



NON-CONVEXITY, DISCOUNTING AND INFINITE HORIZON OPTIMIZATION

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ABSTRACT. We consider the discounted aggregative model of optimal economic growth with non-convex production technology. We review some of the important results on the nature and long run behavior of optimal accumulation paths and indicate how these differ qualitatively from the ones in the classical convex version of the model.

1. INTRODUCTION

Consider the following discounted dynamic programming problem: an economy begins with an initial stock of capital or input x_0 which leads to an output at the end of the period according to a (production) function $f(x_0)$ [where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing and is typically required to satisfy additional properties]. The decision maker, upon observing the output, chooses an action: a fraction $\theta \in [0, 1]$; the amount

$$c_0 = \theta f(x_0)$$

is consumed, leaving

$$x_1 = f(x_0) - c_0$$

as the stock of capital or input x_1 in period 1. Consumption c_0 generates an immediate return or utility $u(c_0)$ according to a (utility) function u [where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing and is often required to satisfy other properties]. The story is repeated over time. A discount factor $\delta \in (0, 1)$ is given. The decision maker is interested in choosing a sequence of actions so as to maximize the discounted intertemporal sum of returns or utilities

$$\sum_{t=0}^{\infty} \delta^t u(c_t).$$

This dynamic optimization problem lies at the heart of the well known one-sector model of optimal economic growth - an important building block in dynamic economics that has been analyzed extensively to understand problems of intertemporal resource allocation including macroeconomic growth and capital accumulation, management of natural resources that are harvested over time and capital accumulation by firms. Based on the (undiscounted) problem of dynamic savings and consumption analyzed by Frank Ramsey (1928), the discounted one sector optimal growth model was first systematically examined by Cass (1965) and Koopmans

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(1965). The classical Cass-Koopmans version of the model and much of the subsequent literature on macroeconomic growth focused on the case where the optimization problem is convex i.e., the utility function u and the production function f are both concave (in addition to various other restrictions).

Concavity of the production function implies that the productivity of (or, the rate of return on investment in) capital is decreasing in capital stock and is maximized at zero. Indeed, in a large section of the literature, it is assumed that productivity of capital is infinitely large at zero. Such assumptions on the technology however imply that the model cannot be used to understand problems of macroeconomic growth in economies characterized by significant degrees of “increasing returns” so that productivity of capital may be low when total capital stock is small and may increase sharply as the capital stock expands beyond a threshold.¹

Moreover, in applications of the one sector growth model to optimal management of biological and other renewable resources, the production function needs to capture features of biological reproduction or natural growth of species. The latter may often be characterized by “depensation” - productivity is low when the population size is small and the species grows faster after the population attains a moderate size (see, Clark, 1990). For the model to be relevant to such applications, we need to allow for production functions that are non-concave (for example, S-shaped).

In this article, we focus on a class of “non-classical” (a term used by Leonid Hurwicz²) models of one sector optimal growth in which the production function f is not restricted to be concave and review the basic results on the long run behavior of optimal sequences of actions.

Section 2 contains a formal specification of the assumptions and a set of basic results on the dynamic optimization exercise. Section 3 discusses certain qualitative properties of optimal programs such as monotonicity and convergence of capital stocks and output over time. Section 4 discusses issues related to the existence of a non-trivial optimal steady state. In section 5, we provide an exposition of results related to the characterization of long run (limiting) behavior of optimal capital stocks and the effect of discounting (including a characterization of extinction and conservation in the long run). Section 6 contains a discussion of turnpike properties of optimal capital sequences as discounting vanishes i.e., $\delta \rightarrow 1$. Section 7 concludes with some indication of the directions in which the literature has extended the basic model outlined in Section 2.

2. PRELIMINARIES

Consider the following one sector model of optimal economic growth. Time is discrete and is indexed by $t \in Z_+$, where $Z_+ = \{0, 1, 2, \dots\}$ is the set of all non-negative integers. The economy begins with an initial stock of *capital or input* $x_0 \geq 0$. Capital depreciates fully every period. At the beginning of every time

¹Increasing returns may reflect scale economies and indivisibilities in the production technology and can explain economic phenomena such as low level “poverty traps” in the process of economic development, the role of the “big push” in moving poor economies out of poverty traps and the persistence of gaps between “clubs” of rich and poor countries.

²Hurwicz (1972).

period $t \in Z_+$, the current stock of capital or input $x_t \geq 0$ generates current *output*

$$y_t = f(x_t)$$

where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the production function. An amount $c_t \in [0, y_t]$ is consumed at the end of period t . An amount

$$x_{t+1} = y_t - c_t$$

is invested in capital formation and constitutes the stock of capital or input at the beginning of period $t + 1$. *Consumption* c_t in period t generates *utility* $u(c_t)$ where u is the (one-period) utility function.

Given an initial capital stock $x_0 \geq 0$, a non-negative sequence $\mathbf{x} = \{x_t\}_{t=0}^\infty$ is said to be a *capital path* if

$$0 \leq x_{t+1} \leq f(x_t), \forall t \in Z_+.$$

Every capital path $\mathbf{x} = \{x_t\}_{t=0}^\infty$ generates a corresponding *consumption path* which is a non-negative sequence $\mathbf{c} = \{c_t\}_{t=0}^\infty$ defined by

$$c_t = f(x_t) - x_{t+1}, t \in Z_+.$$

The dynamic optimization problem is as follows: given a discount factor $\delta \in (0, 1)$, and an initial capital stock $x_0 \geq 0$,

$$(2.1) \quad \begin{aligned} & \text{maximize} \quad \sum_{t=0}^{\infty} \delta^t u(c_t) \\ & \text{subject to} \quad : \quad c_t = f(x_t) - x_{t+1}, x_t \geq 0, \forall t \in Z_+. \end{aligned}$$

A solution $\{x_t, c_t\}_{t=0}^\infty$ to the maximization problem (2.1) is said to be an *optimal path* and, in particular, we refer to $\{x_t\}_{t=0}^\infty$ as an optimal capital path and to $\{c_t\}_{t=0}^\infty$ as an optimal consumption path.

It may be noted that the model has been re-interpreted as illustrating the intertemporal decentralization of consumption and capital investment decisions by an appropriate system of “dual” prices supporting an optimal path (see, Majumdar, 1988). However, in the non-convex models that we review in this article, this decentralization interpretation often breaks down.

Observe that the formal optimization problem stresses the role of consumption explicitly in generating utility. The optimization problem can, however, be written in an equivalent “reduced form” (used first in Gale, 1967) as: given a discount factor $\delta \in (0, 1)$, and an initial capital $x_0 \geq 0$,

$$\begin{aligned} & \text{maximize} \quad \sum_{t=0}^{\infty} \delta^t w(x_t, x_{t+1}) \\ & \text{subject to} \quad : \quad 0 \leq x_{t+1} \leq f(x_t), \forall t \in Z_+. \end{aligned}$$

where w is the reduced form utility function defined on $\{(x, x') : 0 \leq x' \leq f(x), x \geq 0\}$ by

$$w(x, x') = u(f(x) - x').$$

See, Mitra (2000) for a useful survey of the literature (when f is concave) using the “reduced form”.

The following assumptions on the utility function u will be retained throughout the paper:

U.1. $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous, strictly increasing and strictly concave on \mathbb{R}_+ , continuously differentiable on \mathbb{R}_{++} with $u'(c) > 0, u''(c) < 0$ for all $c > 0$.

U.2. $\lim_{c \rightarrow 0} u'(c) = +\infty$.

U.1 imposes standard smoothness, strict concavity and monotonicity assumptions on the utility function. In particular, the assumption of strict concavity rules out environments where utility may be linear in consumption (globally or in segments) in which case, certain monotonicity, interiority and convergence properties of optimal paths need not hold (see, Majumdar and Mitra, 1983, Kamihigashi and Roy, 2006).

U.2 imposes the standard Uzawa-Inada condition on the utility function that requires marginal utility to be infinite at zero consumption; it ensures that optimal paths are interior i.e., optimal consumption and capital are strictly positive every period, if $x_0 > 0$.

The following assumptions on the production function f will be retained throughout the paper:

T.1. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous on \mathbb{R}_+ and continuously differentiable on \mathbb{R}_{++} with $f'(x) > 0$ for all $x > 0$.

T.2. $f(0) = 0; f'(0) = \limsup_{x \rightarrow 0} f'(x) = \liminf_{x \rightarrow 0} f'(x) > 1$; there exists $K > 0$ such that

$$f(K) = K, f(x) < x \text{ for all } x > K.$$

Without loss of generality, we assume that:

$$x_0 \in [0, K].$$

T.1 imposes standard monotonicity and smoothness requirements on the production technology. Many of the results described in the paper have been shown to hold with non-smooth and discontinuous production functions (Mitra and Ray 1984, Kamihigashi and Roy 2006, 2007).

T.2 imposes restriction on the behavior of the production function at zero and at infinity. In particular, it requires that the technology is “productive” at small levels of capital stock so that expansion of output and capital is technologically feasible when the stock of capital is sufficiently small.³ It also requires that the economy exhibit “bounded growth” i.e., capital and output paths are uniformly bounded. This rules out sustained growth of the economy in the long run; however, many of the qualitative properties of optimal paths described in this article can be easily shown to hold in a framework that allows for unbounded expansion of capital and consumption (Kamihigashi and Roy, 2007). It should be observed here that in applications of the model to management of biological species and other renewable resources, “bounded growth” is a natural assumption as the stocks of such resources are eventually limited by the carrying capacity of the ecosystem.

³In applications of the model to problems of optimal management of biological species, this assumption may be restrictive; there are many species that are incapable of sustaining their population (even in the absence of any human intervention) when their biomass falls below a certain threshold. This is often referred to as “critical depensation”. See, Clark (1990), Bhattacharya and Majumdar (2007, Chapter 16).

Note that unlike the classical models of economic growth, the production function f is *not* required to be concave (though assumption $T.2$ does rule out a production function that is globally *convex*). Indeed, this potential non-concavity of the production function, distinguishes the class of non-classical growth models reviewed in this article from the rest of the literature. One implication of this is that, informally stated, the feasible set of the dynamic optimization problem (more precisely, the set of capital and consumption paths from any given initial capital stock) is potentially non-convex.

Further, unlike much of the classical optimal growth literature, the marginal productivity at zero is allowed to be finite; there is no “Uzawa-Inada condition” on the production function. Indeed, the literature on growth with non-convex technology focuses on capital accumulation in economic environments where productivity can be relatively small in a neighborhood of zero as a consequence of increasing returns or due to low natural growth of certain biological species when the stock or biomass is too small (depensation) etc.

The canonical illustration of a non-concave production function that satisfies $T.1$ and $T.2$ is a smooth S-shaped “Knightian” production function with slope greater than 1 near zero that is initially convex and eventually behaves like a concave “Solow” production function. However, the assumptions allow for multiple convex-concave segments, for instance, a production function that may be obtained as a (non-concave) outer envelope of multiple concave production functions, each corresponding to a different production technology, where the economy switches from one technology to another as capital stock expands (Nguyen et al, 2005).

The existence of a solution to the above dynamic optimization problem follows from standard results in the dynamic programming literature (our optimization problem can be studied as a discounted dynamic programming problem with bounded immediate return function and compact action space⁴) but may also be obtained by a more direct compactness argument on the space of feasible paths (Majumdar, 1975). Let $V(x)$ denote the value function i.e., the value of the maximand (2.1) generated by an optimal path from initial capital stock $x_0 = x$. It is easy to verify that under our assumptions, $V(x)$ is continuous and strictly increasing on \mathbb{R}_+ and, using standard dynamic programming arguments, $V(x)$ satisfies the functional equation of dynamic programming (the optimality equation):

$$(2.2) \quad V(x) = \max_{0 \leq x' \leq f(x)} [u(f(x) - x') + \delta V(x')].$$

Let $H(x)$ be the set of solutions to the maximization problem on the right hand side of (2.2) i.e.,

$$(2.3) \quad H(x) = \{x' \in [0, f(x)] : V(x) = u(f(x) - x') + \delta V(x')\}.$$

⁴See, for example, Maitra (1968). Take the interval $[0, K]$ to be the state space. Interpreting the action each period to be the fraction of available output that is consumed, we can take the interval $[0, 1]$ to be the action space. The utility function is continuous and therefore bounded on $[0, K]$.

Then, $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-empty, upper-hemicontinuous correspondence. Standard stationary dynamic programming arguments can be used to show that a program $\{x_t\}$ is optimal if, and only if,

$$(2.4) \quad x_{t+1} \in H(x_t), \forall t \geq 0.$$

In the rest of this article, we shall refer to H as the *optimal investment policy correspondence*.

Using (U.2), one can show that for every $x > 0$,

$$(2.5) \quad x' \in H(x) \Rightarrow 0 < x' < f(x).$$

From (2.4) and (2.5), we have that given $x_0 > 0$, every optimal program $\{x_t\}$ is interior i.e.,

$$0 < x_{t+1} < f(x_t), \forall t \geq 0,$$

so that every period, capital and consumption are both strictly positive.

In the classical optimal growth model where the production function f is concave, the set of consumption and capital paths from any initial capital stock is convex so that strict concavity of the utility function implies that the value function is concave, that there is a unique solution to the maximization problem on the right hand side of (2.2) and therefore, $H(x)$ is a continuous function. In contrast, in our framework, the potential non-concavity of the value function implies that there may not exist any continuous selection from the optimal investment policy correspondence; optimal consumption and investment in capital may be discontinuous with respect to variation in the current capital stock.

Another important point of contrast is that in the classical growth model, concavity of the value function can be used to show that if optimal consumption is strictly positive at any level of capital, then the value function is differentiable at that point. Indeed, under an assumption like (U.2), the value function is globally differentiable in the classical framework.

One implication of the potential non-concavity of the value function in our non-classical set up is that the value function is not necessarily differentiable everywhere. However, it has been shown that:

$$\{x \geq 0 : V(x) \text{ is not differentiable at } x\}$$

is at most countable (see, Dechert and Nishimura, 1983, Kamihigashi and Roy, 2007). It can also be shown that if $\{x_t\}$ is an optimal capital path from initial capital stock $x_0 > 0$, the value function is differentiable at every capital stock $x_t, t \geq 1$ (see Amir, Mirman and Perkins, 1991, Askri and Le Van, 1998).

Finally, one can show that the Ramsey-Euler equation holds. In particular, given $x_0 > 0$, for any optimal capital path $\{x_t\}$,

$$(2.6) \quad u'(f(x_t) - x_{t+1}) = \delta f'(x_{t+1})u'(f(x_{t+1}) - x_{t+2}), t \in Z_+.$$

The Ramsey-Euler essentially provides a first order necessary condition for an interior optimal path. In the classical framework, in conjunction with a transversality condition⁵, (2.6) is also sufficient for optimality of a given capital path. This no

⁵In our framework, the transversality condition is satisfied by an input program $\{x_t\}$ provided $\delta^t u'(f(x_t) - x_{t+1}) \rightarrow 0$ as $t \rightarrow \infty$.

longer holds in the non-classical framework. In particular, even if an interior capital path satisfies the Ramsey-Euler equation (2.6) and consumption is bounded away from zero along the path so that the transversality condition is satisfied, it need not be optimal.

3. MONOTONICITY AND CONVERGENCE OF OPTIMAL PATHS

In the classical model with concave production function, the (unique) optimal policy is one where both optimal consumption as well as investment are increasing in current output and hence, in the capital stock. As a consequence, optimal paths of capital, consumption and output over time are monotonic sequences; being bounded sequences, they converge to a steady state.

As mentioned above, in the non-classical model, optimal policy need not be unique. However, one can show the following “monotonicity property” of the optimal investment policy correspondence $H(x)$.

Lemma 3.1. $\forall x \geq 0, \forall y > x, \forall x' \in H(x), \forall y' \in H(y), y' \geq x'$.

The lemma states that it is never optimal to invest less from a higher level of capital stock. In other words, under an optimal policy, the marginal propensity to consume is less than one. Thus, the classical property of optimal investment being an increasing function of current output and capital stock continues to hold. The basic argument behind this result has to do with the fact that as long as the utility function is strictly concave, current output $y_t = f(x_t)$ available at the beginning of period t and the current investment x_{t+1} are complementary; for any level of investment x_{t+1} , an increase in the available current output y_t has no effect on future state or utility; it affects only the current consumption and as long as the utility function is strictly concave, an increase in output must reduce the opportunity cost of increasing investment (by reducing the marginal utility from consumption sacrificed). Therefore, an increase in the capital stock x_t (and hence output y_t) always increases (weakly) the current investment x_{t+1} . Non-concavity of the production and value functions do not affect this argument. This important result was formally established by Majumdar and Nermuth (1982) and Dechert and Nishimura (1983).⁶

An immediate consequence of Lemma 3.1 is that every optimal capital path is necessarily monotonic⁷.

Proposition 3.2. *Let $\{x_t\}$ be an optimal capital path. Then either $x_t \leq x_{t+1} \forall t \geq 0$, or $x_t \geq x_{t+1} \forall t \geq 0$.*

As optimal capital paths are bounded, they necessarily converge.

A capital stock $x \geq 0$ is said to be an optimal steady state if there exists an optimal capital path $\{x_t\}_{t=0}^\infty$ from initial capital stock $x_0 = x$ such that $x_t = x$, for all $t \in Z_+$.

For any optimal steady state x , it must be true that $f(x) \geq x$ so that $x \in [0, K]$. Trivially, $x = 0$ is an optimal steady state.

⁶Amir, Mirman and Perkins (1991) relate this to the second order necessary condition for the maximization problem on the right hand side of the functional equation (2.2).

⁷This result also follows from an argument used in the working paper version of Mitra and Ray (1984).

Using (2.2), (2.3) and (2.4), for any optimal capital path $\{x_t\}$

$$V(x_t) = u(f(x_t) - x_{t+1}) + \delta V(x_{t+1})$$

Using continuity of u , f and V , we have that if $x = \lim_{t \rightarrow \infty} x_t$, then

$$V(x) = u(f(x) - x) + \delta V(x)$$

so that x is an optimal steady state. Thus,

Proposition 3.3. *Every optimal capital path converges to an optimal steady state.*

Monotonicity of optimal investment as stated in Lemma 3.1 implies that for any two optimal capital paths $\{x_t\}, \{x'_t\}$, $x_0 < x'_0$ implies that $x_t \leq x'_t$ for every $t \in \mathbb{Z}_+$ i.e., every optimal capital path is bounded below by optimal capital paths from lower initial capital stocks and bounded above by optimal capital paths from higher initial capital stocks.

An important economic implication of this is that the limit (optimal steady state) to which an optimal capital path converges is weakly increasing in the level of initial capital stock. In particular, if an optimal capital path converges to zero i.e., extinction occurs from a certain initial capital stock, then every optimal capital path from a smaller initial capital stock must converge to zero. On the other hand, if the optimal capital path from a certain initial capital stock converges to a strictly positive optimal steady state, all optimal capital paths from higher initial capital stocks must converge to the same, or a higher, strictly positive optimal steady state.

In bio-economic applications to management of species, this implies that (loosely speaking) extinction is more likely to be optimal from small population size rather than large population size; this, in turn, provides the foundation for the concept of a critical level such that extinction is optimal from initial capital stocks below this level and conservation is optimal from initial capital stocks above this level. In bio-economics, this critical level is referred to as a critical safe standard of conservation.

In macroeconomic applications, one can similarly define a critical capital stock below which optimal capital paths converge to a steady state with a relatively small level of sustained consumption (not necessarily zero) - a “low level poverty equilibrium” and if the initial capital is above this level (for example, through inflow of foreign capital or aid), the economy may move out of the poverty trap.⁸

Finally, it should be mentioned that unlike the classical model, in the non-classical model with non-concave production function, optimal *consumption* may be non-monotonic in current output and hence, in the capital stock. The curvature of the value and production functions play very important roles in determining the effect of an increase in current output on the “marginal” present and future return from consumption. In particular, current output and consumption are complementary if f (and therefore, V) is concave. In a non-concave framework, they may not be complementary. Consider, for example, an S -shaped production function where the marginal productivity is low at small levels of capital stock but is significantly higher beyond a threshold level. As economic intuition would suggest, the low return on investment provides incentive for high propensity to consume when the

⁸In a growth model with non-concave production function, Majumdar and Mitra (1995) show that international trade may allow an economy to escape such a poverty trap

output available for investment is small. However, at higher levels of output, it may be optimal to increase the fraction invested sharply as productivity of capital could be much higher. As a result, optimal consumption may be smaller at a higher levels of current output (over a certain range) - a richer economy may optimally consume less than a poorer one.

One implication of this is that even though optimal capital paths are necessarily monotone over time and therefore convergent, optimal consumption paths may exhibit non-monotone dynamics and need not converge.

4. NON-ZERO OPTIMAL STEADY STATES

If a capital stock $x > 0$ is an optimal steady state, then there exists an optimal capital path from initial capital $x_0 = x$ that is stationary i.e.,

$$x_t = x > 0, c_t = f(x) - x > 0, \forall t \in Z_+.$$

From the Ramsey-Euler equation (2.6), we have the following necessary condition for a non-zero optimal steady state:

$$(4.1) \quad \delta f'(x) = 1.$$

In the classical model with a concave production function, (4.1) is sufficient to assert that the stationary path from x satisfies the Ramsey-Euler equation and the transversality condition and hence, is optimal, so that (4.1) is also a sufficient condition for an *optimal* steady state (so long as $0 < x < f(x)$). In the non-classical model, (4.1) is necessary but *not* sufficient for $x > 0$ to be an optimal steady state.

For $x \geq 0$, define:

$$\Gamma(x) = \delta f(x) - x.$$

Kamihigashi and Roy (2007) call this the “gain function”; it captures the discounted net return when x units of capital are invested to generate $f(x)$ units of output next period. This function was used by Majumdar and Nermuth (1982), Dechert and Nishimura (1983) and Mitra and Ray (1984) to examine properties of optimal steady states.

Note that under our assumptions $\Gamma(x)$ is a continuously differentiable function on \mathbb{R}_{++} and

$$\Gamma'(x) = \delta f'(x) - 1.$$

If $f'(x) > \frac{1}{\delta}$ i.e., the marginal productivity of investment exceeds the discount *rate*, then $\Gamma'(x) > 0$ and the opposite holds if $f'(x) < \frac{1}{\delta}$. More generally, whether the gain function is increasing or decreasing over a certain range depends on whether the technology over that range of investment exhibits higher productivity relative to the discount rate.

Observe that for any capital path $\{x_t\}$ and the associated consumption path $\{c_t\}$,

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta^t c_t \\ = & \sum_{t=0}^{\infty} \delta^t [f(x_t) - x_{t+1}] \\ = & f(x_0) - x_1 + \delta[f(x_1) - x_2] + \dots + \delta^t [f(x_t) - x_{t+1}] + \dots \end{aligned}$$

$$= f(x_0) + \sum_{t=0}^{\infty} \delta^t \Gamma(x_{t+1})$$

so that $\Gamma(x_{t+1})$ is the contribution of x_{t+1} towards the present discounted sum of consumption over time.

Refining the core arguments made in Majumdar and Nermuth (1982) and Dechert and Nishimura (1983), Kamihigashi and Roy (2007) establish the following useful result:

Lemma 4.1. *Let $\{x_t\}$ be an optimal capital path that is nonstationary i.e., $x_t \neq x_0$ for some $t > 0$. Then, there exists $\tau > 0$, such that $\Gamma(x_0) < \Gamma(x_\tau)$.*

The lemma states that an optimal capital path that is non-stationary must move in a direction in which higher gain will be available at some point in the future. One should remark here that concavity of the utility function plays an important role in this result.

The following result (Kamihigashi and Roy, 2007) is an immediate consequence of Lemma 4.1 and provides a sufficient condition for a non-zero optimal steady state:

Proposition 4.2. *Suppose there exists $\hat{x} > 0$ such that $\Gamma(\hat{x}) \geq \Gamma(x)$ for all $x \geq 0$. Then, \hat{x} is an optimal steady state.*

If, under the hypothesis of Proposition 4.2, \hat{x} is not an optimal steady state, then there is an optimal capital path $\{x_t\}$ from initial stock $x_0 = \hat{x}$ that is non-stationary and along this path, $\Gamma(x_0) \geq \Gamma(x_t)$ for all $t \geq 0$, contradicting Lemma 4.1.

Proposition 4.2 clarifies that as long as the gain function is maximized at a strictly positive capital stock, there is a non-zero optimal steady state and the set of optimal steady states includes all global maximizers of the gain function (note that Γ is non-concave and can be maximized at multiple capital stocks). Note that (4.1) is the first order necessary condition for an interior maximum of the gain function $\Gamma(x)$.

It is easy to check that Γ attains a maximum in $[0, K]$. In order to ensure the existence of a non-zero optimal steady state, it is sufficient to ensure that the gain function Γ is *not* maximized at zero. Now, by definition

$$\Gamma(0) = 0$$

and so, as long as there is some $\tilde{x} > 0$ such that $\Gamma(\tilde{x}) > 0$, the function $\Gamma(x)$ attains an interior maximum. Let $\Gamma'(0)$ defined by

$$\Gamma'(0) = \delta f'(0) - 1$$

where $f'(0)$ is as defined in T.2. A sufficient condition for $\Gamma(x)$ to attain an interior maximum is that $\Gamma'(0) > 0$; however, as f is non-concave, Γ may be non-monotonic and even if $\Gamma'(0) < 0$, there may be a stock \tilde{x} large enough for which $\Gamma(\tilde{x}) > 0$.

Proposition 4.3. *Suppose that there exists $\tilde{x} > 0$ such that $\Gamma(\tilde{x}) > 0$, i.e.,*

$$(4.2) \quad \delta f(\tilde{x}) > \tilde{x}$$

Then, a non-zero optimal steady state exists. A sufficient condition for this is given by:

$$(4.3) \quad \delta f'(0) > 1.$$

Proposition 4.3 outlines sufficient conditions for the existence of a non-zero optimal steady state. The condition (4.2) is often referred to as a “delta-productivity” condition; Mitra and Ray (1984) were the first to show that this ensures the existence of a non-zero optimal steady state for a general non-concave production function. This was further generalized by Kamihigashi and Roy (2007). For the case of S-shaped production functions, more precise sufficient conditions for the existence of a non-zero optimal steady state in terms of the slope of the production function at zero and at the point of maximum average product were provided by Majumdar and Mitra (1982) and Dechert and Nishimura (1983).

Finally, Nguyen et al (2005) demonstrate explicitly the possibility of multiple non-zero optimal steady states that act as local attractors.

5. LONG RUN BEHAVIOR OF OPTIMAL PATHS & THE ROLE OF DISCOUNTING

In the classical model of optimal growth with a strictly concave production function⁹, if

$$\delta < \frac{1}{f'(0)},$$

then every optimal path converges to a unique non-zero optimal steady state (the modified golden rule) characterized by (4.1); otherwise, every optimal path converges to zero. In particular, the long run destiny is independent of initial state. The comparison of marginal productivity at zero¹⁰ with the discount rate determines the long run destiny of optimal paths globally.

In the non-classical model with non-concave production function, the marginal productivity at zero conveys no information about productivity at higher levels of investment. Therefore, the long run destiny of optimal paths is likely to depend on the initial capital stock (hysteresis) and the qualitative characterization of the limiting behavior of optimal capital paths is likely to depend on the productivity of capital over the entire relevant domain of the production function (and not just at zero).

5.1. Heavy Discounting and Extinction. As mentioned earlier, every optimal capital path must converge to an optimal steady state. Therefore, optimal capital paths from all initial capital stocks converge to zero (i.e., *global extinction* is optimal) if, and only if, a non-zero optimal steady state does not exist. Using the necessary condition (4.1) for a non-zero optimal steady state, it follows that if

$$(5.1) \quad \delta < \frac{1}{f'(x)} \text{ for all } x \in (0, K],$$

then zero is the unique optimal steady state and therefore all optimal capital paths must converge to zero. This result was established by Majumdar and Mitra (1982) and Dechert and Nishimura (1983) and later generalized by Kamihigashi and Roy

⁹If the production function is weakly concave, then $\delta < \frac{1}{f'(0)}$ implies that optimal paths converge to some non-zero optimal steady state but there may be a continuum of such states. If $\delta > \frac{1}{f'(0)}$, then every optimal path converges to zero.

¹⁰Bio-economists refer to this as the “intrinsic growth rate of the specie”.

(2007). Note that (5.1) implies that $\Gamma'(x) < 0$ at every $x > 0$ so that the gain from investment attains a global maximum at zero.

Unlike the classical model,

$$(5.2) \quad \delta < \frac{1}{f'(0)}$$

is not sufficient for global extinction to be optimal. However, no matter how high the productivity at higher levels of investment, if (5.2) holds i.e., the marginal productivity at zero is smaller than the discount rate, optimal paths converge to zero from initial capital stocks that are small enough i.e., extinction is optimal from stocks lying in a neighborhood of zero. This was established by Dechert and Nishimura (1983) for the case of S-shaped production functions and generalized by Kamihigashi and Roy (2007). It may be mentioned here that the proof of this result uses the assumption that marginal utility of consumption is infinite at zero. Observe that (5.2) implies that $\Gamma'(0) < 0$ so that Γ is a decreasing function in a neighborhood of zero and the gain function attains a local maximum at zero.

5.2. Intermediate Discounting and Critical Stock. For a non-concave production function, condition (5.2) does not imply any restriction on productivity and gain from investment at higher stocks. In particular, as long as the average product $\frac{f(x)}{x}$ is not maximized at zero, (5.2) is perfectly consistent with the technology being “delta-productive” i.e., $\delta f(\tilde{x}) > \tilde{x}$ (i.e., the gain from investment $\Gamma(\tilde{x}) > 0$) at some $\tilde{x} > 0$ which (using Proposition 4.3) implies the existence of a non-zero optimal steady state. In that case, optimal capital paths from initial capital stocks that are large enough are bounded away from zero; for instance, optimal capital paths from initial capital stocks higher than the stock at which Γ attains global maximum are bounded below by the latter.

Let γ be the maximum average product i.e.,

$$\gamma = \sup_{x>0} \left[\frac{f(x)}{x} \right].$$

In view of the discussion in the previous sub-section and the monotonicity of optimal capital paths in initial capital stock (see Lemma 3.1), it is easy to check that if

$$\frac{1}{f'(0)} > \delta > \frac{1}{\gamma},$$

then there exists a *critical stock* $x^* \in (0, K)$ such that every optimal capital path from initial stock $x_0 < x^*$ necessarily converges to zero while every optimal capital path from initial stock $x_0 > x^*$ is bounded below by x^* . In the bioeconomics literature, x^* is referred to as “the minimum safe standard of conservation” - it is optimal to conserve a biological specie in the long run if the biomass or population size is above this safe standard.

Extensive analysis of the behavior of optimal capital paths under intermediate discounting for the specific case of *S-shaped production functions* is contained in Majumdar and Mitra (1982, 1983) and Dechert and Nishimura (1983). Majumdar and Mitra (1982) argue that every optimal capital path from an initial capital stock that is above the level at which average product is maximized, converges to the (unique) strictly positive optimal steady state that lies in the concave part of the

production function. Dechert and Nishimura (1983) show that this holds as long as the initial stock lies above the smallest stock in the concave part of the production function where the average product equals $\frac{1}{\delta}$.

For more general non-concave production functions, Kamihigashi and Roy (2007) show the following:

Proposition 5.1. *Suppose there exists $\bar{x} > 0$ such that*

$$(5.3) \quad \Gamma(x) \leq \Gamma(\bar{x}), \forall x \in [0, \bar{x}].$$

Then, every optimal capital path from $x_0 \geq \bar{x}$ is bounded below by \bar{x} . A sufficient condition for (5.3) is that

$$(5.4) \quad \delta \geq \frac{1}{f'(x)}, \forall x \in [0, \bar{x}].$$

5.3. Mild Discounting. If

$$(5.5) \quad \delta > \frac{1}{f'(0)},$$

then for every $x_0 > 0$ that is sufficiently close to zero, $\delta > \frac{1}{f'(x)}, \forall x \in [0, x_0]$ so that using Proposition 5.1 (in particular, condition (5.4)) we have that every optimal path from x_0 is bounded below by x_0 itself and hence, bounded away from zero. As optimal capital paths are increasing in initial capital stock, it follows that from any initial capital stock $x_0 > 0$, extinction is never optimal and indeed, every optimal capital path converges to a non-zero optimal steady state. This is what bio-economists refer to as “global conservation”.

This result was first established by Majumdar and Mitra (1982) for the case of S-shaped production functions. Note that for S-shape functions (that are initially strictly convex and then strictly concave), (5.5) implies that there is a unique level of capital that meets the necessary condition (4.1) for a non-zero optimal steady state; it lies in the concave part of the production function and is the unique global maximizer of the gain function $\Gamma(x)$. So, in this case, every optimal capital path converges to a unique non-zero optimal steady state (modified golden rule); there is no initial state dependence.

For more general non-concave production functions, however, (5.5) is consistent with multiple non-zero optimal steady states that are locally stable and the limit to which optimal capital paths converge may depend on the initial capital stock.

6. BEHAVIOR AS DISCOUNTING VANISHES

The analysis in the previous section indicates that unless discounting is so heavy that all optimal capital paths converge to zero, the non-classical model always allows for the possibility that the limit of optimal capital paths is sensitive to initial states and therefore, optimal capital stocks from two distinct initial capital stocks may remain distant from each other in the long run (i.e., the paths may not “approach” each other asymptotically). This contrasts sharply with the fact that in the classical model with a strictly concave production function, the optimal capital path from

every positive initial capital stock converges to the same optimal steady state and as long as

$$\delta > \frac{1}{f'(0)}$$

this steady state capital stock (the “modified golden rule”) is the one where the gain from investment attains its (unique) maximum.

Further, in the classical model, as $\delta \rightarrow 1$, the unique optimal steady state converges to the “golden rule” stock i.e., the unique level of capital that maximizes $f(x) - x$, the level of constant consumption sustainable over time from initial capital stock x .

One of the interesting properties of optimal paths in our non-classical model is that as discounting vanishes or more precisely, for δ close enough to 1, the optimal capital path from any given initial capital stock converges to a small neighborhood of a golden rule stock and in that sense, the model behaves similarly to a classical model.

For ease of exposition, consider the case where the function $[f(x) - x]$ attains its maximum on $[0, K]$ at a unique capital stock x^* i.e., there is a unique golden rule capital stock. Under our assumption that $f'(0) > 1$, we have that $x^* \in (0, K)$. Suppose, further, that $f(x) - x > 0$ for all $x \in (0, x^*)$.

Fix any initial stock $x_0 > 0$. For each $\delta \in (0, 1)$, let $\{x_t^\delta\}$ be an optimal path from x_0 when the discount factor is equal to δ . Then, the neighborhood turnpike property that holds is as follows:

$$\lim_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta = x^*.$$

Indeed, for any given initial stock, for δ close enough to 1, every optimal capital path converges to an optimal steady state which is the global maximizer of the gain function $\delta f(x) - x$ and this optimal steady state converges to the golden rule capital stock x^* as $\delta \uparrow 1$.

Majumdar and Nermuth (1982) were the first to establish this neighborhood turnpike property of optimal paths for the general non-convex model; Majumdar and Mitra (1982) established a similar result for the case of S-shaped production functions. A more general version of this result that, among other things, allows for multiple stocks that maximize the function $f(x) - x$ as well as non-smooth and discontinuous f , is contained in Kamihigashi and Roy (2007).

7. EXTENSIONS

In this final section, we briefly indicate the main directions in which the literature has extended the basic one sector deterministic non-classical optimal growth model discussed in the previous sections.

First, the literature has analyzed versions of the model where the utility function is concave, but not necessarily strictly concave and in particular, the case of a linear utility function. The latter is particularly relevant to models of optimal natural resource exploitation where the net marginal benefit from harvesting is constant when the resource is sold at a given market price and the harvesting technology

exhibits constant returns. With linear utility, one can no longer ensure that optimal capital paths are interior; more importantly, optimal capital paths need not necessarily be monotone. Clark (1971) and Majumdar and Mitra (1983) analyze the behavior of optimal paths for linear utility and S-shaped production functions; they focus, in particular, on conditions for extinction, conservation and the existence of a minimum safe standard of conservation. Mitra and Ray (1984) analyze a more general model with concave utility and fairly general non-concave production function; they study the existence of non-trivial optimal steady states and show that optimal capital paths approach optimal steady states asymptotically. Kamihigashi and Roy (2006) extend the existing results (for the case of linear utility) to a framework that allows for discontinuous production functions (as well as irreversibility in investment). In particular, they show that every optimal capital path is monotone until it reaches a steady state; further, it either converges to zero or reaches a positive steady state in finite time and possibly jumps among different steady states afterwards. They also establish a neighborhood turnpike result about the limiting behavior of optimal paths as discounting vanishes.

Second, while the basic model in the previous sections assumes that capital depreciates fully every period, the literature has discussed more general versions of the model allowing for partial depreciation and irreversibility in capital formation (that puts a possibly non-zero lower bound on the feasible range of capital stock next period, given current capital stock) - see, among others, Majumdar and Nermuth (1982), Kamihigashi and Roy (2006, 2007).

Third, in a large class of economic situations (such as natural resource extraction), the utility or immediate return may depend on the level of current capital stock (or output) in addition to current consumption and this may lead to non-monotone and other complex dynamics of optimal paths (see, Majumdar and Mitra, 1994). Olson and Roy (1996) characterize the long run behavior of optimal paths in such a dynamic optimization problem with non-concave production function.

Fourth, the *multi-sector undiscounted* version of the model (with non-convex technology) has been analyzed. Among other issues, the existence of an optimal path is analyzed by Majumdar and Peleg (1992) and conditions for the existence of a non-trivial optimal steady state are provided by Mitra (1992).

Finally, the model has been extended to a framework where the production technology is both non-convex and stochastic. In the stochastic model, the output in every period depends not only on the accumulated capital stock but also on the realization of a random production shock. In the rest of this section, we briefly summarize this literature.

Majumdar, Mitra and Nyarko (1989) were the first to comprehensively analyze this model with independent and identically distributed (i.i.d.) random shocks. They showed that most of the qualitative properties of the (stationary) optimal decision rule obtained in the deterministic framework can be extended to the stochastic case. In particular, optimal investment is increasing in the current level of output.

In the stochastic model, fluctuations in capital and output can occur over time due to random shock. The concept of a steady state is that of an invariant distribution and one is interested in the convergence of capital and output in distribution to

such a steady state. Majumdar et al (1989) show that non-convexity in production technology may lead to multiple invariant distributions and the distribution to which the stochastic process of optimal capital stocks converges may depend on the initial condition (this is the stochastic analogue of the result about initial state dependence of limiting capital stock in the deterministic model). They also show that if the production technology is “sufficiently stochastic”, then there exists a globally stable invariant distribution (despite the non-convexity).

More recently, Nishimura, Rudnicki and Stachurski (2006) analyze the stochastic optimal growth model with multiplicative i.i.d. random shocks whose common distribution has a density function that is strictly positive on \mathbb{R}_{++} so that from any capital stock, it is possible for the realized current output to be arbitrarily close to zero as well as arbitrarily large. Under restrictions on the expectation of the random shock, they show that the Markov process of optimal capital stocks either converges to zero from every initial state or there is a globally stable non-zero steady state (they identify conditions for these events).¹¹ In a similar framework, Nishimura and Stachurski (2005) use the Euler equation to analyze the stability of the stochastic optimal capital process; in particular, they use the marginal utilities as Foster-Lyapunov functions in order to obtain stability.

The literature on non-convex stochastic growth also develops “turnpike results” under which optimal capital processes approach each other asymptotically as discounting vanishes. In a model with non-convex (and non-stationary) technology, Joshi (1997) shows that, under a strong “value loss” condition that is uniform with respect to time and state, as discounting vanishes, the asymptotic distance between optimal paths from two distinct initial states converges to zero with probability one. However, the uniform value loss condition is not very transparent in terms of its implications for the primitives of the model.

As in the deterministic version of the model, one of the interesting questions in stochastic literature relates to the possibility of extinction i.e., of optimal paths converging to zero. Assuming a bounded growth production function and i.i.d. random shocks that have compact support, Mitra and Roy (2006) develop sufficient conditions on the preferences and technology that ensure that optimal capital stocks are bounded away from zero with probability one (from all positive initial stocks as well as from stocks above a critical level) and conditions under which extinction occurs with probability one from all initial stocks¹². In contrast to the conditions for extinction and conservation discussed in the previous section for the deterministic model that are entirely in terms of comparison of the discount factor to the productivity of capital, the conditions in the stochastic case involve the marginal utility function - one compares the discount rate to expected “welfare-modified” return on investment (marginal productivity).¹³

¹¹To place their results in context, their assumption on the density function automatically satisfies the “very stochastic” assumption in Majumdar et al (1989) that ensures the existence of a globally stable invariant distribution.

¹²See, also Kamihigashi (2006).

¹³Olson and Roy (2000) characterize conditions for avoidance of extinction in a version of the stochastic growth model where the utility depends on both consumption and capital stock (or, output).

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