,\mathcal{L})$ satisfies the following properties:

Assumption 2.5. $\lim_{x\to 0} v'(x)x = +\infty$ and $\lim_{x\to +\infty} v'(x)x = 0$, or $\lim_{x\to 0} v'(x)x = 0$ and $\lim_{x\to +\infty} v'(x)x = +\infty$.[2]

ii) A linearly homogeneous utility function $U(c,\mathcal{L})$. Assumption 2.4 then always holds and beside Assumption 2.3, we impose the following properties:

Assumption 2.6. For all $c,\mathcal{L} > 0$, $\lim_{c/\mathcal{L}\to 0} U_2(c,\mathcal{L})/U_1(c,\mathcal{L}) = 0$ and $\lim_{c/\mathcal{L}\to +\infty} U_2(c,\mathcal{L})/U_1(c,\mathcal{L}) = +\infty$.

Building on the linear homogeneity, we introduce the share of consumption within total utility $\alpha(c,\mathcal{L}) \in (0,1)$ defined as follows:

$$\alpha(c,\mathcal{L}) = \frac{U_1(c,\mathcal{L})c}{U(c,\mathcal{L})} \tag{2.7}$$

that will be useful to characterize the steady state.[3]

iii) A King-Plosser-Rebelo [13] (KPR) formulation such that

$$U(c,\mathcal{L}) = \frac{[cv(\mathcal{L})]^{1-\theta}}{1-\theta} \tag{2.8}$$

which is compatible with both balanced growth and stationary worked hours. Let us define $h(\mathcal{L}) = v'(\mathcal{L})/v(\mathcal{L})$ and

$$\psi(\mathcal{L}) = \mathcal{L}h(\mathcal{L}), \quad \eta(\mathcal{L}) = \frac{\mathcal{L}h'(\mathcal{L})}{h(\mathcal{L})} \tag{2.9}$$

Beside Assumption 2.4, we introduce the following restrictions:

Assumption 2.7. $v(\mathcal{L})$ is a positive increasing function with $\theta \geq 0$, $\eta(\mathcal{L}) \leq -\psi(\mathcal{L})(1-\theta)$ and $\eta(\mathcal{L}) \leq \psi(\mathcal{L})(1-1/\theta)$. Moreover $\lim_{\mathcal{L}\to 0} h(\mathcal{L}) = +\infty$ and $\lim_{\mathcal{L}\to +\infty} h(\mathcal{L}) = 0$

Assumption 2.7 implies that Assumption 2.3 holds.[4] Notice that $\psi(\mathcal{L}) > 0$ can be interpreted as the elasticity of the utility of leisure and $\eta(\mathcal{L}) < 0$ is linked to the elasticity of the labor supply with respect to the wage rate ϵ_{lw}, namely $\eta(\mathcal{L}) = -\mathcal{L}/(l\epsilon_{lw})$.[5]

iv) A Greenwood-Hercovitz-Huffman [9] (GHH) formulation such that

$$U(c,\mathcal{L}) = u(c + G(\mathcal{L}/B)) \tag{2.10}$$

with $u(.)$ and $G(.)$ some increasing and concave functions, and $B > 0$ a normalization constant. Such a specification then satisfies Assumption 2.3 and implies that

[1] The constant B is used to prove the existence of a normalized steady state which remains invariant with respect to preference parameters such that the elasticity of intertemporal substitution in consumption or the elasticity of the labor supply with respect to wage.

[2] If $v(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma \geq 0$ the inverse of the elasticity of labor, the first part of Assumption 2.5 is satisfied when $\gamma > 1$ while the second part holds if $\gamma \in [0,1)$.

[3] The share of leisure within total utility is similarly defined as $1 - \alpha(c,\mathcal{L}) \in (0,1)$.

[4] See Hintermaier [10,11] and Pintus [19].

[5] This expression is obtained from the total differenciation of equation (2.19) given below.

the marginal rate of substitution between consumption and leisure depends on the latter only as

$$\frac{U_2(c,\mathcal{L})}{U_1(c,\mathcal{L})} = G'(\mathcal{L}/B)/B$$

Beside Assumption 2.4, we also impose the following properties:

Assumption 2.8. $\lim_{x\to 0} G'(x)x = +\infty$ and $\lim_{x\to+\infty} G'(x)x = 0$, or $\lim_{x\to 0} G'(x)x = 0$ and $\lim_{x\to+\infty} G'(x)x = +\infty$.[6]

Since $N(t) = 1$ for all $t \geq 0$, we get $L(t) = l(t)$ and $C(t) = c(t)$. The intertemporal maximization program of the representative agent is thus given as follows:

$$(2.11)\qquad \begin{array}{ll} \max\limits_{c(t),l(t),K(t)} & \displaystyle\int_{t=0}^{+\infty} e^{-\rho t} U(c(t), \ell - l(t)) \\ s.t. & \dot{K}(t) = L(t)Af(x(t))e(\bar{K}(t), \bar{L}(t)) - \delta K(t) - C(t) \\ & K(0) = k_0, \{\bar{K}(t), \bar{l}(t)\}_{t\geq 0} \text{ given} \end{array}$$

where $\rho > 0$ denotes the discount factor. We introduce the Hamiltonian in current value:

$$\mathcal{H} \ = U(c(t), \ell - l(t)) + \lambda(t)\Big[L(t)Af(x(t))e(\bar{K}(t), \bar{L}(t)) - \delta K(t) - C(t)\Big]$$

with $\lambda(t)$ the shadow price of capital $K(t)$. Considering the prices (2.1)-(2.2), we derive the following first order conditions

$$(2.12)\qquad U_1(c(t), \ell - l(t)) = \lambda(t)$$

$$(2.13)\qquad U_2(c(t), \ell - l(t)) = \lambda(t)w(t)$$

$$(2.14)\qquad \dot{\lambda}(t) = -\lambda(t)\left[r(t) - \rho - \delta\right]$$

Any solution needs also to satisfy the transversality condition

$$(2.15)\qquad \lim_{t\to+\infty} e^{-\rho t} U_1(c(t), \ell - l(t))K(t) \ = \ 0$$

All firms being identical, the competitive equilibrium conditions imply that $\bar{K}(t) = K(t)$ and $\bar{l}(t) = l(t)$. Under Assumptions 2.3 and 2.4, solving equations (2.12)-(2.13) with respect to $c(t)$ and $l(t)$ gives consumption demand and labor supply functions $c(K(t), \lambda(t))$ and $l(K(t), \lambda(t))$. Using (2.1)-(2.2), we get equilibrium values for the rental rate of capital $r(t)$ and the wage rate $w(t)$:

$$(2.16)\qquad \begin{aligned} r(t) &= Af'(x(t))e(K(t), l(K(t), \lambda(t)) \equiv r(K(t), \lambda(t)) \\ w(t) &= A[f(x(t)) - x(t)f'(x(t))]e(K(t), l(K(t), \lambda(t)) \equiv w(K(t), \lambda(t)) \end{aligned}$$

with $x(t) = K(t)/l(K(t), \lambda(t))$. From the capital accumulation equation (2.6) and (2.14), we finally derive the following system of differential equations in K and λ:

$$(2.17)\qquad \begin{aligned} \dot{K}(t) &= l(K(t), \lambda(t))Af(x(t))e(K(t), l(K(t), \lambda(t))) - \delta K(t) \\ &\quad - c(K(t), \lambda(t)) \\ \dot{\lambda}(t) &= -\lambda(t)\left[r(K(t), \lambda(t)) - \rho - \delta\right] \end{aligned}$$

An intertemporal equilibrium is then a path $\{K(t), \lambda(t)\}_{t\geq 0}$, with $K(0) = k_0 > 0$, that satisfies equations (2.17) and the transversality condition (2.15).

[6]If $G(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma \geq 0$ the inverse of the elasticity of the function, the first part of Assumption 2.8 is satisfied when $\gamma > 1$ while the second part holds if $\gamma \in [0, 1)$.

2.3. **Steady state.** A steady state is a 4-uple (K^*, l^*, x^*, c^*) such that $x^* = K^*/l^*$ and:

$$Af'(x^*)e(x^*l^*, l^*) = \delta + \rho, \quad c^* = l^*Af(x^*)e(x^*l^*, l^*) - \delta x^*l^* \tag{2.18}$$

$$U_2(c^*, \ell - l^*) = A[f(x^*) - x^*f'(x^*)]e(x^*l^*, l^*)U_1(c^*, \ell - l^*) \tag{2.19}$$

We use the scaling parameter A in order to give conditions for the existence of a normalized steady state (NSS in the sequel) such that $x^* = 1$ and $l^* = \bar{l}$ with $\bar{l} \in (0, \ell)$.

Proposition 2.9. *Let Assumptions 2.1-2.2 hold and $A = A^* \equiv (\delta + \rho)/f'(1)e(\bar{l}, \bar{l})$. Then a NSS satisfying $(K^*, l^*, x^*, c^*) = (\bar{l}, \bar{l}, 1, \bar{l}(\delta(1 - s(1)) + \rho)/s(1))$ is a solution of (2.18)-(2.19) if one of the following sets of conditions holds:*

- i) *$U(c, \mathcal{L}) = u(c) + v(\mathcal{L}/B)$, Assumptions 2.3, 2.4, 2.5 hold, $v'(\mathcal{L}/B) + (\mathcal{L}/B)v''(\mathcal{L}/B) \neq 0$ and $B = B^*$ with B^* the unique solution of $v'((\ell - \bar{l})/B)/B = u'(\bar{l}(\delta(1 - s(1)) + \rho)/s(1))w(\bar{l}, \bar{l})$.*
- ii) *$U(c, \mathcal{L})$ is linearly homogeneous and Assumptions 2.3, 2.6 hold.*
- iii) *$U(c, \mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1 - \theta)$ and Assumptions 2.4, 2.7 hold.*
- iv) *$U(c, \mathcal{L}) = u(c + G(\mathcal{L}/B))$, Assumptions 2.4, 2.8 hold, $G'(\mathcal{L}/B) + (\mathcal{L}/B)G''(\mathcal{L}/B) \neq 0$ and $B = B^*$ with B^* the unique solution of $G'((\ell - \bar{l})/B)/B = w(\bar{l}, \bar{l})$.*

Proof. See Appendix 5.1. □

In the rest of the paper, we evaluate all the shares and elasticities previously defined at the NSS. From (2.3), (2.4), (2.5) and (2.7), we consider indeed $s(1) = s$, $\sigma(1) = \sigma$, $\varepsilon_{eK}(\bar{l}, \bar{l}) = \varepsilon_{eK}$, $\varepsilon_{eL}(\bar{l}, \bar{l}) = \varepsilon_{eL}$, $\alpha(\bar{c}, \ell - \bar{l}) = \alpha$, $\psi(\ell - \bar{l}) = \psi$ and $\eta(\ell - \bar{l}) = \eta$.

Remark 1. When the utility function is linear homogeneous or assume the KPR formulation, we do not need to introduce a normalization constant B. Indeed, considering the prices (2.16) and the shares and elasticities defined by (2.3), (2.7) and (2.9), the first order condition (2.19) can be written as follows

$$\frac{\bar{l}}{\ell - \bar{l}} = \frac{\alpha}{1-\alpha}\frac{(1-s)(\delta+\rho)}{\delta(1-s)+\rho} \tag{2.20}$$

for a linear homogeneous utility function, and

$$\frac{\bar{l}}{\ell - \bar{l}} = \frac{(1-s)(\delta+\rho)}{\psi[\delta(1-s)+\rho]} \tag{2.21}$$

for a KPR utility function. Hence, choosing a particular value for the stationary labor supply $\bar{l} \in (0, \ell)$ implies to consider a particular value for the share of consumption into total utility $\alpha \in (0, 1)$ or for the elasticity of the utility of leisure $\psi > 0$. Notice however that ψ has to satisfy Assumption 2.7.

Remark 2: Using a continuity argument we derive from Proposition 2.9 that there exists an intertemporal equilibrium for any initial capital stock k_0 in the neighborhood of K^*. Notice also that Proposition 2.9 ensures the existence and uniqueness of the NSS. However, the presence of externalities implies that one or two other steady states may co-exist.

3. Indeterminacy with small externalities

Let us linearize the dynamical system (2.17) around the NSS. We introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,\mathcal{L})}{U_{11}(c,\mathcal{L})c}, \quad \epsilon_{\mathcal{L}c} = -\frac{U_2(c,\mathcal{L})}{U_{21}(c,\mathcal{L})c}, \quad \epsilon_{c\mathcal{L}} = -\frac{U_1(c,\mathcal{L})}{U_{12}(c,\mathcal{L})\mathcal{L}}, \quad \epsilon_{\mathcal{L}\mathcal{L}} = -\frac{U_2(c,\mathcal{L})}{U_{22}(c,\mathcal{L})\mathcal{L}},$$

However, it is more convenient to write the linearized dynamical system in terms of elasticities with respect to labor. Let $\tilde{U}(c,l) \equiv U(c,\ell - l)$ a decreasing and concave function with respect to l. We get $\tilde{U}_1(c,l) = U_1(c,\ell - l)$, $\tilde{U}_2(c,l) = -U_2(c,\ell - l)$, $\tilde{U}_{12}(c,l) = -U_{12}(c,\ell - l)$, $\tilde{U}_{22}(c,l) = U_{22}(c,\ell - l)$ and thus:

$$\epsilon_{lc} = -\frac{\tilde{U}_2}{\tilde{U}_{21}c} = \epsilon_{\mathcal{L}c}, \quad \epsilon_{cl} = -\frac{\tilde{U}_1}{\tilde{U}_{12}l} = -\epsilon_{c\mathcal{L}}\frac{\ell - l}{l}, \quad \epsilon_{ll} = -\frac{\tilde{U}_2}{\tilde{U}_{22}l} = -\epsilon_{\mathcal{L}\mathcal{L}}\frac{\ell - l}{l} \tag{3.1}$$

Since $\tilde{U}(c,l)$ is decreasing and concave with respect to l, the elasticity ϵ_{ll} is negative. Notice that an additively separable utility function is characterized by $\epsilon_{cl} = \epsilon_{lc} = \infty$.

We first introduce through the following Lemma a useful relationship between ϵ_{lc} and ϵ_{cl}.[7]

Lemma 3.1. *Let Assumptions 2.1 and 2.3 hold. Then at the NSS*

$$\epsilon_{cl} = -\frac{\delta(1-s)+\rho}{(1-s)(\delta+\rho)}\epsilon_{lc} \tag{3.2}$$

Considering all these elasticities evaluated at the NSS together with Lemma 3.1, we get the following Proposition:

Proposition 3.2. *Under Assumptions 2.1-2.3, the characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D}$$

with

$$\begin{aligned}
\mathcal{D} = \; & \frac{\delta+\rho}{\Delta}\Big\{\varepsilon_{eL}\left[\frac{\delta(1-s)+\rho}{s\sigma}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} - 1\right) - \frac{\rho(1-\sigma)}{\sigma}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right)\right] \\
& + \varepsilon_{eK}\left[\frac{(1-s)(\delta+\rho)(1-\sigma)}{s\sigma}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right) + \frac{\delta(1-s)+\rho}{s}\left(\frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} - \frac{1}{\sigma}\right)\right] \\
& + \frac{(1-s)[\delta(1-s)+\rho]}{s\sigma}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} - \frac{1}{\epsilon_{ll}} + \frac{1}{\epsilon_{cl}}\right)\Big\} \\
\mathcal{T} = \; & \rho + \frac{\varepsilon_{eL}(\delta+\rho)}{\Delta}\left(\frac{\sigma-1}{\sigma\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right) - \frac{\varepsilon_{eK}(\delta+\rho)}{s\Delta}\Big[(1-s)\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right) + \frac{s}{\epsilon_{cc}\sigma} \\
& - \left(\frac{1}{\epsilon_{cc}\epsilon_{ll}} - \frac{1}{\epsilon_{cl}\epsilon_{lc}}\right)\Big]
\end{aligned}$$

and

$$\Delta = \frac{1}{\epsilon_{cc}}\left(\frac{1}{\epsilon_{ll}} + \varepsilon_{eL} - \frac{s}{\sigma}\right) - \frac{1}{\epsilon_{cl}\epsilon_{lc}}$$

Proof. See Appendix 5.3. □

In order to have aggregate demand functions for capital and labor which are decreasing respectively with respect to the rental rate and the wage rate, we introduce the following assumption on the size of externalities:

Assumption 3.3. $\varepsilon_{eK} < (1-s)/\sigma$ and $\varepsilon_{eL} < s/\sigma$.

[7] A similar relationship has been obtained by Hintermaier [10, 11].

Notice that concavity of the utility function implies

$$\frac{1}{\epsilon_{cc}\epsilon_{ll}} - \frac{1}{\epsilon_{cl}\epsilon_{lc}} \leq 0 \tag{3.3}$$

and under Assumption 3.3 we derive $\Delta < 0$. Moreover, Assumption 2.4 implies

$$\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}} \geq 0 \tag{3.4}$$

Our aim is to discuss the local indeterminacy properties of equilibria, i.e. the existence of a continuum of equilibrium paths starting from the same initial capital stock and converging to the NSS. Our model consists in one predetermined variable, the capital stock K, and one forward variable, the shadow price λ of capital. Any solution from (2.17) that converges to the NSS satisfies the transversality condition and is an equilibrium. Therefore, given $K(0)$, if there is more than one initial price $\lambda(0)$ in the stable manifold of the NSS, the equilibrium path from $K(0)$ will not be unique. In particular, if J has two eigenvalues with negative real parts, there will be a continuum of converging paths and thus a continuum of equilibria.

Definition 3.4. If the locally stable manifold of the NSS is two-dimensional, then the NSS is locally indeterminate.

Therefore, the NSS is locally indeterminate if and only if $\mathcal{D} > 0$ and $\mathcal{T} < 0$.

We provide a first general result from (3.3), (3.4) and a direct inspection of $\mathcal{T}$. Indeed, on the one hand, if there is no externality coming from labor, i.e. $\varepsilon_{eL} = 0$, then $\mathcal{T} > 0$. On the other hand, even with externalities from capital and labor, $\mathcal{T} > 0$ if $1 - 1/\sigma \leq \epsilon_{cc}/\epsilon_{lc}$. We then get the following result:

Proposition 3.5. *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 3.3, the NSS is locally determinate in the following cases:*

i) *When* $\varepsilon_{eL} = 0$.[8]

ii) *When* $1/\sigma \geq 1 - \epsilon_{cc}/\epsilon_{lc}$.

In case i), we show that for any utility function satisfying standard assumptions, local indeterminacy is ruled out if there is no externality coming from labor. We generalize a conclusion already shown in discrete-time models with linear homogeneous preferences.[9] We also prove that Theorem 4 of Hintermaier [10] (p.14), which claims that in a one-sector model with Cobb-Douglas technology and no externalities in labor, there are non-separable preferences consistent with indeterminacy if capital externalities are high enough, is not compatible with the concavity and normality assumptions. Notice though that his existence result is obtained through numerical simulations which do not allow to determine the precise formulation of the utility function.

In case ii), we show that local indeterminacy requires a large enough elasticity of capital-labor substitution. Notice that this bound can be lower or larger than 1 depending on whether ϵ_{lc} is negative or positive. In any case, we conclude that local

[8]In an OLG model, we also show in Lloyd-Braga, Nourry and Venditti [16] that when capital externalities only enter the technology and the homogeneous utility function is characterized by a large share of young agents' consumption over the wage income, the steady state is locally determinate.

[9]See Lloyd-Braga, Nourry and Venditti [15].

indeterminacy is ruled out if the production function is close enough to a Leontief technology.

In the following, we will first concentrate on the consideration of a Cobb-Douglas technology, i.e. $\sigma = 1$, as this case has been widely studied in the literature.[10] This will represent a benchmark formulation from which we will derive more general results with $\sigma > 0$.

3.1. **Cobb-Douglas technology.** When $\sigma = 1$ we derive from Proposition 3.2 simplified expressions for $\mathcal{D}$ and $\mathcal{T}$:

$$\begin{aligned}\mathcal{D} &= \tfrac{\delta+\rho}{\Delta}\Big\{\varepsilon_{eL}\tfrac{\delta(1-s)+\rho}{s}\left(\tfrac{1}{\epsilon_{cc}} - \tfrac{1}{\epsilon_{lc}} - 1\right) + \varepsilon_{eK}\tfrac{\delta(1-s)+\rho}{s}\left(\tfrac{1}{\epsilon_{ll}} - \tfrac{1}{\epsilon_{cl}} - 1\right) \\ &\quad + \tfrac{(1-s)[\delta(1-s)+\rho]}{s}\left(\tfrac{1}{\epsilon_{cc}} - \tfrac{1}{\epsilon_{lc}} - \tfrac{1}{\epsilon_{ll}} + \tfrac{1}{\epsilon_{cl}}\right)\Big\} \\ \mathcal{T} &= \rho - \tfrac{\varepsilon_{eL}(\delta+\rho)}{\Delta\epsilon_{lc}} - \tfrac{\varepsilon_{eK}(\delta+\rho)}{s\Delta}\Big[(1-s)\left(\tfrac{1}{\epsilon_{cc}} - \tfrac{1}{\epsilon_{lc}}\right) + \tfrac{s}{\epsilon_{cc}} - \left(\tfrac{1}{\epsilon_{cc}\epsilon_{ll}} - \tfrac{1}{\epsilon_{cl}\epsilon_{lc}}\right)\Big]\end{aligned} \tag{3.5}$$

We first show as in Hintermaier [10, 11] that when the utility function is either additively separable, or satisfies the KPR or GHH formulation, then local indeterminacy is ruled out. Indeed if $U(c, \mathcal{L}) = u(c) + v(\mathcal{L}/B)$, we get

$$\mathcal{T} = \rho - \tfrac{\varepsilon_{eK}(\delta+\rho)}{\Delta s\epsilon_{cc}}\left(1 - \tfrac{1}{\epsilon_{ll}}\right) > 0$$

Moreover if $U(c, \mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1-\theta)$, we easily derive that

$$\tfrac{1}{\epsilon_{cc}} = \tfrac{1}{\epsilon_{lc}} + 1 \tag{3.6}$$

and thus

$$\mathcal{D} = \tfrac{\delta+\rho}{\Delta}\tfrac{\delta(1-s)+\rho}{s}\left(\tfrac{1}{\epsilon_{ll}} - \tfrac{1}{\epsilon_{cl}} - 1\right)(\varepsilon_{eK} - (1-s))$$

Mixing (3.3) and (3.6) we get

$$\tfrac{1}{\epsilon_{ll}} - \tfrac{1}{\epsilon_{cl}} < -\tfrac{\epsilon_{cc}}{\epsilon_{cl}} < 1 \tag{3.7}$$

so that $\mathcal{D} < 0$. We also conclude from (3.5) that $\mathcal{T} > 0$ for any utility function satisfying Assumption 2.4 and $\epsilon_{lc} \geq 0$. But such a property is satisfied when $U(c, \mathcal{L}) = u(c + G(\mathcal{L}/B))$ as we easily get in this case that $\epsilon_{cc} = \epsilon_{lc} > 0$. We have then proved:

Proposition 3.6. *Under Assumptions 2.1, 2.2, 2.3 and 3.3, let $\sigma = 1$. Then the NSS is locally determinate in the following cases:*
i) *When $U(c, \mathcal{L}) = u(c) + v(\mathcal{L}/B)$.*
ii) *When $U(c, \mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1-\theta)$ and Assumptions 2.4 and 2.7 holds.*
iii) *When $U(c, \mathcal{L})$ satisfies Assumption 2.4 and $\epsilon_{lc} \geq 0$.*

The consideration of weak externalities through Assumption 3.3 has strong consequences on the local stability properties of the NSS. Notice indeed that in the particular case of additively separable preferences, a necessary condition for $\mathcal{T} < 0$ is $\Delta > 0$, but this property requires an increasing labor demand function, i.e. $\varepsilon_{eL} > s$, as shown in Benhabib and Farmer [4]. In the case of a KPR utility function initially

[10] See Benhabib and Farmer [2], Bennett and Farmer [4], Hintermaier [10, 11].

considered by Bennett and Farmer [4], we easily show as in Hintermaier [10,11] that local indeterminacy with a negatively sloped labor demand function is ruled out as soon as we impose concavity of preferences. We also provide a new conclusion showing that local indeterminacy cannot occur for any type of utility function with a negative cross derivative U_{12}.

Proposition 3.6 gives conditions to rule out local indeterminacy. We can now focus on conditions ensuring the existence of multiple equilibrium paths. We consider a linear homogeneous utility function. Using the Euler Theorem, we know that $U_{12} = -(c/\mathcal{L})U_{11}$ and $U_{22} = (c/\mathcal{L})^2 U_{11}$. We then derive from (2.7), (2.20) and (3.1):

$$\epsilon_{lc} = -\epsilon_{cc}\tfrac{1-\alpha}{\alpha}, \ \epsilon_{cl} = \epsilon_{cc}\tfrac{1-\alpha}{\alpha}\tfrac{\delta(1-s)+\rho}{(1-s)(\delta+\rho)}, \ \epsilon_{ll} = -\epsilon_{cc}\left(\tfrac{1-\alpha}{\alpha}\right)^2\tfrac{\delta(1-s)+\rho}{(1-s)(\delta+\rho)} \tag{3.8}$$

so that $1/\epsilon_{cc}\epsilon_{ll} - 1/\epsilon_{cl}\epsilon_{lc} = 0$ and $\Delta = (\varepsilon_{eL} - s)/\epsilon_{cc}$. Notice also that $\epsilon_{lc} < 0$, $\epsilon_{cl} > 0$ and $\epsilon_{ll} < 0$.

Remark 3: From a total differenciation of equation (2.19), we can define the elasticity of the labor supply with respect to the real wage as follows:

$$\tfrac{dl}{dw}\tfrac{w}{l} \equiv \epsilon_{lw} = -\alpha\epsilon_{ll} > 0 \tag{3.9}$$

Thus, for any $\alpha \in (0,1)$, ϵ_{lw} can be equivalently appraised through ϵ_{ll}.

As Proposition 3.5 shows that capital externalities do not play a positive role for the existence of local indeterminacy, we also assume that $\varepsilon_{eK} = 0$. This allows to consider a mild amount of increasing returns. We then get:

$$\begin{aligned} \mathcal{D} &= \tfrac{\delta+\rho}{\varepsilon_{eL}-s}\tfrac{\delta(1-s)+\rho}{s(1-\alpha)}\left\{\varepsilon_{eL}\left[1-\epsilon_{cc}(1-\alpha)\right] + \tfrac{1-s}{1-\alpha}\tfrac{\delta(1-s)+\rho(1-\alpha s)}{\delta(1-s)+\rho}\right\} \\ \mathcal{T} &= \rho + \tfrac{\varepsilon_{eL}(\delta+\rho)\alpha}{(\varepsilon_{eL}-s)(1-\alpha)} \end{aligned} \tag{3.10}$$

It follows that $\mathcal{D} > 0$ if and only if

$$\epsilon_{cc} > \frac{\varepsilon_{eL} + \frac{1-s}{1-\alpha}\frac{\delta(1-s)+\rho(1-\alpha s)}{\delta(1-s)+\rho}}{\varepsilon_{eL}(1-\alpha)} \equiv \underline{\epsilon}_{cc} \tag{3.11}$$

and $\mathcal{T} < 0$ if and only if

$$\alpha > \tfrac{\rho(s-\varepsilon_{eL})}{\delta\varepsilon_{eL}+\rho s} \equiv \underline{\alpha} \tag{3.12}$$

Recall now from Remark 1 and equation (2.20) that the normalized stationary value for labor $\bar{l}$ is obtained for a given value of α. Such value needs therefore to be chosen so as to satisfy condition (3.12). However, for such a given value of α, we need to be able to choose a value of ϵ_{cc} that satisfies condition (3.11). But ϵ_{cc} and α are linked through the elasticity of substitution between consumption and leisure. Indeed, denoting this elasticity as

$$\phi(c,\mathcal{L}) = \frac{\frac{U_2(c/\mathcal{L},1)/U_1(c/\mathcal{L},1)}{c/\mathcal{L}}}{\frac{\partial U_2(c/\mathcal{L},1)/U_1(c/\mathcal{L},1)}{\partial c/\mathcal{L}}}$$

and using (2.7) and (2.20), we derive at the NSS

$$\phi = \epsilon_{cc}(1-\alpha)$$

Therefore, condition (3.11) can be satisfied if and only if

$$\phi > 1 + \frac{(1-s)[\delta(1-s)+\rho(1-\alpha s)]}{\varepsilon_{eL}(1-\alpha)[\delta(1-s)+\rho]} \equiv \underline{\phi} \tag{3.13}$$

As a result, with Cobb-Douglas preferences, where $\phi = 1$, local indeterminacy is ruled out. We have then proved:[11]

Proposition 3.7. *Let $U(c, \mathcal{L})$ be linear homogeneous, Assumptions 2.1, 2.2, 2.3, 2.6 and 3.3 hold, $\sigma = 1$ and $\varepsilon_{eK} = 0$. Then the NSS is locally indeterminate if and only if $\alpha > \underline{\alpha}$ and $\epsilon_{cc} > \underline{\epsilon}_{cc}$ (or equivalently $\phi > \underline{\phi}$).*

Local indeterminacy requires a large enough share of consumption within total utility and a large enough elasticity of intertemporal substitution in consumption. It is worth noticing that $\underline{\epsilon}_{cc} > 1$. Using (3.8) and (3.9), this last restriction implies that the elasticity of labor supply is also large enough. Notice that as usual, if the amount of labor externalities goes toward zero, the lower bound on the elasticity of intertemporal substitution in consumption goes toward $+\infty$.

A puzzeling question remains however: Why is it possible to easily get local indeterminacy with linear homogeneous preferences while local determinacy always occurs with KPR preferences ? One basic reason explains this fact: Consider $\psi > 0$ and $\eta < 0$ as defined in Assumption 2.7 and evaluated at the NSS. Total differenciation of equation (2.19) gives

$$\epsilon_{lw} = -\frac{\ell - \bar{l}}{\eta \bar{l}}$$

and using (2.21) we obtain

$$\epsilon_{lw} = -\frac{\psi}{\eta}\frac{\delta(1-s)+\rho}{(1-s)(\delta+\rho)}$$

Recall now that Assumption 2.7 requires $\eta \leq \psi(1 - 1/\theta)$, with $1/\theta = \epsilon_{cc}$. We derive therefore from the previous equality that

$$\frac{\delta(1-s)+\rho}{\epsilon_{lw}(1-s)(\delta+\rho)} \geq \epsilon_{cc} - 1$$

But as shown in Proposition 3.7 and using (3.8), local indeterminacy requires large enough values for both ϵ_{cc} and ϵ_{lw}.

We finally discuss the existence of a Hopf bifurcation which is associated with the existence of two complex roots on the imaginary axis, i.e. with parameters' values for which $\mathcal{T} = 0$ and $\mathcal{D} > 0$. Notice indeed that when $\alpha = \underline{\alpha}$ as defined by (3.12), then $\mathcal{T} = 0$. Let us then denote

$$\underline{\epsilon}_{cc}^{\alpha} \equiv \frac{\varepsilon_{eL} + \frac{1-s}{1-\underline{\alpha}}\frac{\delta(1-s)+\rho(1-\underline{\alpha}s)}{\delta(1-s)+\rho}}{\varepsilon_{eL}(1-\underline{\alpha})} \tag{3.14}$$

If $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\alpha}$, we get $\mathcal{D} > 0$ when $\alpha = \underline{\alpha}$ and the following result is derived:

Proposition 3.8. *Let $U(c, \mathcal{L})$ be linear homogeneous, Assumptions 2.1, 2.2, 2.3, 2.6 and 3.3 hold, $\sigma = 1$ and $\varepsilon_{eK} = 0$. Consider the bound $\underline{\alpha}$ as defined by (3.12) and assume that $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\alpha}$. Then when α crosses $\underline{\alpha}$ from above, there generically exists a Hopf bifurcation giving rise to locally indeterminate (resp. locally unstable) periodic orbits for any α in a left (resp. right) neighborhood of $\underline{\alpha}$.*

[11]A similar conclusion has been reached in a discrete-time setting by Lloyd-Braga, Nourry and Venditti [15].

The existence of local indeterminacy appears then to be intimately associated with periodic cycles.

3.2. **General technology.** Now we consider the general model with $\sigma \neq 1$.

3.2.1. *Additively separable preferences.* Consider first the case of additively separable preferences. We derive from Proposition 3.2:

$$\begin{aligned}\mathcal{D} &= \tfrac{\delta+\rho}{\Delta}\Big\{\varepsilon_{eL}\left[\tfrac{\delta(1-s)+\rho}{s\sigma}\left(\tfrac{1}{\epsilon_{cc}}-1\right)-\tfrac{\rho(1-\sigma)}{\sigma\epsilon_{cc}}\right]+\varepsilon_{eK}\Big[\tfrac{(1-s)(\delta+\rho)(1-\sigma)}{s\sigma\epsilon_{cc}} \\ &\quad + \tfrac{\delta(1-s)+\rho}{s}\left(\tfrac{1}{\epsilon_{ll}}-\tfrac{1}{\sigma}\right)\Big]+\tfrac{(1-s)[\delta(1-s)+\rho]}{s\sigma}\left(\tfrac{1}{\epsilon_{cc}}-\tfrac{1}{\epsilon_{ll}}\right)\Big\} \\ \mathcal{T} &= \rho+\tfrac{\varepsilon_{eL}(\delta+\rho)(\sigma-1)}{\sigma\Delta\epsilon_{cc}}-\tfrac{\varepsilon_{eK}(\delta+\rho)}{s\Delta\epsilon_{cc}}\left(1-s+\tfrac{s}{\sigma}-\tfrac{1}{\epsilon_{ll}}\right)\end{aligned}$$

Notice that as suggested by Propositions 3.5 and 3.6, $\mathcal{T} > 0$ when $\sigma \leq 1$. Moreover, under Assumption 3.3, we easily get when labor is inelastic

$$\lim_{\epsilon_{ll}\to 0_-} \mathcal{D} < 0$$

Finally, based again on Proposition 3.5, if we assume that there are no capital externalities, i.e. $\varepsilon_{eK} = 0$, we get $\mathcal{D} < 0$ when $\sigma > 1$ and $\epsilon_{cc} \leq 1$.[12] We have then proved:

Proposition 3.9. *Let $U(c, \mathcal{L}) = u(c) + v(\mathcal{L}/B)$ and Assumptions 2.1, 2.2, 2.3 and 3.3 hold. Then the NSS is locally determinate in the following cases:*
i) *When $\sigma \leq 1$.*
ii) *When $\epsilon_{ll} = 0$.*[13]
iii) *When $\varepsilon_{eK} = 0$, $\sigma > 1$ and $\epsilon_{cc} \leq 1$.*

We show that with additively separable preferences, local indeterminacy requires a large enough (larger than 1) elasticity of capital-labor substitution, endogenous labor and a large enough elasticity of intertemporal substitution in consumption. Let us now focus on clear-cut conditions for the existence of local indeterminacy. When $\varepsilon_{eK} = 0$, we can again simplify $\mathcal{D}$ and $\mathcal{T}$ as follows

$$\begin{aligned}\mathcal{D} &= \tfrac{\delta+\rho}{\Delta s\sigma}\Big\{\tfrac{1}{\epsilon_{cc}}\Big[\varepsilon_{eL}\left[\delta(1-s)+\rho[1-s(1-\sigma)]\right]+(1-s)[\delta(1-s)+\rho]\Big] \\ &\quad - \varepsilon_{eL}[\delta(1-s)+\rho]-\tfrac{(1-s)[\delta(1-s)+\rho]}{\epsilon_{ll}}\Big\} \\ \mathcal{T} &= \tfrac{1}{\sigma\Delta\epsilon_{cc}}\left\{\varepsilon_{eL}\left[(\delta+\rho)(\sigma-1)+\rho\sigma\right]+\rho\sigma\left(\tfrac{1}{\epsilon_{ll}}-\tfrac{s}{\sigma}\right)\right\}\end{aligned}$$

As $\epsilon_{ll} > 0$, assuming $\sigma > 1$ implies $\mathcal{D} > 0$ if and only if

$$\epsilon_{ll} < -\tfrac{1-s}{\varepsilon_{eL}} \equiv \epsilon_{ll}^1 \text{ and } \epsilon_{cc} > \frac{\varepsilon_{eL}\left(\frac{\delta(1-s)+\rho}{s\sigma}+\frac{\rho(\sigma-1)}{\sigma}\right)+\frac{(1-s)[\delta(1-s)+\rho]}{s\sigma}}{\frac{\delta(1-s)+\rho}{s\sigma}\left(\varepsilon_{eL}+\frac{1-s}{\epsilon_{ll}}\right)} \equiv \underline{\epsilon}_{cc} > 1 \quad (3.15)$$

and $\mathcal{T} < 0$ if and only if

[12] It is worth noticing however that when $\varepsilon_{eK} > 0$, $\mathcal{D}$ can be positive with $\sigma > 1$ and $\epsilon_{cc} \leq 1$ provided ε_{eK} is large enough. We do not explore this case as we want to consider extremely small externalities.

[13] Boldrin and Rustichini [5] and Kehoe [12] provide a similar conclusion in a discrete-time model.

(3.16) $$\epsilon_{ll} < -\frac{\rho}{\varepsilon_{eL}(\delta+2\rho)} \equiv \epsilon_{ll}^2 \text{ and } \sigma > \frac{\varepsilon_{eL}(\delta+\rho)+s\rho}{\frac{\rho}{\epsilon_{ll}}+\varepsilon_{eL}(\delta+2\rho)} \equiv \underline{\sigma} > 1$$

Let us denote $\bar{\epsilon}_{ll} = \min\{\epsilon_{ll}^1, \epsilon_{ll}^2\}$.[14] Proceeding as in Remark 3, when the utility function is additively separable, the elasticity of the labor supply with respect to the wage rate is given by $\epsilon_{lw} = -\epsilon_{ll}$. We then define $\underline{\epsilon}_{lw} \equiv -\bar{\epsilon}_{ll}$, and we get:[15]

Proposition 3.10. *Let $U(c,\mathcal{L}) = u(c)+v(\mathcal{L}/B)$, Assumptions 2.1, 2.2, 2.3 and 3.3 hold, and $\varepsilon_{eK} = 0$. Then, for any given $\epsilon_{lw} > \underline{\epsilon}_{lw}$, the NSS is locally indeterminate if and only if $\sigma > \underline{\sigma} > 1$ and $\epsilon_{cc} > \underline{\epsilon}_{cc} > 1$.*

Notice that the bounds $\underline{\epsilon}_{lw}$ and $\underline{\sigma}$ converge to $+\infty$ when the amount of labor externalities ε_{eL} approaches zero. Moreover, as usual in one-sector models, local indeterminacy is based on a large enough elasticity of the labor supply.

We can again focus on the existence of a Hopf bifurcation. Notice indeed that when $\epsilon_{lw} > \underline{\epsilon}_{lw}$ and $\sigma = \underline{\sigma}$ as defined by (3.16), then $\mathcal{T} = 0$. Let us then denote

(3.17) $$\underline{\epsilon}_{cc}^{\underline{\sigma}} \equiv \frac{\varepsilon_{eL}\left(\frac{\delta(1-s)+\rho}{s\underline{\sigma}} + \frac{\rho(\underline{\sigma}-1)}{\underline{\sigma}}\right) + \frac{(1-s)[\delta(1-s)+\rho]}{s\underline{\sigma}}}{\frac{\delta(1-s)+\rho}{s\underline{\sigma}}\left(\varepsilon_{eL}+\frac{1-s}{\epsilon_{ll}}\right)}$$

If $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\underline{\sigma}}$, we get $\mathcal{D} > 0$ when $\sigma = \underline{\sigma}$ and the following result is derived:

Proposition 3.11. *Let $U(c,\mathcal{L}) = u(c) + v(\mathcal{L}/B)$, Assumptions 2.1, 2.2, 2.3 and 3.3 hold, and $\varepsilon_{eK} = 0$. Consider the bound $\underline{\sigma}$ as defined by (3.16) and assume that $\epsilon_{lw} > \underline{\epsilon}_{lw}$ and $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\underline{\sigma}}$. Then when σ crosses $\underline{\sigma}$ from above, there generically exists a Hopf bifurcation giving rise to locally indeterminate (resp. locally unstable) periodic orbits for any σ in a left (resp. right) neighborhood of $\underline{\sigma}$.*

Again the existence of local indeterminacy appears to be intimately associated with periodic cycles. Notice however that such a conclusion is closely related to Assumption 3.3 restricting the size of externalities. Indeed, if Assumption 3.3 does not hold, as shown in Benhabib and Farmer [2] under a Cobb-Douglas, local indeterminacy can arise provided the externalities are large enough to generate a positively sloped labour demand function, but a Hopf bifurcation cannot occur since the characteristic roots are bifurcating through an infinite real part.

3.2.2. *Linear homogeneous preferences.* Consider now the case of linear homogeneous preferences. Using (3.8), we derive from Proposition 3.2:

$$\begin{aligned}\mathcal{D} =& \frac{\delta+\rho}{\left(\varepsilon_{eL}-\frac{s}{\sigma}\right)s\sigma(1-\alpha)}\Big\{\varepsilon_{eL}\Big[[\delta(1-s)+\rho]\left[1-\epsilon_{cc}(1-\alpha)\right] - s\rho(1-\sigma)\Big]\\ &+ \varepsilon_{eK}\left[\frac{(1-s)(\delta+\rho)(1-\alpha-\sigma)}{1-\alpha} - [\delta(1-s)+\rho]\epsilon_{cc}(1-\alpha)\right]\\ &+ \frac{1-s}{1-\alpha}[\delta(1-s)+\rho(1-\alpha s)]\Big\}\\ \mathcal{T} =& \rho + \frac{(\delta+\rho)}{\varepsilon_{eL}-\frac{s}{\sigma}}\left\{\varepsilon_{eL}\left(\frac{\sigma-(1-\alpha)}{\sigma(1-\alpha)}\right) - \frac{\varepsilon_{eK}}{s}\left[\frac{1-s}{1-\alpha}+\frac{s}{\sigma}\right]\right\}\end{aligned}$$

Notice that under Assumption 3.3, if $\sigma < 1-\alpha$, then $\mathcal{T} > 0$.

[14]Notice that $\epsilon_{ll}^1 < \epsilon_{ll}^2$ if $s < 1/2$.

[15]The same kind of conclusions has been obtained by Pintus [18] in a discrete-time setting.

Proposition 3.12. *Let $U(c,\mathcal{L})$ be linear homogeneous and Assumptions 2.1, 2.2, 2.3 and 3.3 hold. Then the NSS is locally determinate if $\sigma < 1-\alpha$.*

Let us now focus on the existence of local indeterminacy. When $\varepsilon_{eK} = 0$, we have $\mathcal{D} > 0$ if and only if

$$\epsilon_{cc} > \frac{\varepsilon_{eL}[\delta(1-s)+\rho[1-s(1-\sigma)]]+\frac{1-s}{1-\alpha}[\delta(1-s)+\rho(1-\alpha s)]}{\varepsilon_{eL}(1-\alpha)[\delta(1-s)+\rho]} \equiv \underline{\epsilon}_{cc} \tag{3.18}$$

and $\mathcal{T} < 0$ if and only if

$$\sigma > (1-\alpha)\frac{s\rho+\varepsilon_{eL}(\delta+\rho)}{\varepsilon_{eL}[\rho(1-\alpha)+\delta+\rho]} \equiv \underline{\sigma} \tag{3.19}$$

Notice that as a non-unitary elasticity of capital-labor substitution σ is considered, the sign of $\mathcal{T}$ is more conveniently discussed with respect to σ than with respect to α. Recall now that ϵ_{cc} and α are linked through the elasticity of substitution between consumption and leisure ϕ, namely $\phi = \epsilon_{cc}(1-\alpha)$. It follows that condition (3.18) can be satisfied if and only if

$$\phi > \frac{\varepsilon_{eL}[\delta(1-s)+\rho[1-s(1-\sigma)]]+\frac{1-s}{1-\alpha}[\delta(1-s)+\rho(1-\alpha s)]}{\varepsilon_{eL}[\delta(1-s)+\rho]} \equiv \underline{\phi} \tag{3.20}$$

We have then proved:[16]

Proposition 3.13. *Let $U(c,\mathcal{L})$ be linear homogeneous, Assumptions 2.1, 2.2, 2.3, 2.6 and 3.3 hold, and $\varepsilon_{eK} = 0$. Then the NSS is locally indeterminate if and only if $\sigma > \underline{\sigma}$ and $\epsilon_{cc} > \underline{\epsilon}_{cc}$ (or equivalently $\phi > \underline{\phi}$).*

We prove that with linear homogeneous preferences, local indeterminacy requires a large enough elasticity of capital-labor substitution and a large enough elasticity of intertemporal substitution in consumption (or equivalently a large enough elasticity of substitution between consumption and leisure). As suggested by (3.8), this last condition implies also a large enough elasticity of the labor supply. As shown by Proposition 3.7, the lower bound $\underline{\sigma}$ can be less than one and thus the critical bound $\underline{\phi}$ on the elasticity of substitution between consumption and leisure can be also less than one. It follows that if the elasticity of capital-labor substitution is less than unity, local indeterminacy becomes compatible with a linearly homogeneous Cobb-Douglas utility function. Notice however that if we assume $s \leq 1/2$, this is no longer true as $\underline{\phi}$ becomes larger than one.[17]

As in Section 3.1, the Hopf bifurcation is closely related to the existence of local indeterminacy. Notice that when $\sigma = \underline{\sigma}$ as defined by (3.19), then $\mathcal{T} = 0$. Let us then denote

$$\underline{\epsilon}_{cc}^{\underline{\sigma}} \equiv \frac{\varepsilon_{eL}[\delta(1-s)+\rho[1-s(1-\underline{\sigma})]]+\frac{1-s}{1-\alpha}[\delta(1-s)+\rho(1-\alpha s)]}{\varepsilon_{eL}(1-\alpha)[\delta(1-s)+\rho]} \tag{3.21}$$

If $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\underline{\sigma}}$, we still get $\mathcal{D} > 0$ when $\sigma = \underline{\sigma}$ and the following result is derived:

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[16] A similar conclusion has been reached in a discrete-time setting by Lloyd-Braga, Nourry and Venditti [15].

[17] Indeed, when $s < 1/2$, conditions (3.19) and (3.20) cannot hold simultaneously with Assumption 3.3.

Proposition 3.14. *Let $U(c,\mathcal{L})$ be linear homogeneous, Assumptions 2.1, 2.2, 2.3, 2.6 and 3.3 hold, and $\varepsilon_{eK} = 0$. Consider the bound $\underline{\sigma}$ as defined by (3.19) and assume that $\epsilon_{cc} > \underline{\epsilon}_{cc}^{\sigma}$. Then when σ crosses $\underline{\sigma}$ from above, there generically exists a Hopf bifurcation giving rise to locally indeterminate (resp. locally unstable) periodic orbits for any σ in a left (resp. right) neighborhood of $\underline{\sigma}$.*

This Proposition therefore extends Proposition 3.8 to the case of a general production function.

3.2.3. *KPR preferences.* Consider now a KPR utility function. Using (3.6), we derive from Proposition 3.2:

$$\mathcal{D} = \frac{\delta+\rho}{\Delta}\Big\{ -\varepsilon_{eL}\frac{\rho(1-\sigma)}{\sigma} + \frac{\delta(1-s)+\rho}{s}\left(\frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} - 1\right)\left(\varepsilon_{eK} - \frac{1-s}{\sigma}\right) - \varepsilon_{eK}\frac{(1-\sigma)\rho}{\sigma}\Big\}$$
$$\mathcal{T} = \rho + \frac{\varepsilon_{eL}(\delta+\rho)}{\Delta}\left(\frac{\sigma\epsilon_{cc}-1}{\sigma\epsilon_{cc}}\right) - \frac{\varepsilon_{eK}(\delta+\rho)}{s\Delta}\Big[1 - s + \frac{s}{\epsilon_{cc}\sigma} - \left(\frac{1}{\epsilon_{cc}\epsilon_{ll}} - \frac{1}{\epsilon_{cl}\epsilon_{lc}}\right)\Big]$$

Assumption 3.3 and (3.7) imply that $\mathcal{D} < 0$ when $\sigma \geq 1$. Moreover, when $\sigma < 1$, we get $\mathcal{T} > 0$ as soon as $\epsilon_{cc} \leq 1$ or $\sigma \leq 1/\epsilon_{cc}$. Consider then $\epsilon_{cc} > 1$ and $\sigma \in (1/\epsilon_{cc}, 1)$. Building again on Proposition 3.5, assume that $\varepsilon_{eK} = 0$ in order to keep externalities as small as possible.[18] As shown by (2.21), normalyzing the steady state with $(\ell - \bar{l})/\bar{l} = a > 0$ gives

$$\psi = \frac{a(1-s)(\delta+\rho)}{[\delta(1-s)+\rho]} \tag{3.22}$$

We also derive

$$\epsilon_{cc} = \frac{1}{\theta}, \quad \epsilon_{lc} = -\frac{1}{1-\theta}, \quad \epsilon_{cl} = \frac{\delta(1-s)+\rho}{(1-\theta)(1-s)(\delta+\rho)}, \quad \epsilon_{ll} = \frac{a[\delta(1-s)+\rho]}{\eta[\delta(1-s)+\rho]+(1-\theta)a(1-s)(\delta+\rho)}$$

Notice that $\epsilon_{cc} > 1$ is equivalent to $\theta < 1$ and implies $\epsilon_{lc} < 0$ and $\epsilon_{cl} > 0$. Moreover, Assumption 2.7 implies

$$\theta > \underline{\theta} \equiv \frac{a(1-s)(\delta+\rho)}{a(1-s)(\delta+\rho)-\eta[\delta(1-s)+\rho]} \in (0,1) \tag{3.23}$$

and we derive under this restriction that $\epsilon_{ll} < 0$. It follows also that $\sigma > \theta > \underline{\theta}$ is a necessary condition for local indeterminacy.

We get after simplifications

$$\begin{aligned} \mathcal{D} &= \frac{\delta+\rho}{\Delta\sigma}\left\{-\varepsilon_{eL}\rho(1-\sigma) + \frac{(a-\eta)(1-s)[\delta(1-s)+\rho]}{sa}\right\} \\ \mathcal{T} &= \frac{1}{\sigma\Delta}\Big\{\varepsilon_{eL}\left[(\delta+\rho)(\sigma-\theta) + \rho\sigma\theta\right] \\ &- \rho\sigma\left[\frac{\theta s}{\sigma} + \frac{\theta[a(1-s)(\delta+\rho)-\eta[\delta(1-s)+\rho]]-a(1-s)(\delta+\rho)}{a[\delta(1-s)+\rho]}\right]\Big\} \end{aligned} \tag{3.24}$$

We easily derive that $\mathcal{D} > 0$ if and only if

$$\varepsilon_{eL} > \frac{(a-\eta)(1-s)[\delta(1-s)+\rho]}{as\rho(1-\sigma)} \equiv \varepsilon_{eL}^1 \tag{3.25}$$

But Assumption 3.3 requires that $\varepsilon_{eL}^1 < s/\sigma$ and this inequality is satisfied if and only if

$$\sigma < \frac{s^2 a\rho}{(a-\eta)(1-s)[\delta(1-s)+\rho]+s^2 a\rho} \equiv \bar{\sigma} \tag{3.26}$$

[18]It is also clear that as $\mathcal{T}$ is an increasing function of ε_{eK}, externalities from capital favor the occurrence of local determinacy.

It is easy to notice that $\bar{\sigma} < s$. Recall now that as we have assumed $\sigma > \underline{\theta}$, we need $\underline{\theta} < \bar{\sigma}$. Straightforward computations show that this last inequality finally requires $s > \underline{s} > 1/2$.

The following Proposition summarizes all the above results:

Proposition 3.15. *Let $U(c, \mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1-\theta)$ and Assumptions 2.1, 2.2, 2.4, 2.7 and 3.3 hold. Then the NSS is locally determinate in the following cases:*[19]

i) *When $\sigma \geq 1$.*
ii) *When $\sigma < 1$ and $\theta \geq 1$.*
iii) *When $\theta < 1$ and $\sigma \in (0, \theta]$.*
iv) *When $\varepsilon_{eK} = 0$, $\theta < 1$ and $\sigma \in [s, 1)$.*
v) *When $\varepsilon_{eK} = 0$, $\theta < 1$ $\sigma \in (0, s)$ and $s < \underline{s}$ with $\underline{s} > 1/2$.*

When KPR preferences are considered local indeterminacy requires drastically different conditions on the elasticity of capital-labor substitution than with additively separable or linear homogeneous preferences. Indeed, in case i) we show that the elasticity of capital-labor substitution needs to be lower than unity here while it has to be larger than unity under the specifications of Propositions 3.10 and 3.13. In cases ii) and iii), we prove at the same time that the elasticity of capital-labor substitution needs to be larger than the inverse of the elasticity of intertemporal substitution in consumption which is restricted to be larger than 1. In case iv), we also show that although labor externalities are considered, local indeterminacy necessarily requires an elasticity of capital-labor substitution lower than the share of capital in total income. Notice at this point that this conclusion appears to be very similar to those obtained within standard models which are free of externalities, i.e. overlapping generations models with endogenous labor,[20] and infinite horizon models with heterogeneous agents and financial constraint.[21] Finally, in case v), we show that all these restrictions on σ are compatible only if the share of capital s is strictly larger than 1/2.

To summarize, local indeterminacy requires the following conditions: $\theta < s$, $\sigma \in (\theta, s) \subset (0, 1)$ and $s > 1/2$. Duffy and Papageorgiou [7] have recently proved that the elasticity of capital-labor substitution within developed countries is significantly larger than unity. Moreover, it is a well-established fact that the share of capital in OECD coutries is generically lower than 1/2. Proposition 3.16 then implies that local indeterminacy is extremely unlikely when a KPR utility function is considered.

3.2.4. *GHH preferences.* Consider finally a GHH utility function. Let us denote

$$\epsilon^G_{\mathcal{L}\mathcal{L}} = -\frac{G'(\mathcal{L}/B)}{G''(\mathcal{L}/B)(\mathcal{L}/B)} > 0$$

the elasticity of the function $G(\mathcal{L}/B)$. We easily get from (3.1) that

$$\epsilon_{cc} = \epsilon_{lc} \text{ and } \frac{1}{\epsilon_{ll}} = \frac{1}{\epsilon_{cl}} + \frac{1}{\epsilon^G_{ll}} \text{ with } \epsilon^G_{ll} = -\epsilon^G_{\mathcal{L}\mathcal{L}}\frac{\ell - \bar{l}}{\bar{l}} < 0$$

We then derive from Proposition 3.2

[19] Similar results as in cases i), iii) and v) have been reached in a discrete-time setting by Pintus [19].

[20] See Cazzavillan and Pintus [6], Lloyd-Braga [14], Nourry and Venditti [17] and Reichlin [21].

[21] See Grandmont, Pintus and de Vilder [8] and Woodford [22].

$$\mathcal{T} = \rho - \frac{\varepsilon_{eL}(\delta+\rho)}{\epsilon_{cc}\Delta} - \frac{\varepsilon_{eK}(\delta+\rho)}{\epsilon_{cc}s\Delta}\left(\frac{s}{\sigma} - \frac{1}{\epsilon_{ll}^G}\right) > 0 \qquad (3.27)$$

We have thus proved:

Proposition 3.16. *Let $U(c, \mathcal{L}) = u(c + G(\mathcal{L}/B)$ and Assumptions 2.1, 2.2, 2.4, 2.8 and 3.3 hold. Then the NSS is always locally determinate.*

This Proposition shows that the GHH specification is even worse than the KPR specification as local indeterminacy is completely ruled out as soon as we restrict the externalities to satisfy Assumption 3.3. Notice indeed that if on the contrary we allow for large external effects leading to an increasing labor demand function, then we derive from (3.27) that local indeterminacy might be obtained as $\mathcal{T}$ might be negative.

4. Concluding comments

In this paper we have studied a Ramsey-type aggregate model with four different formulations for preferences which are widely used in the literature, a general technology, endogenous labor and factor-specific productive external effects arising from average capital and labor. First, we have shown under minimal retrictions on the fundamentals that indeterminacy cannot arise when there are only capital externalities but that it does when there are only labor external effects. Second, we have proved that only the additively-separable and linear homogeneous specifications for the utility function allow to get local indeterminacy under small externalities and plausible restrictions on the main parameters, namely, the elasticity of capital-labor substitution, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply needs to be large enough. However, a Cobb-Douglas technology appears to be compatible with local indeterminacy with a linear homogeneous utility function while this cannot be the case with an additively separable one. Third, we have shown that the existence of sunspot fluctuations is intimately related to the occurrence of periodic cycles through a Hopf bifurcation. Fourth, we have proved that the existence of multiple equilibria is ruled out when KPR or GHH preferences are considered as soon as plausible restrictions on the main parameters are imposed. These results then show that the existence of local indeterminacy is the outcome of a complex interplay between preferences and technology.

5. Appendix

5.1. **Proof of Proposition 2.9.** Consider equations (2.18) and (2.19): $(x^*, l^*, c^*) = (1, \bar{l}, \bar{c})$ is a steady state if there exists a value for A such that:

$$\bar{c} = \bar{l}Af(1)e(\bar{l}, \bar{l}) - \delta\bar{l}, \quad \frac{U_2(\bar{c}, \ell-\bar{l})}{U_1(\bar{c}, \ell-\bar{l})} = A[f(1) - f'(1)]e(\bar{l}, \bar{l}), \quad Af'(1)e(\bar{l}, \bar{l}) = \delta + \rho$$

Solving the third equation gives

$$A = \frac{\delta+\rho}{f'(1)e(\bar{l}, \bar{l})} \equiv A^*$$

and considering $A = A^*$ into the first and second equations implies

$$\bar{c} = \bar{l}\frac{\delta(1-s)+\rho}{s} \equiv \bar{l}\mathcal{C}, \quad \frac{U_2(\bar{l}\mathcal{C}, \ell-\bar{l})}{U_1(\bar{l}\mathcal{C}, \ell-\bar{l})} \equiv g(\bar{l}) = \frac{(1-s)(\delta+\rho)}{s} \qquad (5.1)$$

with $s = s(1)$.

i) Consider the case of an additively separable utility function such that $U(c,\mathcal{L}) = u(c) + v(\mathcal{L}/B)$, with $B > 0$ a normalization constant. We get

$$g(\bar{l}) = \frac{v'((\ell-\bar{l})/B)}{Bu'(\bar{l}\mathcal{C})} \equiv \tilde{g}(B)$$

If $v'(\mathcal{L}/B) + (\mathcal{L}/B)v''(\mathcal{L}/B) \neq 0$ then $\tilde{g}'(B) \neq 0$ and Assumption 2.5 implies that there exists a unique value B^* of B such that when $B = B^*$, $\bar{l}$ satisfies equation (5.1).

ii) Consider the case of a linear homogeneous utility function. Under Assumptions 2.3 and 2.4 we get $\lim_{\bar{l}\to 0} g(\bar{l}) = 0$ and $\lim_{\bar{l}\to \ell} g(\bar{l}) = +\infty$ with $g'(\bar{l}) > 0$. It follows that there exists a unique NSS with $x^* = 1$ and $l^* = \bar{l} \in (0, \ell)$.

iii) Consider a KPR utility function such that $U(c,\mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1-\theta)$. We then get

$$g(\bar{l}) = \frac{cv'(\ell-\bar{l})}{v(\ell-\bar{l})} = ch(\ell-\bar{l})$$

Equation (5.1) can thus be written as

$$\bar{l}h(\ell-\bar{l}) \equiv \tilde{g}(\bar{l}) = \frac{(1-s)(\delta+\rho)}{\delta(1-s)+\rho}$$

and Assumption 2.7 implies $\lim_{\bar{l}\to 0} \tilde{g}(\bar{l}) = 0$, $\lim_{\bar{l}\to \ell} \tilde{g}(\bar{l}) = +\infty$ and $\tilde{g}'(\bar{l}) > 0$. Therefore there exists a unique NSS with $x^* = 1$ and $l^* = \bar{l} \in (0, \ell)$.

iv) Consider finally a GHH utility function such that $U(c,\mathcal{L}) = u(c + G(\mathcal{L}/B))$, with $B > 0$ a normalization constant. We get

$$g(\bar{l}) = G'((\ell-\bar{l})/B)/B \equiv \tilde{g}(B)$$

If $G'(\mathcal{L}/B) + (\mathcal{L}/B)G''(\mathcal{L}/B) \neq 0$ then $\tilde{g}'(B) \neq 0$ and Assumption 2.8 implies that there exists a unique value B^* of B such that when $B = B^*$, $\bar{l}$ satisfies equation (5.1). □

5.2. **Proof of Lemma 3.1.** Using (3.1) and the first order conditions (2.12) and (2.13), we get $\epsilon_{cl} = -\epsilon_{lc}(c/wl)$. But using the expression of w at the NSS given in (2.16) together with (2.3) we find $wl = (1-s)(c+\delta K)$. Recall then that at the NSS, $c = \bar{l}Af(1)e(\bar{l},\bar{l}) - \delta\bar{l}$. We then derive using again (2.3)

$$\frac{c+\delta K}{K} = \frac{\delta+\rho}{s}, \quad \frac{c}{K} = \frac{\delta(1-s)+\rho}{s} \tag{5.2}$$

The result follows. □

5.3. **Proof of Proposition 3.2.** Consider the first order conditions (2.12) and (2.13). Under Assumptions 2.3 and 2.4, solving with respect to $c(t)$ and $l(t)$ gives consumption demand and labor supply functions $c(K(t),\lambda(t))$ and $l(K(t),\lambda(t))$. Using (3.1), the implicit function Theorem allows to get the partial derivatives of these functions evaluated at the NSS

$$\frac{dc}{dK} = \frac{c}{K\Delta}\frac{\varepsilon_{eK}+\frac{s}{\sigma}}{\epsilon_{cl}}, \quad \frac{dc}{d\lambda} = -\frac{c}{\lambda\Delta}\left(\frac{1}{\epsilon_{ll}} + \varepsilon_{eL} - \frac{s}{\sigma} - \frac{1}{\epsilon_{cl}}\right)$$

$$\frac{dl}{dK} = -\frac{l}{K\Delta}\frac{\varepsilon_{eK}+\frac{s}{\sigma}}{\epsilon_{cc}}, \quad \frac{dl}{d\lambda} = -\frac{l}{\lambda\Delta}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right)$$

with

$$\Delta = \frac{1}{\epsilon_{cc}}\left(\frac{1}{\epsilon_{ll}} + \varepsilon_{eL} - \frac{s}{\sigma}\right) - \frac{1}{\epsilon_{cl}\epsilon_{lc}}$$

From these results and (2.16) we also derive at the NSS

$$\frac{dr}{dK} = \frac{r}{K}\left[\varepsilon_{eK} - \frac{1-s}{\sigma} - \left(\varepsilon_{eL} + \frac{1-s}{\sigma}\right)\frac{\varepsilon_{eK}\frac{s}{\sigma}}{\epsilon_{cc}\Delta}\right], \quad \frac{dr}{d\lambda} = -\frac{r}{\lambda\Delta}\left(\varepsilon_{eL} + \frac{1-s}{\sigma}\right)\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right)$$

Consider then the system of differential equations in K and λ:

$$\dot{K}(t) = l(K(t),\lambda(t))Af(x(t))e(K(t),l(K(t),\lambda(t))) - \delta K(t) - c(K(t),\lambda(t))$$
$$\dot{\lambda}(t) = -\lambda(t)\left[r(K(t),\lambda(t)) - \rho - \delta\right]$$

Linearization around the NSS using (5.2) and the above results gives

$$\frac{d\dot{K}}{dK} = \rho - \frac{(\delta+\rho)(1-s+\varepsilon_{eL})}{s\Delta\epsilon_{cc}}\left(\varepsilon_{eK} + \frac{s}{\sigma}\right) + \varepsilon_{eK}\frac{\delta+\rho}{s} - \frac{\delta(1-s)+\rho}{s\Delta\epsilon_{cl}}\left(\varepsilon_{eK} + \frac{s}{\sigma}\right)$$
$$\frac{d\dot{K}}{d\lambda} = \frac{K}{\lambda\Delta}\left[\frac{\delta(1-s)+\rho}{s}\left(\frac{1}{\epsilon_{ll}} + \varepsilon_{eL} - \frac{s}{\sigma} - \frac{1}{\epsilon_{cl}}\right) - \frac{(\delta+\rho)(1-s+\varepsilon_{eL})}{s}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right)\right]$$
$$\frac{d\dot{\lambda}}{dK} = -\frac{\lambda}{K}(\delta+\rho)\left[\varepsilon_{eK} - \frac{(1-s)}{\sigma} - \left(\varepsilon_{eL} + \frac{(1-s)}{\sigma}\right)\frac{\varepsilon_{eK}+\frac{s}{\sigma}}{\epsilon_{cc}\Delta}\right]$$
$$\frac{d\dot{\lambda}}{d\lambda} = \frac{\delta+\rho}{\Delta}\left(\varepsilon_{eL} + \frac{(1-s)}{\sigma}\right)\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{lc}}\right)$$

The expression of the characteristic polynomial follows after tedious computations and straightforward simplifications based on Lemma 3.1. □

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