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H_∞ CHEAP CONTROL FOR A CLASS OF LINEAR SYSTEMS WITH STATE DELAYS

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Abstract. An infinite horizon H_∞ cheap control problem with a given performance level for a linear system with point-wise and distributed state delays is considered. By a proper transformation of the control variable, this problem is converted to an H_∞ control problem for a singularly perturbed system with state delays. For the latter problem, considered in the sequel as an original one, two methods of asymptotic analysis and solution are proposed.

1. Introduction

For many decades, controlled systems with disturbances (uncertainties) in dynamics are investigated extensively. One of the main objectives in studying such systems is a design of a robust controller, i.e., a feedback control independent of the disturbance and providing a desirable property of the closed-loop system regardless disturbance realizations from a given set. Two main cases of disturbances are considered in the literature: (i) disturbances with bounded realizations in an Euclidean space; (ii) disturbances with square-integrable realizations. In the second case, an H_∞ problem is studied (as a rule) for the controlled system.

The H_∞ control problem has been studied for systems without and with delays in the state variables in a number of works (see e.g. [1, 3, 7, 11, 12, 22]). In both cases, the solution of the H_∞ control problem can be reduced to a solution of a game-theoretic Riccati equation. In the case of an undelayed system, the Riccati equation is finite dimensional (matrix one), while in the case of a delayed system, it is infinite dimensional (operator one). The operator Riccati equation can be reduced to a hybrid system of three matrix equations of Riccati type. Solution of this system is a very complicated task.

The H_∞ cheap control problem is an H_∞ problem with a small control cost (with respect to a state cost and a disturbance cost) in the cost functional (performance index). It should be noted that a performance index with a small control cost (cheap control performance index) arises in many topics of control theory. For instance, it arises in the regularization method of a singular optimal control [2], in studying the limitations of optimal regulators and filters [5, 25, 34], in analysis of control problems with a high control gain [24, 42], in the investigation of inverse control problems [26], in the design of a robust control for systems with disturbances [38], and some others.

Control problems with a cheap control performance index for systems without disturbances (uncertainties) were investigated in the literature. The case of systems

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with an undelayed dynamics was treated more extensively (see e.g. [4, 21, 24, 28, 29, 33, 35] and references therein). The case of systems with a delayed dynamics was studied less extensively (see [13, 16–18]). In both cases, an optimal control problem was analyzed.

In this paper, a system with point-wise and distributed state delays, and with a square-integrable disturbance is considered. For this system, an H_∞ cheap control problem is formulated and analyzed. For our best knowledge, the H_∞ cheap control problem has not yet been studied in the literature, neither for systems without delays nor for systems with delays in dynamics. However, it should be noted that two-player zero-sum differential games with a cheap control cost of one of the players in the performance index were analyzed in [14, 32, 36, 38]. In these works, the case of an undelayed game dynamics and a cheap control cost for the player, minimizing the performance index, was treated. This circumstance makes the problems, considered in [14, 32, 36, 38], to be close to the H_∞ cheap control problem for a system without delays.

In the present paper, two methods of solution of the considered H_∞ cheap control problem are proposed. The first one is based on an asymptotic solution of a set of Riccati-type matrix equations arising in the solvability conditions for the H_∞ control problem. Using this asymptotic solution, a simplified controller, solving the H_∞ cheap control problem, is constructed. The second method is a direct method of solution of this problem, which does not use its solvability conditions. This method is based on: (i) an equivalent transformation of the H_∞ cheap control problem to a new H_∞ problem for a singularly perturbed controlled system; (ii) an asymptotic decomposition of the resulting problem into two much simpler parameter-free subproblems, the slow and fast ones. It should be noted that the fast state variable of the new H_∞ control problem becomes a control in the slow subproblem. The slow subproblem is an H_∞ control problem for a system with state delays. The fast subproblem does not contain delays, and it is solved analytically. Using controllers, solving the slow and fast subproblems, a composite controller, solving the transformed problem, is designed. The latter yields a controller, solving the original H_∞ cheap control problem.

Several works, related to the present paper, should be mentioned. Thus, in [8, 9, 23, 30, 31, 37, 40], the H_∞ control problem for singularly perturbed systems without delays was studied. The H_∞ control problem for systems with small delays in either the state variable or in the state and control variables was investigated in [11, 12, 27]. The H_∞ control problem for singularly perturbed systems with small state delays was analyzed in [15, 19]. In [10], the robust sampled-data H_∞ control for a singularly perturbed linear uncertain system was studied. Cheap suboptimal control of an integral sliding mode for uncertain systems with state delays and matched bounded uncertainties was analyzed in [20].

The paper is organized as follows. The next section is devoted to a rigorous problem formulation. In Section 3, an asymptotic solution of the set of Riccati-type equations, arising in the H_∞ problem solvability conditions, is constructed and justified. Parameter-free solvability conditions of the original H_∞ control problem are derived in Section 4. In Section 5, a simplified controller, solving the original H_∞ control problem, is designed and justified by using the asymptotic solution of the set of Riccati-type equations obtained in Section 3. In Section 6, an auxiliary lemma, formulated in Section 5, is proved. The direct method of constructing a controller, solving the original H_∞ problem, is described in Section 7. Concluding remarks are presented in Section 8.

The following main notations are applied in the paper:
(1) E^n is the n-dimensional real Euclidean space;
(2) $\|\cdot\|$ denotes the Euclidean norm either of a vector or of a matrix;
(3) the prime denotes the transposition of a matrix $A, (A')$ or of a vector $x, (x')$;
(4) $L^2[b,c;E^n]$ is the Hilbert space of n-dimensional vector-valued functions $v(t)$ defined, measurable and square-integrable on the interval $[b,c)$, the inner product in this space is $(v(\cdot),w(\cdot))_{L^2} = \int_b^c v'(t)w(t)dt$, and the norm is $\|v(\cdot)\|_{L^2} = \sqrt{(v(\cdot),v(\cdot))_{L^2}}$;
(5) I_n is the n-dimensional identity matrix;
(6) $\mathrm{col}(x,y)$, where $x \in E^n, y \in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x,y) = (x', y')'$.

2. Problem Formulation

2.1. H_∞ Cheap Control Problem.

Consider the controlled system

$$dx(t)/dt = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \tag{2.1}$$

$$dy(t)/dt = A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h)$$
$$+ \int_{-h}^{0} G_{21}(\tau)x(t+\tau)d\tau + Bu(t) + F_2 w(t), \tag{2.2}$$

where $t > 0$; $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^m$, (u is a control), $w(t) \in E^q$, (w is a disturbance); $h > 0$ is a given constant time delay; A_{ij}, $(i,j = 1,2)$, H_{i1}, $G_{i1}(\tau)$, F_i, $(i = 1,2)$ and B are given time-invariant matrices of corresponding dimensions; B has the full rank; the matrix-valued functions $G_{i1}(\tau)$, $(i = 1,2)$ are piece-wise continuous for $\tau \in [-h, 0]$.

Assuming that $w(t) \in L^2[0,+\infty;E^q]$, we consider the following functional

$$J_\varepsilon(u,w) = \int_0^{+\infty} \Big[x'(t)D_1x(t) + y'(t)D_2y(t) + \varepsilon^2 u'(t)u(t) - \gamma^2 w'(t)w(t)\Big]dt, \tag{2.3}$$

where D_1 is symmetric positive-semi-definite, while D_2 is symmetric positive-definite matrices; $\gamma > 0$ is a given constant; ε is a small positive parameter.

The H_∞ control problem with a performance level γ for the system (2.1)-(2.2) is to find a controller $u^*[x(\cdot),y(\cdot)](t)$ that internally stabilizes this system and ensures the inequality $J_\varepsilon(u^*,w) \le 0$ along trajectories of (2.1)-(2.2) for all $w(t) \in L^2[0,+\infty;E^q]$ and for $x(t) = 0,\ t \le 0,\ y(0) = 0$. The presence of a small multiplier ε^2 in the control cost of the functional (2.3) means that this problem is the H_∞ cheap control problem.

By the control transformation

$$u(t) = (1/\varepsilon)v(t), \tag{2.4}$$

where v is a new control, this H_∞ cheap control problem becomes

$$dx(t)/dt = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \tag{2.5}$$

$$\varepsilon dy(t)/dt = \varepsilon \left\{ A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h) + \int_{-h}^{0} G_{21}(\tau)x(t+\tau)d\tau \right\}$$

$$+Bv(t) + \varepsilon F_2 w(t), \quad t > 0, \tag{2.6}$$

$$x(0) = 0, \quad t \le 0; \qquad y(0) = 0, \tag{2.7}$$

$$J(v,w) = \int_0^{+\infty} \left[x^{'}(t)D_1 x(t) + y^{'}(t)D_2 y(t) + v^{'}(t)v(t) - \gamma^2 w^{'}(t)w(t) \right] dt. \tag{2.8}$$

It should be noted that the system (2.5)-(2.6) is singularly perturbed [24]. The state variables $x(\cdot)$ and $y(\cdot)$ are the slow and fast ones, respectively. It is seen that in this system, the slow state variable is with a delay, while the fast state variable is delay free.

In the sequel, we deal with the H_∞ control problem consisting of the system (2.5)-(2.6), the initial conditions (2.7) and the cost functional (2.8). This problem is called the original H_∞ control problem (OHICP). It is clear that once a controller of the OHICP is obtained, the respective controller of the H_∞ problem (2.1)-(2.2),(2.3) is obtained directly by using the equation (2.4).

2.2. **Solvability Conditions.** Consider the following $(n+m) \times (n+m)$-matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix}, \quad G(\tau) = \begin{pmatrix} G_{11}(\tau) & 0 \\ G_{21}(\tau) & 0 \end{pmatrix}, \tag{2.9}$$

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad S(\varepsilon) = \gamma^{-2}\mathcal{F}\mathcal{F}^{'} - \varepsilon^{-2}\mathcal{B}\mathcal{B}^{'}, \tag{2.10}$$

where

$$\mathcal{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}. \tag{2.11}$$

By using (2.10) and (2.11), the matrix $S(\varepsilon)$ can be represented in the block form

$$S(\varepsilon) = \begin{pmatrix} \gamma^{-2}F_1F_1^{'} & \gamma^{-2}F_1F_2^{'} \\ \gamma^{-2}F_2F_1^{'} & \gamma^{-2}F_2F_2^{'} - \varepsilon^{-2}BB^{'} \end{pmatrix} \stackrel{\triangle}{=} \begin{pmatrix} S_1 & S_2 \\ S_2^{'} & S_3(\varepsilon) \end{pmatrix}. \tag{2.12}$$

Consider the following hybrid set of Riccati-type algebraic, ordinary differential and partial differential equations for the matrices P, $Q(\tau)$ and $R(\tau,\rho)$ in the domain $\mathcal{D} = \{(\tau,\rho) : -h \le \tau \le 0, \ -h \le \rho \le 0\}$:

$$PA + A^{'}P + PS(\varepsilon)P + Q(0) + Q^{'}(0) + D = 0, \tag{2.13}$$

$$dQ(\tau)/d\tau = \left(A + S(\varepsilon)P\right)^{'} Q(\tau) + PG(\tau) + R(0,\tau), \tag{2.14}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R(\tau,\rho) = G^{'}(\tau)Q(\rho) + Q^{'}(\tau)G(\rho) + Q^{'}(\tau)S(\varepsilon)Q(\rho). \tag{2.15}$$

The matrices $Q(\tau)$ and $R(\tau,\rho)$ satisfy the boundary conditions

$$Q(-h) = PH, \quad R(-h,\tau) = H^{'}Q(\tau), \quad R(\tau,-h) = Q^{'}(\tau)H. \tag{2.16}$$

It is seen that the matrix-valued functions $Q(\tau)$ and $R(\tau,\rho)$ are present in the set (2.13)-(2.15) with deviating arguments. The problem (2.13)-(2.16) is, in general, of a high dimension. Moreover, due to the expression for $S(\varepsilon)$ (see (2.12)), this problem is ill-posed for $\varepsilon \to +0$.

Let, for some $\varepsilon > 0$, the triplet $\{P(\varepsilon), Q(\tau,\varepsilon), R(\tau,\rho,\varepsilon)\}$ be a solution of (2.13)-(2.16) in the domain $\mathcal{D}$. Consider the linear systems

$$dz(t)/dt = [A - \varepsilon^{-2}\mathcal{B}\mathcal{B}^{'}P(\varepsilon)]z(t) + Hz(t-h)$$

$$+ \int_{-h}^{0}[G(\tau) - \varepsilon^{-2}\mathcal{B}\mathcal{B}^{'}Q(\tau,\varepsilon)]z(t+\tau)d\tau, \quad t > 0, \tag{2.17}$$

In the sequel, we call the system (2.17) to be exponentially stable for a given $\varepsilon > 0$, if for this ε, and any given $\varphi_z(\tau) \in L^2[-h,0;E^{n+m}]$ and $\varphi_0 \in E^{n+m}$, its solution $z(t,\varepsilon)$ with the initial conditions

$$z(\tau) = \varphi_z(\tau), \quad \tau \in [-h,0); \qquad z(0) = \varphi_0 \tag{2.18}$$

satisfies the inequality

$$\|z(t,\varepsilon)\| \le c(\varepsilon)\exp(-\mu(\varepsilon)t)\Big(\|\varphi_0\| + \|\varphi_z\|_{L^2}\Big), \qquad t \ge 0, \tag{2.19}$$

where $c(\varepsilon) > 0$ and $\mu(\varepsilon) > 0$ are some constants.

Remark 2.1. Note that, by virtue of [6] (Theorem 5.3), the system (2.17) is exponentially stable for a given $\varepsilon > 0$, if and only if all roots $\lambda = \lambda(\varepsilon)$ of its characteristic equation

$$\det\Big[A - \varepsilon^{-2}\mathcal{B}\mathcal{B}^{'}P(\varepsilon) + \exp(-\lambda h)H$$

$$+ \int_{-h}^{0}\exp(\lambda\tau)[G(\tau) - \varepsilon^{-2}\mathcal{B}\mathcal{B}^{'}Q(\tau,\varepsilon)]d\tau - \lambda I_{n+m}\Big] = 0 \tag{2.20}$$

lie inside the left-hand half-plane.

Lemma 2.2. *Let, for a given $\varepsilon > 0$, there exist a solution $\{P(\varepsilon), Q(\tau,\varepsilon), R(\tau,\rho,\varepsilon)\}$ of (2.13)-(2.16) such that*

$$P^{'}(\varepsilon) = P(\varepsilon), \qquad R^{'}(\tau,\rho,\varepsilon) = R(\rho,\tau,\varepsilon), \tag{2.21}$$

and the system (2.17) is exponentially stable. Then, for this ε, the controller

$$v^*[x(\cdot),y(\cdot)](t) = -\varepsilon^{-1}\mathcal{B}^{'}\left[P(\varepsilon)z(t) + \int_{-h}^{0}Q(\tau,\varepsilon)z(t+\tau)d\tau\right], \quad z = col(x,y) \tag{2.22}$$

solves the OHICP.

Proof. The lemma is a direct technical extension of the result of [11] (Lemma 1 and its proof) where the case of only a point-wise state delay in the controlled system has been considered. □

2.3. **Objectives of the paper.** The objectives of this paper are the following:
(i) to construct and justify an asymptotic solution of the set (2.13)-(2.16);
(ii) to derive ε-free conditions, which guarantee the existence of the controller (2.22) solving the OHICP for all sufficiently small $\varepsilon > 0$;
(iii) to obtain a controller much simpler than (2.22), which is constructed independently of ε while solves the OHICP for all sufficiently small $\varepsilon > 0$.

3. Zero-order asymptotic solution of (2.13)-(2.16)

3.1. **Transformation of (2.13)-(2.16).** In order to remove the singularities at $\varepsilon = 0$ from the right-hand sides of the equations (2.13)-(2.15), we represent the solution $\{P(\varepsilon), Q(\tau,\varepsilon), R(\tau,\rho,\varepsilon)\}$ of (2.13)-(2.16) in the block form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P_2'(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \qquad Q(\tau,\varepsilon) = \begin{pmatrix} Q_1(\tau,\varepsilon) & Q_2(\tau,\varepsilon) \\ \varepsilon Q_3(\tau,\varepsilon) & \varepsilon Q_4(\tau,\varepsilon) \end{pmatrix}, \tag{3.1}$$

$$R(\tau,\rho,\varepsilon) = \begin{pmatrix} R_1(\tau,\rho,\varepsilon) & R_2(\tau,\rho,\varepsilon) \\ R_2'(\rho,\tau,\varepsilon) & R_3(\tau,\rho,\varepsilon) \end{pmatrix}, \tag{3.2}$$

where $P_j(\varepsilon),\ R_j(\tau,\rho,\varepsilon),\ (j = 1,2,3)$ are matrices of the dimensions $n\times n,\ n\times m,\ m\times m$, respectively; $Q_i(\tau,\varepsilon),\ (i = 1,...,4)$ are matrices of the dimensions $n\times n,\ n\times m,\ m\times n,\ m\times m$, respectively.

By substituting (3.1)-(3.2), as well as the block representations for the matrices A, H, $G(\tau)$, $S(\varepsilon)$ and D (see (2.9),(2.10),(2.12)) into (2.13)-(2.16), one obtains the following system (in this new system, for simplicity, we omit the designation of the dependence of the unknown matrices on ε):

$$P_1A_{11} + A_{11}'P_1 + \varepsilon P_2A_{21} + \varepsilon A_{21}'P_2' + P_1S_1P_1 + \varepsilon P_2S_2'P_1 + \varepsilon P_1S_2P_2'$$
$$+\varepsilon^2 P_2S_3(\varepsilon)P_2' + Q_1(0) + Q_1'(0) + D_1 = 0, \tag{3.3}$$

$$P_1A_{12} + \varepsilon P_2A_{22} + \varepsilon A_{11}'P_2 + \varepsilon A_{21}'P_3 + \varepsilon P_1S_1P_2 + \varepsilon^2 P_2S_2'P_2 + \varepsilon P_1S_2P_3$$
$$+\varepsilon^2 P_2S_3(\varepsilon)P_3 + Q_2(0) + \varepsilon Q_3'(0) = 0, \tag{3.4}$$

$$\varepsilon P_2'A_{12} + \varepsilon A_{12}'P_2 + \varepsilon P_3A_{22} + \varepsilon A_{22}'P_3 + \varepsilon^2 P_2'S_1P_2 + \varepsilon^2 P_3S_2'P_2 + \varepsilon^2 P_2'S_2P_3$$
$$+\varepsilon^2 P_3S_3(\varepsilon)P_3 + \varepsilon Q_4(0) + \varepsilon Q_4'(0) + D_2 = 0, \tag{3.5}$$

$$dQ_1(\tau)/d\tau = A_{11}'Q_1(\tau) + \varepsilon A_{21}'Q_3(\tau) + P_1S_1Q_1(\tau) + \varepsilon P_2S_2'Q_1(\tau)$$
$$+\ \varepsilon P_1S_2Q_3(\tau) + \varepsilon^2 P_2S_3(\varepsilon)Q_3(\tau) + P_1G_{11}(\tau) + \varepsilon P_2G_{21}(\tau) + R_1(0,\tau), \tag{3.6}$$

$$dQ_2(\tau)/d\tau = A_{11}'Q_2(\tau) + \varepsilon A_{21}'Q_4(\tau) + P_1S_1Q_2(\tau) + \varepsilon P_2S_2'Q_2(\tau)$$
$$+\varepsilon P_1S_2Q_4(\tau) + \varepsilon^2 P_2S_3(\varepsilon)Q_4(\tau) + R_2(0,\tau), \tag{3.7}$$

$$\varepsilon dQ_3(\tau)/d\tau = A_{12}'Q_1(\tau) + \varepsilon A_{22}'Q_3(\tau) + \varepsilon P_2'S_1Q_1(\tau) + \varepsilon P_3S_2'Q_1(\tau)$$
$$+\varepsilon^2 P_2'S_2Q_3(\tau) + \varepsilon^2 P_3S_3(\varepsilon)Q_3(\tau) + \varepsilon P_2'G_{11}(\tau) + \varepsilon P_3G_{21}(\tau) + R_2'(\tau,0), \tag{3.8}$$

$$\varepsilon dQ_4(\tau)/d\tau = A_{12}'Q_2(\tau) + \varepsilon A_{22}'Q_4(\tau) + \varepsilon P_2'S_1Q_2(\tau) + \varepsilon P_3S_2'Q_2(\tau)$$
$$+\varepsilon^2 P_2'S_2Q_4(\tau) + \varepsilon^2 P_3S_3(\varepsilon)Q_4(\tau) + R_3(0,\tau), \tag{3.9}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_1(\tau,\rho) = G_{11}'(\tau)Q_1(\rho) + Q_1'(\tau)G_{11}(\rho)$$
$$+\varepsilon G_{21}'(\tau)Q_3(\rho) + \varepsilon Q_3'(\tau)G_{21}(\rho) + Q_1'(\tau)S_1Q_1(\rho)$$
$$+\varepsilon Q_3'(\tau)S_2'Q_1(\rho) + \varepsilon Q_1'(\tau)S_2Q_3(\rho) + \varepsilon^2 Q_3'(\tau)S_3(\varepsilon)Q_3(\rho), \tag{3.10}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_2(\tau,\rho) = G_{11}'(\tau)Q_2(\rho) + \varepsilon G_{21}'(\tau)Q_4(\rho) + Q_1'(\tau)S_1Q_2(\rho)$$
$$+\varepsilon Q_3'(\tau)S_2'Q_2(\rho) + \varepsilon Q_1'(\tau)S_2Q_4(\rho) + \varepsilon^2 Q_3'(\tau)S_3(\varepsilon)Q_4(\rho), \tag{3.11}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_3(\tau,\rho) = Q_2'(\tau)S_1Q_2(\rho) + \varepsilon Q_4'(\tau)S_2'Q_2(\rho)$$
$$+\varepsilon Q_2'(\tau)S_2Q_4(\rho) + \varepsilon^2 Q_4'(\tau)S_3(\varepsilon)Q_4(\rho), \tag{3.12}$$

$$Q_1(-h) = P_1H_{11} + \varepsilon P_2H_{21}, \qquad Q_2(-h) = 0, \tag{3.13}$$

$$Q_3(-h) = P_2'H_{11} + P_3H_{21}, \qquad Q_4(-h) = 0, \tag{3.14}$$

$$R_1(-h,\tau) = H_{11}'Q_1(\tau) + \varepsilon H_{21}'Q_3(\tau), \quad R_1(\tau,-h) = Q_1'(\tau)H_{11} + \varepsilon Q_3'(\tau)H_{21}, \tag{3.15}$$

$$R_2(-h,\tau) = H_{11}'Q_2(\tau) + \varepsilon H_{21}'Q_4(\tau), \qquad R_2(\tau,-h) = 0, \tag{3.16}$$

$$R_3(-h,\tau) = R_3(\tau,-h) = 0. \tag{3.17}$$

It is verified directly that we can set

$$Q_2(\tau) \equiv 0, \quad Q_4(\tau) \equiv 0, \quad R_2(\tau,\rho) \equiv 0, \quad R_3(\tau,\rho) \equiv 0, \quad (\tau,\rho) \in \mathcal{D} \tag{3.18}$$

without a formal contradiction with the system (3.3)-(3.17). In the sequel, we seek the solution of this system satisfying the condition (3.18).

By substitution (3.18) into (3.3)-(3.17), the latter is reduced to the system

$$P_1A_{11} + A_{11}'P_1 + \varepsilon P_2A_{21} + \varepsilon A_{21}'P_2' + P_1S_1P_1 + \varepsilon P_2S_2'P_1 + \varepsilon P_1S_2P_2'$$
$$+\varepsilon^2 P_2S_3(\varepsilon)P_2' + Q_1(0) + Q_1'(0) + D_1 = 0, \tag{3.19}$$

$$P_1A_{12} + \varepsilon P_2A_{22} + \varepsilon A_{11}'P_2 + \varepsilon A_{21}'P_3 + \varepsilon P_1S_1P_2 + \varepsilon^2 P_2S_2'P_2 + \varepsilon P_1S_2P_3$$
$$+\varepsilon^2 P_2S_3(\varepsilon)P_3 + \varepsilon Q_3'(0) = 0, \tag{3.20}$$

$$\varepsilon P_2'A_{12} + \varepsilon A_{12}'P_2 + \varepsilon P_3A_{22} + \varepsilon A_{22}'P_3 + \varepsilon^2 P_2'S_1P_2 + \varepsilon^2 P_3S_2'P_2 + \varepsilon^2 P_2'S_2P_3$$
$$+\varepsilon^2 P_3S_3(\varepsilon)P_3 + D_2 = 0, \tag{3.21}$$

$$dQ_1(\tau)/d\tau = A_{11}'Q_1(\tau) + \varepsilon A_{21}'Q_3(\tau) + P_1S_1Q_1(\tau) + \varepsilon P_2S_2'Q_1(\tau)$$
$$+ \varepsilon P_1S_2Q_3(\tau) + \varepsilon^2 P_2S_3(\varepsilon)Q_3(\tau) + P_1G_{11}(\tau) + \varepsilon P_2G_{21}(\tau) + R_1(0,\tau), \tag{3.22}$$

$$\varepsilon dQ_3(\tau)/d\tau = A_{12}'Q_1(\tau) + \varepsilon A_{22}'Q_3(\tau) + \varepsilon P_2'S_1Q_1(\tau) + \varepsilon P_3S_2'Q_1(\tau)$$
$$+\varepsilon^2 P_2'S_2Q_3(\tau) + \varepsilon^2 P_3S_3(\varepsilon)Q_3(\tau) + \varepsilon P_2'G_{11}(\tau) + \varepsilon P_3G_{21}(\tau), \tag{3.23}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_1(\tau,\rho) = G_{11}'(\tau)Q_1(\rho) + Q_1'(\tau)G_{11}(\rho)$$
$$+\varepsilon G_{21}'(\tau)Q_3(\rho) + \varepsilon Q_3'(\tau)G_{21}(\rho) + Q_1'(\tau)S_1Q_1(\rho)$$
$$+\varepsilon Q_3'(\tau)S_2'Q_1(\rho) + \varepsilon Q_1'(\tau)S_2Q_3(\rho) + \varepsilon^2 Q_3'(\tau)S_3(\varepsilon)Q_3(\rho), \tag{3.24}$$

$$Q_1(-h) = P_1 H_{11} + \varepsilon P_2 H_{21}, \tag{3.25}$$

$$Q_3(-h) = P_2^{'} H_{11} + P_3 H_{21}, \tag{3.26}$$

$$R_1(-h,\tau) = H_{11}^{'} Q_1(\tau) + \varepsilon H_{21}^{'} Q_3(\tau), \quad R_1(\tau,-h) = Q_1^{'}(\tau) H_{11} + \varepsilon Q_3^{'}(\tau) H_{21}. \tag{3.27}$$

The system (3.19)-(3.27) represents a singularly perturbed boundary-value problem for a hybrid set of equations, which contains matrix algebraic, and ordinary and partial differential equations of Riccati type. Moreover, the unknown matrices $Q_1(\tau)$, $Q_3(\tau)$ and $R_1(\tau,\rho)$ are with deviating arguments in this set. This problem is considered in the domain $\mathcal{D}$ with a non-smooth boundary. In order to construct the asymptotic solution of this problem, we adapt the idea of the boundary function method [39].

3.2. **Formal asymptotic solution of (3.19)-(3.27).** We seek the zero-order asymptotic solution of the problem (3.19)-(3.27) in the form

$$\{\bar{P}_{j0}, Q_{l0}(\tau,\varepsilon), R_{10}(\tau,\rho,\varepsilon)\}, \quad j = 1,2,3, \quad l = 1,3, \tag{3.28}$$

where the matrices $\bar{P}_{j0}$ are independent of ε, while the matrices $Q_{l0}(\tau,\varepsilon)$ and $R_{10}(\tau,\rho,\varepsilon)$ have the form

$$Q_{l0}(\tau,\varepsilon) = \bar{Q}_{l0}(\tau) + Q_{l0}^{\tau}(\eta), \quad l = 1,3, \quad \eta = (\tau+h)/\varepsilon, \tag{3.29}$$

$$R_{10}(\tau,\rho,\varepsilon) = \bar{R}_{10}(\tau,\rho) + R_{10}^{\tau}(\eta,\rho) + R_{10}^{\rho}(\tau,\zeta) + R_{10}^{\tau,\rho}(\eta,\zeta), \quad \zeta = (\rho+h)/\varepsilon. \tag{3.30}$$

Here the terms with the bar are so called outer solution, the terms with the superscript "τ" are the boundary layer correction in a neighborhood of the boundary $\tau = -h$, the term with the superscript "ρ" is the boundary layer correction in a neighborhood of the boundary $\rho = -h$, and the term with the superscript "τ,ρ" is the boundary layer correction in a neighborhood of the corner point $(\tau = -h, \rho = -h)$. Equations and conditions for the asymptotic solution are obtained by substituting (3.28),(3.29) and (3.30) into (3.19)-(3.27) and equating coefficients for the same power of ε on both sides of the resulting equations, separately for the outer solution and for the boundary layer corrections of each type. The boundary layer corrections are assumed (in accordance with the boundary function method [39]) to be considerable only in small neighborhoods of the respective boundaries. Such an assumption on the behavior of each boundary layer correction, yields an additional condition for its obtaining.

3.3. **Obtaining $Q_{10}^{\tau}(\eta)$ and $R_{10}^{\tau}(\eta,\rho)$, $R_{10}^{\rho}(\tau,\zeta)$, $R_{10}^{\tau,\rho}(\eta,\zeta)$.** Due to the above mentioned procedure of obtaining equations for the terms of the asymptotic solution, we obtain the following equation for $Q_{10}^{\tau}(\eta)$:

$$dQ_{10}^{\tau}(\eta)/d\eta = 0, \quad \eta \geq 0. \tag{3.31}$$

In order to obtain a single solution of this equation, we need an additional condition. By such a condition, we use (due to the boundary function method [39]) a reasonable requirement that the boundary layer correction is considerable only in some right-hand neighborhood of $\eta = 0$, and it tends to zero while $\eta \to +\infty$, i.e.,

$$\lim_{\eta\to+\infty} Q_{10}^{\tau}(\eta) = 0. \tag{3.32}$$

Using this requirement, one directly has from (3.31)

$$Q^{\tau}_{10}(\eta) = 0 \quad \forall \eta \geq 0. \tag{3.33}$$

For $R^{\tau}_{10}(\eta,\rho)$, $R^{\rho}_{10}(\tau,\zeta)$, $R^{\tau,\rho}_{10}(\eta,\zeta)$, the following equations are obtained:

$$\partial R^{\tau}_{10}(\eta,\rho)/\partial\eta = 0, \quad \eta \geq 0, \tag{3.34}$$

$$\partial R^{\rho}_{10}(\tau,\zeta)/\partial\zeta = 0, \quad \zeta \geq 0, \tag{3.35}$$

$$(\partial/\partial\eta + \partial/\partial\zeta) R^{\tau,\rho}_{10}(\eta,\zeta) = 0, \quad \eta \geq 0, \quad \zeta \geq 0. \tag{3.36}$$

To obtain single solutions of these equations, we use (similarly to (3.32)) the additional conditions

$$\lim_{\eta \to +\infty} R^{\tau}_{10}(\eta,\rho) = 0, \quad \rho \in [-h, 0], \tag{3.37}$$

$$\lim_{\zeta \to +\infty} R^{\rho}_{10}(\tau,\zeta) = 0, \quad \tau \in [-h, 0], \tag{3.38}$$

$$\lim_{\eta+\zeta \to +\infty} R^{\tau,\rho}_{10}(\eta,\zeta) = 0. \tag{3.39}$$

The equations (3.34)-(3.36) subject to the conditions (3.37)-(3.39) yield the unique solutions

$$R^{\tau}_{10}(\eta,\rho) = 0 \quad \forall (\eta,\rho) \in [0,+\infty) \times [-h,0], \tag{3.40}$$

$$R^{\rho}_{10}(\tau,\zeta) = 0 \quad \forall (\tau,\zeta) \in [-h,0] \times [0,+\infty), \tag{3.41}$$

$$R^{\tau,\rho}_{10}(\eta,\zeta) = 0 \quad \forall (\eta,\zeta) \in [0,+\infty) \times [0,+\infty). \tag{3.42}$$

3.4. **Obtaining the outer solution.**

3.4.1. *Equations and conditions for the outer solution.* By using (2.12), we have the following equations and conditions for the outer solution in the domain $\mathcal{D}$:

$$\begin{aligned}\bar{P}_{10}A_{11} + A'_{11}\bar{P}_{10} + \gamma^{-2}\bar{P}_{10}F_1F'_1\bar{P}_{10} - \bar{P}_{20}BB'\bar{P}'_{20}\\ +\bar{Q}_{10}(0) + \bar{Q}'_{10}(0) + D_1 = 0,\end{aligned} \tag{3.43}$$

$$\bar{P}_{10}A_{12} - \bar{P}_{20}BB'\bar{P}_{30} = 0, \tag{3.44}$$

$$-\bar{P}_{30}BB'\bar{P}_{30} + D_2 = 0, \tag{3.45}$$

$$\begin{aligned}d\bar{Q}_{10}(\tau)/d\tau = A'_{11}\bar{Q}_{10}(\tau) + \gamma^{-2}\bar{P}_{10}F_1F'_1\bar{Q}_{10}(\tau) - \bar{P}_{20}BB'\bar{Q}_{30}(\tau)\\ +\bar{P}_{10}G_{11}(\tau) + \bar{R}_{10}(0,\tau),\end{aligned} \tag{3.46}$$

$$A'_{12}\bar{Q}_{10}(\tau) - \bar{P}_{30}BB'\bar{Q}_{30}(\tau) = 0, \tag{3.47}$$

$$\begin{aligned}(\partial/\partial\tau + \partial/\partial\rho)\bar{R}_{10}(\tau,\rho) = G'_{11}(\tau)\bar{Q}_{10}(\rho) + \bar{Q}'_{10}(\tau)G_{11}(\rho)\\ +\gamma^{-2}\bar{Q}'_{10}(\tau)F_1F'_1\bar{Q}_{10}(\rho) - \bar{Q}'_{30}(\tau)BB'\bar{Q}_{30}(\rho),\end{aligned} \tag{3.48}$$

$$\bar{Q}_{10}(-h) = \bar{P}_{10}H_{11}, \tag{3.49}$$

$$\bar{R}_{10}(-h,\tau) = H'_{11}\bar{Q}_{10}(\tau), \quad \bar{R}_{10}(\tau,-h) = \bar{Q}'_{10}(\tau)H_{11}. \tag{3.50}$$

Since the matrix B is invertible and the matrix D_2 is positive definite, then, due to [41], the equation (3.45) has the following unique symmetric positive definite solution

$$\bar{P}_{30} = (BB^{'})^{-1/2}\Big((BB^{'})^{1/2}D_2(BB^{'})^{1/2}\Big)^{1/2}(BB^{'})^{-1/2}, \tag{3.51}$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of respective symmetric positive definite matrix, the one "-1/2" denotes the square root of respective inverse matrix.

The equations (3.44) and (3.47) yield, respectively,

$$\bar{P}_{20} = \bar{P}_{10}A_{12}\alpha^{-1}, \tag{3.52}$$

and

$$\bar{Q}_{30}(\tau) = (\alpha^{'})^{-1}A_{12}^{'}\bar{Q}_{10}(\tau). \tag{3.53}$$

where

$$\alpha \stackrel{\triangle}{=} BB^{'}\bar{P}_{30} = (BB^{'})^{1/2}\Big((BB^{'})^{1/2}D_2(BB^{'})^{1/2}\Big)^{1/2}(BB^{'})^{-1/2}. \tag{3.54}$$

Since D_2 is positive definite, all eigenvalues of α are real positive.

Eliminating $\bar{P}_{20}$ and $\bar{Q}_{30}(\tau)$ from the equations (3.43),(3.46) and (3.48) by using (3.51) and (3.52)-(3.53) yields the following set of equations

$$\bar{P}_{10}A_{11} + A_{11}^{'}\bar{P}_{10} + \bar{P}_{10}\bar{S}\bar{P}_{10} + \bar{Q}_{10}(0) + \bar{Q}_{10}^{'}(0) + D_1 = 0, \tag{3.55}$$

$$d\bar{Q}_{10}(\tau)/d\tau = A_{11}^{'}\bar{Q}_{10}(\tau) + \bar{P}_{10}\bar{S}\bar{Q}_{10}(\tau) + \bar{P}_{10}G_{11}(\tau) + \bar{R}_{10}(0,\tau), \tag{3.56}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\bar{R}_{10}(\tau,\rho) = G_1^{'}(\tau)\bar{Q}_{10}(\rho) + \bar{Q}_{10}^{'}(\tau)G_1(\rho) + \bar{Q}_{10}^{'}(\tau)\bar{S}\bar{Q}_{10}(\rho), \tag{3.57}$$

where

$$\bar{S} = \gamma^{-2}F_1F_1^{'} - A_{12}D_2^{-1}A_{12}^{'}. \tag{3.58}$$

Thus, in order to obtain the outer solution, one has to solve the system (3.55)-(3.57) with the boundary conditions (3.49)-(3.50).

3.4.2. *Reduced $\boldsymbol{H_\infty}$ Control Problem and Solution of the Problem (3.49)-(3.50), (3.55)-(3.57).* Setting formally $\varepsilon = 0$ in the OHICP, one obtains the following problem, after a simple rearrangement and a redenoting x, y, w and J by $\bar{x}$, $\bar{y}$, $\bar{w}$ and $\bar{J}$, respectively,

$$d\bar{x}(t)/dt = A_{11}\bar{x}(t) + H_{11}\bar{x}(t-h) + \int_{-h}^{0} G_{11}(\tau)\bar{x}(t+\tau)d\tau$$

$$+A_{12}\bar{y}(t) + F_1\bar{w}(t), \quad t > 0, \tag{3.59}$$

$$\bar{x}(t) = 0, \quad t \leq 0. \tag{3.60}$$

$$\bar{J} \stackrel{\triangle}{=} \int_0^{+\infty} \Big[\bar{x}^{'}(t)D_1\bar{x}(t) + \bar{y}^{'}(t)D_2\bar{y}(t) - \gamma^2\bar{w}^{'}(t)\bar{w}(t)\Big]. \tag{3.61}$$

Since the variable $\bar{y}(t)$ does not satisfy any equation for $t \in [0, +\infty)$, one can choose it to satisfy a desirable property of the system (3.59). This means that the variable $\bar{y}(t)$ can be considered as a control variable in the system (3.59). Thus, the functional (3.61), calculated along trajectories of this system, depends on the control variable $\bar{y}(t)$ and the disturbance $\bar{w}(t) \in L^2[0, +\infty; E^q]$, i.e., $\bar{J} = \bar{J}(\bar{y}, \bar{w})$. For the system (3.59), the H_∞ control problem with a performance level γ can be formulated. Namely, to find a controller $\bar{y}^*[x(\cdot)](t)$ that internally stabilizes this system and ensures the inequality $\bar{J}(\bar{y}^*, \bar{w}) \leq 0$ along trajectories of (3.59)-(3.60) for all $\bar{w}(t) \in L_2[0, +\infty; E^q]$. This H_∞ control problem is called the reduced H_∞ control problem (RHICP) associated with the OHICP.

Let the triplet $\bar{\mathcal{S}} \triangleq \left\{\bar{P}_{10}, \bar{Q}_{10}(\tau), \bar{R}_{10}(\tau, \rho)\right\}$ be a solution of the problem (3.49)-(3.50), (3.55)-(3.57) in the domain $\mathcal{D}$.

Consider the linear systems

$$d\bar{x}(t)/dt = (A_{11} - A_{12}D_2^{-1}A_{12}'\bar{P}_{10})\bar{x}(t) + H_{11}\bar{x}(t-h)$$

$$+ \int_{-h}^{0} [G_{11}(\tau) - A_{12}D_2^{-1}A_{12}'\bar{Q}_{10}(\tau)]\bar{x}(t+\tau)d\tau, \quad t > 0, \tag{3.62}$$

$$d\bar{x}(t) = (A_{11} + \bar{S}\bar{P}_{10})\bar{x}(t) + H_{11}\bar{x}(t-h)$$

$$+ \int_{-h}^{0} [G_{11}(\tau) + \bar{S}\bar{Q}_{10}(\tau)]\bar{x}(t+\tau)d\tau, \quad t > 0. \tag{3.63}$$

In the sequel, we assume:

A1. The problem (3.49)-(3.50), (3.55)-(3.57) has a solution $\bar{\mathcal{S}}$ in the domain $\mathcal{D}$ such that $\bar{P}_{10}' = \bar{P}_{10}$, $\bar{R}_{10}'(\tau, \rho) = \bar{R}_{10}(\rho, \tau)$, and:

(a) the system (3.62) is exponentially stable, i.e. (see [6], Theorem 5.3), all roots λ of its characteristic equation

$$\det\Big[A_{11} - A_{12}D_2^{-1}A_{12}'\bar{P}_{10} + \exp(-\lambda h)H_{11}$$

$$+ \int_{-h}^{0} \exp(\lambda\tau)[G_{11}(\tau) - A_{12}D_2^{-1}A_{12}'\bar{Q}_{10}(\tau)]d\tau - \lambda I_n\Big] = 0 \tag{3.64}$$

lie inside the left-hand half-plane;

(b) the system (3.63) is exponentially stable, i.e., all roots λ of its characteristic equation

$$\det\Big[A_{11} + \bar{S}\bar{P}_{10} + \exp(-\lambda h)H_{11}$$

$$+ \int_{-h}^{0} \exp(\lambda\tau)[G_{11}(\tau) + \bar{S}\bar{Q}_{10}(\tau)]d\tau - \lambda I_n\Big] = 0 \tag{3.65}$$

lie inside the left-hand half-plane.

Similarly to Lemma 2.2, one directly obtains the following lemma.

Lemma 3.1. *Under the assumption A1 (item a), the controller*

$$\bar{y}^*[\bar{x}(\cdot)](t) = -D_2^{-1}A_{12}'\left[\bar{P}_{10}\bar{x}(t) + \int_{-h}^{0} \bar{Q}_{10}(\tau)\bar{x}(t+\tau)d\tau\right] \tag{3.66}$$

solves the RHICP.

3.5. **Obtaining** $Q_{30}^{\tau}(\eta)$**.** Using (3.33), (3.40)-(3.42) and (3.54), we obtain the following differential equation and initial condition for $Q_{30}^{\tau}(\eta)$:

$$dQ_{30}^{\tau}(\eta)/d\eta = -\alpha^{'} Q_{30}^{\tau}(\eta), \quad \eta \geq 0. \tag{3.67}$$

$$Q_{30}^{\tau}(0) = \bar{P}_{20}^{'} H_{11} + \bar{P}_{30} H_{21} - \bar{Q}_{30}(-h). \tag{3.68}$$

Similarly to [18], by using the equations (3.49) and (3.52)-(3.53), one can transform (3.68) as follows

$$Q_{30}^{\tau}(0) = \bar{P}_{30} H_{21}. \tag{3.69}$$

The initial-value problem (3.67),(3.69) has the unique solution

$$Q_{30}^{\tau}(\eta) = \exp(-\alpha^{'}\eta)\bar{P}_{30} H_{21}, \quad \eta \geq 0. \tag{3.70}$$

Since all eigenvalues of the matrix α are real positive, this solution satisfies the inequality

$$\left\| Q_{30}^{\tau}(\eta) \right\| \leq a \exp(-\beta\eta), \quad \eta \geq 0, \tag{3.71}$$

where $a > 0$ and $\beta > 0$ are some constants.

The inequality (3.71) means that the boundary layer correction $Q_{30}^{\tau}(\eta)$ is considerable only in some right-hand neighborhood of $\eta = 0$, and it is exponentially decaying for $\eta \to +\infty$.

3.6. **Justification of the Asymptotic Solution.**

Theorem 3.2. *Under the assumption A1 (item b), there exists a positive number* ε_1^* *such that, for all* $\varepsilon \in (0, \varepsilon_1^*]$*, the problem (3.19)-(3.27) has a solution* $\{P_j(\varepsilon), Q_l(\tau,\varepsilon), R_1(\tau,\rho,\varepsilon),\ j = 1,2,3,\ l = 1,3\}$ *in the domain* $\mathcal{D}$*. For all* $(\tau,\rho,\varepsilon) \in \mathcal{D} \times (0, \varepsilon_1^*]$*, this solution satisfies the symmetry properties*

$$P_1^{'}(\varepsilon) = P_1(\varepsilon), \quad P_3^{'}(\varepsilon) = P_3(\varepsilon), \quad R_1^{'}(\tau,\rho,\varepsilon) = R_1(\rho,\tau,\varepsilon), \tag{3.72}$$

and the inequalities

$$\left\| P_j(\varepsilon) - \bar{P}_{j0} \right\| \leq a\varepsilon, \quad j = 1,2,3, \qquad \left\| R_1(\tau,\rho,\varepsilon) - \bar{R}_{10}(\tau,\rho) \right\| \leq a\varepsilon, \tag{3.73}$$

$$\left\| Q_1(\tau,\varepsilon) - \bar{Q}_{10}(\tau) \right\| \leq a\varepsilon, \qquad \left\| Q_3(\tau,\varepsilon) - Q_{30}(\tau,\varepsilon) \right\| \leq a\varepsilon, \tag{3.74}$$

where $a > 0$ *is some constant independent of* ε*.*

Proof. The theorem is proved very similarly to [18] (Theorem 3.1). □

Theorem 3.2 directly yields the following corollary.

Corollary 3.3. *Under the assumption A1 (item b), for all* $\varepsilon \in (0, \varepsilon_1^*]$*, the problem (3.3)-(3.17) has a solution* $\{P_j(\varepsilon), Q_i(\tau,\varepsilon), R_j(\tau,\rho,\varepsilon),\ j = 1,2,3,\ i = 1,...,4\}$*. The components* $Q_k(\tau,\varepsilon)$*,* $(k = 2,4)$ *and* $R_l(\tau,\rho,\varepsilon)$*,* $(l = 2,3)$ *of this solution satisfy (3.18). The other components of this solution constitute the solution of the problem (3.19)-(3.27) mentioned in Theorem 3.2.*

4. ε-FREE SOLVABILITY CONDITIONS FOR THE OHICP

Consider the controller (2.22) with $P(\varepsilon)$ and $Q(\tau, \varepsilon)$ given by (3.1) where $P_j(\varepsilon)$ $(j = 1, 2, 3)$ and $Q_i(\tau, \varepsilon)$, $(i = 1, ..., 4)$ are the respective components of the solution to the problem (3.3)-(3.17) mentioned in Corollary 3.3.

By substituting the block form of $\mathcal{B}$, $P(\varepsilon)$ and $Q(\tau, \varepsilon)$ (see (2.11),(3.1)) into (2.22) and using (3.18), one obtains after a simple algebra

$$v^*[x(\cdot), y(\cdot)](t) = -B'\Big[P_2'(\varepsilon)x(t) + P_3(\varepsilon)y(t) + \int_{-h}^{0} Q_3(\tau, \varepsilon)x(t+\tau)d\tau\Big]. \tag{4.1}$$

Lemma 4.1. *Let the assumption A1 (item a) be valid. Then, there exists a positive constant* ε_2^*, $(\varepsilon_2^* \le \varepsilon_1^*)$ *such that the system (2.17) is exponentially stable uniformly with respect to* $\varepsilon \in (0, \varepsilon_2^*]$.

Proof. Substituting the block form of $\mathcal{B}$, $P(\varepsilon)$ and $Q(\tau, \varepsilon)$ (see (2.11),(3.1)), as well as $z = \mathrm{col}(x, y)$, into (2.17) and using (3.18) yield after some rearrangement

$$dx(t)/dt = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau, \quad t > 0, \tag{4.2}$$

$$\varepsilon dy(t)/dt = [\varepsilon A_{21} - BB'P_2'(\varepsilon)]x(t) + [\varepsilon A_{22} - BB'P_3(\varepsilon)]y(t)$$

$$+\varepsilon H_{21}x(t-h) + \int_{-h}^{0} [\varepsilon G_{21}(\tau) - BB'Q_3(\tau, \varepsilon)]x(t+\tau)d\tau, \quad t > 0. \tag{4.3}$$

By virtue of Theorem 3.2, the matrices $P_2(\varepsilon)$, $P_3(\varepsilon)$ and $Q_3(\tau, \varepsilon)$ can be represented in the following form, valid for all $\varepsilon \in (0, \varepsilon_1^*]$ and $\tau \in [-h, 0]$,

$$P_2(\varepsilon) = \bar{P}_{20} + O_{P2}(\varepsilon), \qquad P_3(\varepsilon) = \bar{P}_{30} + O_{P3}(\varepsilon), \tag{4.4}$$

$$Q_3(\tau, \varepsilon) = \bar{Q}_{30}(\tau) + Q_{30}^{\tau}((\tau + h)/\varepsilon) + O_{Q3}(\tau, \varepsilon), \tag{4.5}$$

where $O_{P2}(\varepsilon)$, $O_{P3}(\varepsilon)$ and $O_{Q3}(\tau, \varepsilon)$ are known matrix-valued functions satisfying the inequalities

$$\|O_{P2}(\varepsilon)\| \le a\varepsilon, \ \ \|O_{P3}(\varepsilon)\| \le a\varepsilon, \ \ \|O_{Q3}(\tau, \varepsilon)\| \le a\varepsilon, \ \ \varepsilon \in (0, \varepsilon_1^*], \ \tau \in [-h, 0]. \tag{4.6}$$

Let $\varphi_x(\tau) \in L^2[-h, 0; E^n]$, $\varphi_{0x} \in E^n$ and $\varphi_{0y} \in E^m$ be any given. Now, by using the equations (4.4)-(4.5), the inequalities (4.6), the positiveness of all eigenvalues of the matrix $\alpha = BB'\bar{P}_{30}$ and the assumption A1 (item a), one obtains (very similarly to [18] (proof of Lemma 7.1)) the existence of a positive number ε_2^*, $(\varepsilon_2^* \le \varepsilon_1^*)$, such that the unique solution $\mathrm{col}(x(t, \varepsilon), y(t, \varepsilon))$ of the system (4.2)-(4.3) with the initial conditions

$$x(\tau) = \varphi_x(\tau), \quad \tau \in [-h, 0); \quad x(0) = \varphi_{0x}, \quad y(0) = \varphi_{0y} \tag{4.7}$$

satisfies the following inequalities for all $\varepsilon \in (0, \varepsilon_2^*]$:

$$\|x(t, \varepsilon)\| \le a\exp(-\nu t)\Big(\|\varphi_{0x}\| + \|\varphi_{0y}\| + \|\varphi_x\|_{L^2}\Big), \quad t \ge 0, \tag{4.8}$$

$$\|y(t, \varepsilon)\| \le a\exp(-\nu t)\Big(\|\varphi_{0x}\| + \|\varphi_{0y}\| + \|\varphi_x\|_{L^2}\Big), \quad t \ge 0, \tag{4.9}$$

where $a > 0$ and $\nu > 0$ are some constants independent of ε.

The inequalities (4.8)-(4.9) prove the exponential stability of the system (2.17) uniformly with respect to $\varepsilon \in (0, \varepsilon_2^*]$, which completes the proof of the lemma. □

Lemmas 2.2, 4.1, Theorem 3.2 and Corollary 3.3 directly yield the following theorem.

Theorem 4.2. *Let the assumption A1 be satisfied. Then, there exists a number $\varepsilon_0 > 0$ such that:*

(i) *for all $\varepsilon \in (0, \varepsilon_0]$, the set of Riccati-type equations (2.13)-(2.16) has a solution $\{P(\varepsilon), Q(\tau, \varepsilon), R(\tau, \rho, \varepsilon)\}$, $(\tau, \rho) \in \mathcal{D}$ of the form (3.1)-(3.2),(3.18);*

(ii) *this solution satisfies the symmetry properties $P'(\varepsilon) = P(\varepsilon)$, $R'(\tau, \rho, \varepsilon) = R(\rho, \tau, \varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$ and $(\tau, \rho) \in \mathcal{D}$;*

(iii) *this solution provides the system (2.17) to be exponentially stable uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$;*

(iv) *the controller (4.1) solves the OHICP for all $\varepsilon \in (0, \varepsilon_0]$.*

5. Simplified controller for the OHICP

Consider the following $(n+m) \times (n+m)$ block matrices

$$\bar{P}_0(\varepsilon) = \begin{pmatrix} \bar{P}_{10} & \varepsilon \bar{P}_{20} \\ \varepsilon \bar{P}_{20}' & \varepsilon \bar{P}_{30} \end{pmatrix}, \qquad \bar{Q}_0(\tau, \varepsilon) = \begin{pmatrix} \bar{Q}_{10}(\tau) & 0 \\ \varepsilon \bar{Q}_{30}(\tau) & 0 \end{pmatrix}, \tag{5.1}$$

where the matrices $\bar{P}_{j0}$, $(j = 1, 2, 3)$, and $\bar{Q}_{l0}(\tau)$, $(l = 1, 3)$ have been obtained in Section 3.

Consider the following controller for the OHICP

$$\bar{v}_0[x(\cdot), y(\cdot)](t) = -\varepsilon^{-1}\mathcal{B}' \left[\bar{P}_0(\varepsilon)z(t) + \int_{-h}^{0} \bar{Q}_0(\tau, \varepsilon)z(t+\tau)d\tau\right], \ z = \mathrm{col}(x, y). \tag{5.2}$$

This controller is obtained from the OHICP controller (2.22) by replacing there the matrices $P(\varepsilon)$ and $Q(\tau, \varepsilon)$ by the ones $\bar{P}_0(\varepsilon)$ and $\bar{Q}_0(\tau, \varepsilon)$, respectively.

Substituting the block form of the state variable z and of the matrices $\mathcal{B}$, $\bar{P}_0(\varepsilon)$ and $\bar{Q}_0(\tau, \varepsilon)$ (see (2.11) and (5.1)) into (5.2) yields after a simple rearrangement

$$\bar{v}_0[x(\cdot), y(\cdot)](t) = -B' \left(\bar{P}_{20}'x(t) + \bar{P}_{30}y(t) + \int_{-h}^{0} \bar{Q}_{30}(\tau)x(t+\tau)d\tau\right). \tag{5.3}$$

It is seen that the controller $\bar{v}_0[x(\cdot), y(\cdot)](t)$ is independent of ε.

Substituting $v = \bar{v}_0[x(\cdot), y(\cdot)](t)$ into the system (2.5)-(2.6) and the cost functional (2.8), one obtains

$$dx(t)/dt = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h)$$

$$+ \int_{-h}^{0} G_{11}(\tau)x(t+\tau)d\tau + F_1 w(t), \qquad t > 0, \tag{5.4}$$

$$\varepsilon dy(t)/dt = [\varepsilon A_{21} - BB'\bar{P}_{20}']x(t) + [\varepsilon A_{22} - BB'\bar{P}_{30}]y(t) + \varepsilon H_{21}x(t-h)$$

$$+\int_{-h}^{0}[\varepsilon G_{21}(\tau) - BB'\bar{Q}_{30}(\tau)]x(t+\tau)d\tau + \varepsilon F_2 w(t), \qquad t > 0, \tag{5.5}$$

$$J(\bar{v}_0, w) \triangleq \bar{J}_0(w) = \int_0^{+\infty}\Big[x'(t)D_1x(t) + y'(t)D_2y(t)$$

$$+\bar{v}_0'[x(\cdot),y(\cdot)](t)\bar{v}_0[x(\cdot),y(\cdot)](t) - \gamma^2 w'(t)w(t)\Big]dt$$

$$= \int_0^{+\infty}\Big[x'(t)\bar{D}_{P1}x(t) + 2x'(t)\bar{D}_{P2}x(t) + y'(t)\bar{D}_{P3}y(t)$$

$$+2x'(t)\int_{-h}^{0}\bar{D}_{Q1}(\tau)x(t+\tau)d\tau + 2y'(t)\int_{-h}^{0}\bar{D}_{Q2}(\tau)x(t+\tau)d\tau$$

$$+\int_{-h}^{0}\int_{-h}^{0}x'(t+\tau)\bar{D}_{R1}(\tau,\rho)x(t+\rho)d\tau d\rho - \gamma^2 w'(t)w(t)\Big]dt, \tag{5.6}$$

where

$$\bar{D}_{P1} = D_1 + \bar{P}_{20}BB'\bar{P}_{20}', \quad \bar{D}_{P2} = \bar{P}_{20}BB'\bar{P}_{30}, \quad \bar{D}_{P3} = D_2 + \bar{P}_{30}BB'\bar{P}_{30}, \tag{5.7}$$

$$\bar{D}_{Q1}(\tau) = \bar{P}_{20}BB'\bar{Q}_{30}(\tau), \qquad \bar{D}_{Q2}(\tau) = \bar{P}_{30}BB'\bar{Q}_{30}(\tau), \tag{5.8}$$

$$\bar{D}_{R1}(\tau,\rho) = \bar{Q}_{30}'(\tau)BB'\bar{Q}_{30}(\rho). \tag{5.9}$$

Remark 5.1. Since, for any $x(t) \in E^n$, $y(t) \in E^m$ and $v(t) \in E^m$,

$$x'(t)D_1x(t) + y'(t)D_2y(t) + v'(t)v(t) \geq 0, \tag{5.10}$$

then, for any $x(t) \in E^n$, $y(t) \in E^m$ and $x(t+\tau) \in L^2[-h,0;E^n]$,

$$x'(t)D_1x(t) + y'(t)D_2y(t) + v_0'[x(\cdot),y(\cdot)](t)v_0[x(\cdot),y(\cdot)](t) \geq 0, \tag{5.11}$$

i.e.,

$$x'(t)\bar{D}_{P1}x(t) + 2x'(t)\bar{D}_{P2}x(t) + y'(t)\bar{D}_{P3}y(t)$$

$$+2x'(t)\int_{-h}^{0}\bar{D}_{Q1}(\tau)x(t+\tau)d\tau + 2y'(t)\int_{-h}^{0}\bar{D}_{Q2}(\tau)x(t+\tau)d\tau$$

$$+\int_{-h}^{0}\int_{-h}^{0}x'(t+\tau)\bar{D}_{R1}(\tau,\rho)x(t+\rho)d\tau d\rho \geq 0. \tag{5.12}$$

Lemma 5.2. *Under the assumption A1 (item a), there exists a positive constant $\bar{\varepsilon}_1$, such that the system (5.4)-(5.5) is internally exponentially stable uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon}_1]$.*

Proof. In order to prove the lemma, one has to show that the system, obtained from (5.4)-(5.5) by setting there $w(t) \equiv 0$, is exponentially stable uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon}_1]$ with some positive $\bar{\varepsilon}_1$. The latter is proved similarly to Lemma 4.1. □

Consider the following $(n+m)\times(n+m)$ block matrices

$$\bar{A}(\varepsilon)=\begin{pmatrix} A_{11} & A_{12} \\ A_{21}-\varepsilon^{-1}BB'\bar{P}_{20}' & A_{22}-\varepsilon^{-1}BB'\bar{P}_{30}\end{pmatrix}, \tag{5.13}$$

$$\bar{G}(\tau,\varepsilon)=\begin{pmatrix} G_{11}(\tau) & 0 \\ G_{21}(\tau)-\varepsilon^{-1}BB'\bar{Q}_{30}(\tau) & 0\end{pmatrix}, \tag{5.14}$$

$$\bar{D}_P=\begin{pmatrix} \bar{D}_{P1} & \bar{D}_{P2} \\ \bar{D}_{P2}' & \bar{D}_{P3}\end{pmatrix}, \quad \bar{D}_Q(\tau)=\begin{pmatrix} \bar{D}_{Q1}(\tau) & 0 \\ \bar{D}_{Q2}(\tau) & 0\end{pmatrix}, \tag{5.15}$$

$$\bar{D}_R(\tau,\rho)=\begin{pmatrix} \bar{D}_{R1}(\tau,\rho) & 0 \\ 0 & 0\end{pmatrix}, \quad \bar{S}_F=\gamma^{-2}\mathcal{F}\mathcal{F}'. \tag{5.16}$$

Consider the following hybrid system of algebraic, ordinary differential and partial differential equations of Riccati type with respect to $(n+m)\times(n+m)$-matrices $\hat{P}$, $\hat{Q}(\tau)$ and $\hat{R}(\tau,\rho)$ in the domain $\mathcal{D}$:

$$\hat{P}\bar{A}(\varepsilon)+\bar{A}'(\varepsilon)\hat{P}+\hat{P}\bar{S}_F\hat{P}+\hat{Q}(0)+\hat{Q}'(0)+\bar{D}_P=0, \tag{5.17}$$

$$d\hat{Q}(\tau)/d\tau=\left(\bar{A}(\varepsilon)+\bar{S}_F\hat{P}\right)'\hat{Q}(\tau)+\hat{P}\bar{G}(\tau,\varepsilon)+\hat{R}(0,\tau)+\bar{D}_Q(\tau), \tag{5.18}$$

$$(\partial/\partial\tau+\partial/\partial\rho)\hat{R}(\tau,\rho)=\bar{G}'(\tau,\varepsilon)\hat{Q}(\rho)+\hat{Q}'(\tau)\bar{G}(\rho,\varepsilon)$$
$$+\hat{Q}'(\tau)\bar{S}_F\hat{Q}(\rho)+\bar{D}_R(\tau,\rho), \tag{5.19}$$

The system (5.17)-(5.19) is considered subject to the boundary conditions

$$\hat{Q}(-h)=\hat{P}H, \quad \hat{R}(-h,\tau)=H'\hat{Q}(\tau), \quad \hat{R}(\tau,-h)=\hat{Q}'(\tau)H, \tag{5.20}$$

where the matrix H is given in (2.9).

Let the triple $\{\hat{P}(\varepsilon),\hat{Q}(\tau,\varepsilon),\hat{R}(\tau,\rho,\varepsilon)\}$ be a solution of the problem (5.17)-(5.20) for some $\varepsilon\in(0,\bar{\varepsilon}_1]$, where the positive constant $\bar{\varepsilon}_1$ has been introduced in Lemma 5.2.

Lemma 5.3. *Let the assumption A1 (item a) be satisfied. Let, for some $\varepsilon\in(0,\bar{\varepsilon}_1]$, the problem (5.17)-(5.20) has a solution $\{\hat{P}(\varepsilon),\hat{Q}(\tau,\varepsilon),\hat{R}(\tau,\rho,\varepsilon)\}$ such that*

$$\hat{P}'(\varepsilon)=\hat{P}(\varepsilon), \quad \hat{R}'(\tau,\rho,\varepsilon)=\hat{R}(\rho,\tau,\varepsilon), \quad (\tau,\rho)\in\mathcal{D}. \tag{5.21}$$

Then, for this ε, the following inequality is satisfied along trajectories of the system (5.4)-(5.5) with the initial conditions (2.7):

$$\bar{J}_0(w)\le 0 \quad \forall w(t)\in L^2[-h,0;E^q]. \tag{5.22}$$

The proof of the lemma is presented in Section 6.

Lemma 5.4. *Let the assumption A1 (item b) be satisfied. Then, there exists a positive number $\bar{\varepsilon}_2$ such that for all $\varepsilon\in(0,\bar{\varepsilon}_2]$:*

(a) *the problem (5.17)-(5.20) has a solution* $\{\hat{P}(\varepsilon), \hat{Q}(\tau,\varepsilon), \hat{R}(\tau,\rho,\varepsilon)\}$, $(\tau,\rho) \in \mathcal{D}$ *of the block form*

$$\hat{P}(\varepsilon) = \begin{pmatrix} \hat{P}_1(\varepsilon) & \varepsilon\hat{P}_2(\varepsilon) \\ \varepsilon\hat{P}_2'(\varepsilon) & \varepsilon\hat{P}_3(\varepsilon) \end{pmatrix}, \qquad \hat{Q}(\tau,\varepsilon) = \begin{pmatrix} \hat{Q}_1(\tau,\varepsilon) & 0 \\ \varepsilon\hat{Q}_3(\tau,\varepsilon) & 0 \end{pmatrix}, \tag{5.23}$$

$$\hat{R}(\tau,\rho,\varepsilon) = \begin{pmatrix} \hat{R}_1(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.24}$$

where $\hat{P}_j(\varepsilon)$, $(j = 1,2,3)$ *are matrices of the dimensions* $n\times n$, $n\times m$, $m\times m$, *respectively;* $\hat{Q}_l(\tau,\varepsilon)$, $(l = 1,3)$ *are matrices of the dimensions* $n\times n$, $m\times n$, *respectively;* $\hat{R}_1(\tau,\rho,\varepsilon)$ *is a matrix of the dimension* $n \times n$*;*

(b) *the matrices* $\hat{P}_l(\varepsilon)$, $(l = 1,3)$ *and* $\hat{R}_1(\tau,\rho,\varepsilon)$ *satisfy the symmetry properties*

$$\hat{P}_l'(\varepsilon) = \hat{P}_l(\varepsilon), \quad l = 1,3; \qquad \hat{R}_1'(\tau,\rho,\varepsilon) = \hat{R}_1(\rho,\tau,\varepsilon), \quad (\tau,\rho) \in \mathcal{D}; \tag{5.25}$$

(c) *the matrices* $\hat{P}_j(\varepsilon)$, $(j = 1,2,3)$, $\hat{Q}_l(\tau,\varepsilon)$, $(l = 1,3)$ *and* $\hat{R}_1(\tau,\rho,\varepsilon)$ *satisfy the inequalities*

$$\left\|\hat{P}_j(\varepsilon) - \bar{P}_{j0}\right\| \le a\varepsilon, \qquad j = 1,2,3, \tag{5.26}$$

$$\left\|\hat{Q}_1(\tau,\varepsilon) - \bar{Q}_{10}(\tau)\right\| \le a\varepsilon, \qquad \left\|\hat{Q}_3(\tau,\varepsilon) - Q_{30}(\tau,\varepsilon)\right\| \le a\varepsilon, \qquad \tau \in [-h,0], \tag{5.27}$$

$$\left\|\hat{R}_1(\tau,\rho,\varepsilon) - \bar{R}_{10}(\tau,\rho)\right\| \le a\varepsilon, \qquad (\tau,\rho) \in \mathcal{D}, \tag{5.28}$$

where the matrices $\bar{P}_{j0}$, $(j = 1,2,3)$, $\bar{Q}_{10}(\tau)$, $Q_{30}(\tau,\varepsilon)$ *and* $\bar{R}_{10}(\tau,\rho)$ *are the same as in Theorem 3.2;* $a > 0$ *is some constant independent of* ε*.*

Proof. The statements of the lemma are obtained similarly to Theorem 3.2 and Corollary 3.3. □

Lemmas 5.2-5.4 directly yield the following theorem.

Theorem 5.5. *Let the assumption A1 be satisfied. Then, the controller (5.3) solves the OHICP for all* $\varepsilon \in (0, \bar{\varepsilon}_0]$, *where* $\bar{\varepsilon}_0 = \min(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$.

6. Proof of Lemma 5.3

Consider the following functional, depending on a parameter $t \ge 0$, on a vector $\varphi_0 \in E^{n+m}$ and on a function $\varphi_z(\theta) \in L^2[t-h, t; E^{n+m}]$:

$$V[t,\varphi_0,\varphi_z(\theta)] \stackrel{\triangle}{=} \varphi_0'\hat{P}(\varepsilon)\varphi_0 + 2\varphi_0' \int_{t-h}^{t} \hat{Q}(\theta - t,\varepsilon)\varphi_z(\theta)d\theta$$

$$+ \int_{t-h}^{t}\int_{t-h}^{t} \varphi_z'(\theta)\hat{R}(\theta - t, \sigma - t, \varepsilon)\varphi_z(\sigma)d\theta d\sigma. \tag{6.1}$$

By using the block vector $z = \mathrm{col}(x,y)$, one can rewrite the system (5.4)-(5.5) and the cost functional (5.6) in the form

$$dz(t)/dt = \bar{A}(\varepsilon)z(t) + Hz(t-h) + \int_{-h}^{9} \bar{G}(\tau,\varepsilon)z(t+\tau)d\tau + \mathcal{F}w(t), \quad t > 0, \tag{6.2}$$

$$\bar{J}_0(w) = \int_0^{+\infty} \Big[z^{'}(t)\bar{D}_P z(t) + 2z^{'}(t)\int_{-h}^0 \bar{D}_Q(\tau)z(t+\tau)d\tau$$

$$+\int_{-h}^0\int_{-h}^0 z^{'}(t+\tau)\bar{D}_R(\tau,\rho)z(t+\rho)d\tau d\rho - \gamma^2 w^{'}(t)w(t)\Big]dt. \tag{6.3}$$

Let, for any given $w(t) \in L^2[0,+\infty;E^q]$, the vector valued function $z_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)$ be the solution of the system (6.2) subject to the initial conditions (2.18). Let

$$V_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0) \stackrel{\triangle}{=} V[t, z_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0), z_0(\theta;w(\cdot),\varphi_z(\cdot),\varphi_0)], \tag{6.4}$$

where $t \geq 0$ and $\theta \in [t-h,t)$.

Calculating the derivative of $V_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)$ with respect to t and using the equations (5.17)-(5.20) and (6.2) yield after some rearrangement

$$dV_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)/dt = -z_0^{'}(t;w(\cdot),\varphi_z(\cdot),\varphi_0)\bar{D}_P z_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)$$

$$-2z_0^{'}(t;w(\cdot),\varphi_z(\cdot),\varphi_0)\int_{t-h}^t \bar{D}_Q(\theta-t)z_0(\theta;w(\cdot),\varphi_z(\cdot),\varphi_0)d\theta$$

$$-\int_{t-h}^t\int_{t-h}^t z_0^{'}(\theta;w(\cdot),\varphi_z(\cdot),\varphi_0)\bar{D}_R(\theta-t,\sigma-t)z_0(\sigma;w(\cdot),\varphi_z(\cdot),\varphi_0)d\theta d\sigma$$

$$+\gamma^2 w^{'}(t)w(t) - \gamma^2[w(t)-w_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)]^{'}[w(t)-w_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)], \tag{6.5}$$

where

$$w_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0) = \gamma^{-2}\mathcal{F}^{'}\Big[\hat{P}(\varepsilon)z_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)$$

$$+\int_{t-h}^t \hat{Q}(\theta-t,\varepsilon)z_0(\theta;w(\cdot),\varphi_z(\cdot),\varphi_0)d\theta\Big]. \tag{6.6}$$

By changing the integration variables $\theta - t = \tau$ and $\sigma - t = \rho$, the equation (6.5) becomes

$$dV_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)/dt = -\mathcal{Z}_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0) + \gamma^2 w^{'}(t)w(t)$$

$$-\gamma^2[w(t)-w_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)]^{'}[w(t)-w_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)], \tag{6.7}$$

where

$$\mathcal{Z}_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0) = z_0^{'}(t;w(\cdot),\varphi_z(\cdot),\varphi_0)\bar{D}_P z_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0)$$

$$+2z_0^{'}(t;w(\cdot),\varphi_z(\cdot),\varphi_0)\int_{-h}^0 \bar{D}_Q(\tau)z_0(t+\tau;w(\cdot),\varphi_z(\cdot),\varphi_0)d\tau$$

$$+\int_{-h}^0\int_{-h}^0 z_0^{'}(t+\tau;w(\cdot),\varphi_z(\cdot),\varphi_0)\bar{D}_R(\tau,\rho)z_0(t+\rho;w(\cdot),\varphi_z(\cdot),\varphi_0)d\tau d\rho. \tag{6.8}$$

By virtue of Remark 5.1, one has the inequality

$$\mathcal{Z}_0(t;w(\cdot),\varphi_z(\cdot),\varphi_0) \geq 0 \tag{6.9}$$

$$\forall t \geq 0,\ w(\cdot) \in L^2[0,+\infty;E^q],\ \varphi_z(\cdot) \in L^2[-h,0;E^{n+m}],\ \varphi_0 \in E^{n+m}.$$

Setting in (6.7) $w(\cdot) = 0$ and using (6.9) yield

$$dV_0(t; 0, \varphi_z(\cdot), \varphi_0)/dt \le 0 \quad \forall t \ge 0,\ \varphi_z(\cdot) \in L^2[-h, 0; E^{n+m}],\ \varphi_0 \in E^{n+m}. \tag{6.10}$$

Due to Lemma 5.2,

$$\lim_{t\to+\infty} z_0(t; 0, \varphi_z(\cdot), \varphi_0) = 0 \quad \forall \varphi_z(\cdot) \in L^2[-h, 0; E^{n+m}],\ \varphi_0 \in E^{n+m}. \tag{6.11}$$

Integrating the inequality (6.10) from 0 to $+\infty$ and using (6.1), (6.4) and (6.11), we obtain the inequality

$$V_0(0; 0, \varphi_z(\cdot), \varphi_0) \ge 0 \quad \forall \varphi_z(\cdot) \in L^2[-h, 0; E^{n+m}],\ \varphi_0 \in E^{n+m}, \tag{6.12}$$

or

$$\begin{aligned} &\varphi_0' \hat{P}(\varepsilon)\varphi_0 + 2\varphi_0' \int_{-h}^{0} \hat{Q}(\tau, \varepsilon)\varphi_z(\tau)d\tau \\ &+ \int_{-h}^{0}\int_{-h}^{0} \varphi_z'(\tau)\hat{R}(\tau, \rho, \varepsilon)\varphi_z(\rho)d\tau d\rho \ge 0 \\ &\forall \varphi_z(\cdot) \in L^2[-h, 0; E^{n+m}],\ \varphi_0 \in E^{n+m}. \end{aligned} \tag{6.13}$$

The equation (6.7) yields the inequality

$$dV_0(t; w(\cdot), \varphi_z(\cdot), \varphi_0)/dt + \mathcal{Z}_0(t; w(\cdot), \varphi_z(\cdot), \varphi_0) \le \gamma^2 w'(t)w(t), \quad t \ge 0. \tag{6.14}$$

Setting in (6.14) $\varphi_z(\cdot) = 0$, $\varphi_0 = 0$, integrating the resulting inequality from 0 to any fixed $T > 0$ and using (6.1), (6.4) yield

$$V_0(T; w(\cdot), 0, 0) + \int_0^T \mathcal{Z}_0(t; w(\cdot), 0, 0)dt \le \int_0^T \gamma^2 w'(t)w(t)dt. \tag{6.15}$$

By changing in (6.1) the integration variables $\theta - t = \tau$ and $\sigma - t = \rho$, and using (6.4), the value $V_0(T; w(\cdot), 0, 0)$ can be expressed as follows:

$$\begin{aligned} V_0(T; w(\cdot), 0, 0) &= z_0'(T; w(\cdot), 0, 0)\hat{P}(\varepsilon)z_0(T; w(\cdot), 0, 0) \\ &+2z_0'(T; w(\cdot), 0, 0)\int_{-h}^{0} \hat{Q}(\tau, \varepsilon)z_0(T + \tau; w(\cdot), 0, 0)d\tau \\ &+ \int_{-h}^{0}\int_{-h}^{0} z_0'(T + \tau; w(\cdot), 0, 0)\hat{R}(\tau, \rho, \varepsilon)z_0(T + \rho; w(\cdot), 0, 0)d\tau d\rho. \end{aligned} \tag{6.16}$$

By virtue of (6.13), the equation (6.16) yields

$$V_0(T; w(\cdot), 0, 0) \ge 0 \quad \forall T > 0,\ w(\cdot) \in L^2[0, +\infty; E^q]. \tag{6.17}$$

The latter, along with (6.15), implies

$$\int_0^T \mathcal{Z}_0(t; w(\cdot), 0, 0)dt \le \int_0^T \gamma^2 w'(t)w(t)dt \quad \forall T > 0,\ w(\cdot) \in L^2[0, +\infty; E^q]. \tag{6.18}$$

By using the inequalities (6.9) and 6.18), we directly obtain that the integral

$$\int_0^{+\infty} \mathcal{Z}_0(t; w(\cdot), 0, 0)dt$$

converges, and

$$\int_0^{+\infty} \mathcal{Z}_0(t; w(\cdot), 0, 0)dt \le \int_0^{+\infty} \gamma^2 w'(t)w(t)dt \quad \forall w(\cdot) \in L^2[0, +\infty; E^q]. \tag{6.19}$$

The latter, along with (6.3) and (6.8), yields the inequality (5.22), which completes the proof of the lemma.

7. Direct method of constructing a simplified controller for the OHICP

In this section, we propose another method of constructing a simplified controller for the OHICP. This method is not based on the asymptotic solution of the set of Riccati-type matrix equations arising in the solvability conditions for the OHICP, but it is based on an asymptotic decomposition of the OHICP into two much simpler ε-free subproblems, the slow and fast ones.

7.1. **Slow Subproblem.** The slow subproblem is obtained from the OHICP by setting there formally $\varepsilon = 0$ and redenoting x, y, v and J by x_s, y_s, v_s and J_s, respectively. Thus, one obtains

$$dx_s(t)/dt = A_{11}x_s(t) + A_{12}y_s(t) + H_{11}x_s(t-h) + \int_{-h}^0 G_{11}(\tau)x_s(t+\tau)d\tau + F_1 w_s(t), \quad t > 0, \tag{7.1}$$

$$Bv_s(t) = 0, \quad t \in [0, +\infty), \tag{7.2}$$

$$x_s(t) = 0, \quad t \le 0, \tag{7.3}$$

$$J_s = \int_0^{+\infty} \Big[x_s'(t)D_1x_s(t) + y_s'(t)D_2y_s(t) + v_s'(t)v_s(t) - \gamma^2 w_s'(t)w_s(t)\Big]dt. \tag{7.4}$$

Since the matrix B is invertible, the equation (7.2) implies

$$v_s(t) = 0, \quad t \in [0, +\infty). \tag{7.5}$$

Substituting (7.5) into (7.4) yields

$$J_s = \int_0^{+\infty} \Big[x_s'(t)D_1x_s(t) + y_s'(t)D_2y_s(t) - \gamma^2 w_s'(t)w_s(t)\Big]dt. \tag{7.6}$$

Note, that in the system (7.1) (similarly to the system (3.59)), the variable $y_s(t)$ can be considered as a control. The latter means that the functional (7.6), calculated along trajectories of (7.1), (7.3), depends on the control function $y_s(t)$ and the disturbance $w_s(t) \in L^2[0, +\infty; E^q]$, i.e., $J_s = J_s(y_s, w_s)$. Thus, for the system (7.1), initial condition (7.3) and the cost functional (7.6), the H_∞ control problem with a performance level γ can be formulated. Comparing this problem with the RHICP introduced in Section 3.4.2, one can conclude that these problems coincide with each other. Thus, by Lemma 3.1, the controller, solving the H_∞ control problem (7.1),(7.3),(7.6), has the form

$$y_s^*[x_s(\cdot)](t) = -D_2^{-1}A_{12}'\left[\bar{P}_{10}x_s(t) + \int_{-h}^0 \bar{Q}_{10}(\tau)x_s(t+\tau)d\tau\right], \tag{7.7}$$

where the matrices $\bar{P}_{10}$ and $\bar{Q}_{10}(\tau)$ are the respective components of the solution $\bar{\mathcal{S}}$ to the problem (3.49)-(3.50),(3.55)-(3.57), satisfying the assumption A1 (item a).

7.2. **Fast Subproblem.** The fast subproblem is obtained in the following three stages. First, the slow variable $x(\cdot)$ is removed from the equation (2.6) and the cost functional (2.8) of the OHICP. Second, the following transformation of variables is made in the resulting problem:

$$t = \varepsilon\xi, \quad y(\varepsilon\xi) = y_f(\xi), \quad v(\varepsilon\xi) = v_f(\xi),$$

$$w(\varepsilon\xi) = w_f(\xi), \quad J(v(\varepsilon\xi), w(\varepsilon\xi)) = \varepsilon J_f(v_f(\xi), w_f(\xi)), \tag{7.8}$$

where ξ, y_f, v_f, w_f and J_f are new independent variable, state, control, disturbance and cost functional, respectively. Thus, we obtain the system and the cost functional

$$dy_f(\xi)/d\xi = \varepsilon A_{22} y_f(\xi) + Bv_f(\xi) + \varepsilon F_2 w_f(\xi), \quad \xi > 0, \tag{7.9}$$

$$J_f(v_f, w_f) = \int_0^{+\infty} \Big[y_f'(\xi) D_2 y_f(\xi) + v_f'(\xi) v_f(\xi) - \gamma^2 w_f'(\xi) w_f(\xi) \Big] d\xi. \tag{7.10}$$

Finally, neglecting formally the terms with the multiplier ε in (7.9) yields the system

$$dy_f(\xi)/d\xi = Bv_f(\xi), \quad \xi > 0. \tag{7.11}$$

For this system, the H_∞ control problem with a performance level γ can be formulated as follows. To find a controller $v_f^*[y_f(\xi)]$ that stabilizes (7.11) and ensures the inequality $J_f(v_f^*, w_f) \le 0$ along its trajectories for all $w_f(\xi) \in L_2[0, +\infty; E^q]$ and for $y_f(0) = 0$. This H_∞ control problem is called the fast H_∞ control subproblem associated with the OHICP.

Let K be any $m \times m$-matrix such that BK is a Hurwitz matrix. Then, the controller

$$v_f^*[y_f(\xi)] = Ky_f(\xi) \tag{7.12}$$

solves the fast H_∞ control subproblem.

Since the system (7.11) and the cost functional (7.10) are particular cases of the ones (2.5)-(2.6) and (2.8), respectively, we choose the matrix K in accordance with Lemma 2.2. Due to this lemma, the controller, solving the corresponding H_∞ problem, is designed by using a solution of the problem (2.13)-(2.16). For the fast H_∞ control subproblem, (2.13)-(2.16) is reduced to the algebraic matrix Riccati equation with respect to P_f

$$-P_f BB' P_f + D_2 = 0. \tag{7.13}$$

Comparing this equation with the one (3.45), we can conclude that these equations coincide with each other. Therefore, (7.13) has the unique symmetric positive definite solution

$$P_f = \bar{P}_{30}, \tag{7.14}$$

where $\bar{P}_{30}$ is given by (3.51). Based on this solution of (7.13), we choose the gain matrix K in (7.12) as $K = -B'P_f$. Thus, the matrix $BK = -BB'P_f = -\alpha$, where the matrix α is given by (3.54). Since all eigenvalues of α are real positive, then the matrix BK is a Hurwitz one. Hence, the controller

$$v_f^*[y_f(\xi)] = -B^{'}P_f y_f(\xi) \tag{7.15}$$

solves the fast H_∞ control subproblem.

7.3. **Composite Controller for the OHICP.** In this subsection, based on the control $v_s(t)$, given by (7.5), the controller $y_s^*[x_s(\cdot)](t)$, solving the H_∞ control problem (7.1),(7.3),(7.6), and the controller $v_f^*[y_f(\xi)]$, solving the fast H_∞ control subproblem, we construct a composite controller for the OHICP. Then, we show that this controller solves the OHICP for all sufficiently small $\varepsilon > 0$.

The composite controller is obtained in the form

$$v_c[x(\cdot), y(\cdot)](t) = v_s(t) + v_f^*[\tilde{y}(t/\varepsilon)], \tag{7.16}$$

where $\tilde{y}(t/\varepsilon)$ is defined as follows

$$\tilde{y}(t/\varepsilon) \overset{\triangle}{=} y(t) - y_s^*[x(\cdot)](t). \tag{7.17}$$

Substituting (7.5) and (7.15) into (7.16), and using (7.7), (7.14) and (7.17) yield after some rearrangement

$$v_c[x(\cdot), y(\cdot)](t) = -B'\bar{P}_{30}\Big\{y(t)$$

$$+D_2^{-1}A_{12}^{'}\Big[\bar{P}_{10}x(t) + \int_{-h}^{0}\bar{Q}_{10}(\tau)x(t+\tau)d\tau\Big]\Big\}. \tag{7.18}$$

By virtue of the equations (3.45),(3.52) and (3.53), the expression (7.18) can be transformed equivalently as follows

$$v_c[x(\cdot), y(\cdot)](t) = -B^{'}\left(\bar{P}_{20}^{'}x(t) + \bar{P}_{30}y(t) + \int_{-h}^{0}\bar{Q}_{30}(\tau)x(t+\tau)d\tau\right). \tag{7.19}$$

Comparing the expression (7.19) for the composite controller with the expression (5.3) for the ε-free controller, solving the OHICP, one can conclude that these controllers coincide with each other. Thus, the statement of Theorem 5.5 also is valid for the composite controller $v_c[x(\cdot), y(\cdot)](t)$, meaning that this controller solves the OHICP for all sufficiently small $\varepsilon > 0$.

8. Conclusions

In this paper, a linear time-invariant controlled system with point-wise and distributed state delays and a square-integrable disturbance was considered. It is assumed that this system consists of two modes. One of them is controlled directly, while the other is controlled through the first one. Moreover, the case where the state variable of the mode, controlled directly, has no delays is treated. For this system, the infinite horizon H_∞ control problem with a given performance level was studied. The control cost in the cost functional of this problem is assumed to be small with respect to the state and disturbance costs, i.e., the considered problem is the H_∞ cheap control problem. By using a simple control transformation, this problem was converted to the H_∞ control problem for a system with a small multiplier $\varepsilon > 0$ for a part of the derivatives, i.e., to a singularly perturbed system. In this singularly perturbed system, the slow state variable has delays, while the fast state variable has not. For this new H_∞ control problem, considered in the sequel as an original one, two methods of solution were proposed. The first method is based on the asymptotic solution of the set of Riccati-type matrix equations arising in the H_∞ control problem solvability conditions. This method yields the ε-free solvability conditions for the original H_∞ control problem, as well as the ε-free controller, solving this problem for all sufficiently small values $\varepsilon > 0$.

The second method is based on an asymptotic decomposition of the original H_∞ control problem into two much simpler ε-free subproblems, the slow and fast ones. For each of these subproblems the resolving controller was obtained. Then, by using these controllers, the composite controller, solving the original problem, was designed. It was shown that this composite controller coincides with the ε-free controller obtained by the first method.

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