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INFINITE HORIZON CHEAP CONTROL PROBLEM FOR A CLASS OF SYSTEMS WITH STATE DELAYS

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Abstract. An infinite horizon quadratic cheap control of a linear system with point-wise and distributed time delays in the state variable is considered. This optimal control problem is transformed to an optimal control problem of a singularly perturbed system with state delays. The resulting problem is considered in the sequel as an original one. Two methods of suboptimal solution of this problem are proposed.

1. Introduction

The cheap control problem is an optimal control problem with a small control cost (with respect to a state cost) in the cost functional. This problem is of considerable importance in many topics of control theory, for instance, in singular optimal control and its regularization [1], limitations of linear optimal regulators and filters [3], [17], limitations of nonlinear optimal regulators [22], high gain control [15], [28], inverse control problems [18], robust control of systems with disturbances [24], and some others.

The smallness of the control cost yields the singular perturbation [15] in the Hamilton-Jacobi-Bellman equation, as well as in the Hamilton boundary-value problem, associated with the original problem by control optimality conditions.

The cheap control problem for differential equations without delays has been extensively investigated in the literature for both, finite horizon and infinite horizon cost functional, cases (see e.g. [2], [13], [15], [19], [20], [21], [23] and references therein). The cheap control problem with a delayed dynamics was investigated only in few works in the literature [6]- [8], [11], and these works are devoted to the case of a finite horizon cost functional.

In this paper, an infinite horizon linear-quadratic cheap control problem with point-wise and distributed state delays in the dynamics is analyzed. For our best knowledge, the cheap control problem with a delayed dynamics and an infinite horizon cost functional has not yet been studied in the literature.

Two methods of a suboptimal solution of this problem are proposed. The first one is based on an asymptotic solution of a set of Riccati-type matrix equations associated with the original problem by the control optimality conditions. Using this asymptotic solution, an asymptotically suboptimal state-feedback control is constructed for the cheap control problem. The second method is a direct method of suboptimal solution of this problem. This method is based on: (i) an equivalent transformation of the cheap control problem to a control problem with singularly

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perturbed dynamics; (ii) an asymptotic decomposition of the resulting problem into two much simpler parameter-free subproblems, the slow and fast ones. It should be noted that the fast state variable of the control problem, obtained after the transformation, becomes a control in the slow subproblem. The slow subproblem is an optimal control problem with a delayed dynamics. The fast subproblem does not contain delays, and it is solved analytically. Using the optimal feedback controls of the slow and fast subproblems, a suboptimal state-feedback composite control for the transformed problem is designed. The latter yields a suboptimal control for the original cheap control problem.

The paper is organized as follows. The next section is devoted to a rigorous problem formulation. In Section 3, an asymptotic solution of the set of Riccati-type equations, associated with the original control problem by the control optimality conditions, is constructed and justified. Parameter-free conditions of the existence and uniqueness of solution to the original control problem are derived in Section 4. In Section 5, the proof of the theorem, justifying the asymptotic solution of the set of Riccati-type equations obtained in Section 3, is presented. In Section 6, an auxiliary lemma, formulated in Section 3, is proved. In Sections 7 and 8, two suboptimal state-feedback controls for the original control problem are designed and justified by using the asymptotic solution of the set of Riccati-type equations obtained in Section 3. The direct method of suboptimal solution of the original control problem is described in Section 9. Concluding remarks are presented in Section 10.

The following main notations and notions are applied in the paper:

(1) E^n is the n-dimensional real Euclidean space;

(2) $\|\cdot\|$ denotes the Euclidean norm either of a vector or of a matrix;

(3) the prime denotes the transposition of a matrix $A, (A^{'})$ or of a vector $x, (x^{'})$;

(4) $L^2[b,c;E^n]$ is the Hilbert space of n-dimensional vector-valued functions $v(t)$ defined, measurable and square-integrable on the interval $[b,c)$, the inner product in this space is $(v(\cdot),w(\cdot))_{L^2} = \int_b^c v^{'}(t)w(t)dt$, and the norm is $\|v(\cdot)\|_{L^2} = \sqrt{(v(\cdot),v(\cdot))_{L^2}}$;

(5) $L^\infty[b,c;E^n]$ is the space of n-dimensional vector-valued functions $x(t)$ defined, measurable and essentially bounded on the interval $[b,c)$, $\|x(\cdot)\|_\infty = \text{ess}\sup_{t\in[b,c)}\|x(t)\|$ denotes the norm in this space;

(6) $\mathcal{M}[b,c;n,m]$ denotes the Hilbert space of all pairs $f = (f_E, f_L(\cdot)), f_E \in E^n, f_L(\cdot) \in L^2[b,c;E^m]$, the inner product in this space is $(f,g)_{\mathcal{M}} = f_E^{'}g_E + (f_L(\cdot),g_L(\cdot))_{L^2}$, and the norm is $\|f\|_{\mathcal{M}} = \sqrt{(f,f)_{\mathcal{M}}}$;

(7) I_n is the n-dimensional identity matrix;

(8) $\text{Re}\lambda$ is the real part of a complex number λ;

(9) $\text{col}(x,y)$, where $x \in E^n, y \in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\text{col}(x,y) = (x^{'},y^{'})^{'}$;

(10) a self-adjoint operator $\mathcal{F}$, mapping the space $\mathcal{M}[b,c;n;m]$ into itself, is called positive if $(\mathcal{F}f,f)_{\mathcal{M}} \geq 0\ \forall f \in \mathcal{M}[b,c;n;m]$;

(11) a self-adjoint operator $\mathcal{F}$, mapping the space $\mathcal{M}[b,c;n;m]$ into itself, is called uniformly positive if there exists a positive constant ν, such that $(\mathcal{F}f,f)_{\mathcal{M}} \geq \nu\|f\|^2_{\mathcal{M}}$ $\forall f \in \mathcal{M}[b,c;n;m]$.

2. Problem formulation

2.1. **Cheap control problem.** Consider the controlled system

$$dz(t)/dt = Az(t) + Hz(t-h) + \int_{-h}^{0} G(\tau)z(t+\tau)d\tau + Bu(t), \quad t \geq 0, \tag{2.1}$$

where $z(t) \in E^n, u(t) \in E^r, (n \geq r)$, ($u$ is a control); $h > 0$ is a given constant time delay; $A, H, G(\tau)$ and B are given time-invariant matrices of corresponding dimensions; B has full rank r; the matrix-valued function $G(\tau)$ is piece-wise continuous for $\tau \in [-h, 0]$.

Using that rank$B = r$ and results of [16], one can transform (2.1) to an equivalent linear controlled system with state delays, in which the matrix of coefficients for the control has the form $\begin{pmatrix} 0 \\ I_r \end{pmatrix}$. Therefore, in the sequel we assume (without a loss of generality) that

$$B = \begin{pmatrix} 0 \\ I_r \end{pmatrix}. \tag{2.2}$$

The initial conditions for (2.1) have the form

$$z(\tau) = \varphi(\tau), \ \tau \in [-h, 0); \qquad z(0) = \varphi_0, \tag{2.3}$$

where $\varphi(\tau) \in L^2[-h, 0; E^n]$ and $\varphi_0 \in E^n$ are given.

Let partition $z(t), A, H$ and $G(\tau)$ into blocks in the accordance with the block-form of B

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{2.4}$$

$$H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}, \quad G(\tau) = \begin{pmatrix} G_1(\tau) & G_2(\tau) \\ G_3(\tau) & G_4(\tau) \end{pmatrix}, \tag{2.5}$$

where $x(t) \in E^{n-r}, y(t) \in E^r$; A_1, H_1 and $G_1(\tau)$ are of the dimension $(n-r)\times(n-r)$, while A_4, H_4 and $G_4(\tau)$ are of the dimension $r \times r$.

Using (2.2) and (2.4)-(2.5), one can rewrite (2.1) as follows

$$dx(t)/dt = A_1x(t) + A_2y(t) + H_1x(t-h) + H_2y(t-h)$$

$$+ \int_{-h}^{0} \Big[G_1(\tau)x(t+\tau) + G_2(\tau)y(t+\tau)\Big]d\tau, \tag{2.6}$$

$$dy(t)/dt = A_3x(t) + A_4y(t) + H_3x(t-h) + H_4y(t-h)$$

$$+ \int_{-h}^{0} \Big[G_3(\tau)x(t+\tau) + G_4(\tau)y(t+\tau)\Big]d\tau + u(t). \tag{2.7}$$

For the system (2.6)-(2.7) with the initial conditions (2.3) the following performance index is considered

$$J_\varepsilon(u) \triangleq \int_0^{+\infty} \Big[x'(t)D_x x(t) + y'(t)D_y y(t) + \varepsilon^2 u'(t)Mu(t)\Big]dt \to \min_u, \tag{2.8}$$

where D_x, D_y and M are symmetric positive-definite matrices; ε is a small positive parameter. The latter means that the problem of minimizing the cost functional $J_\varepsilon(u)$ along trajectories of the system (2.6)-(2.7) with the initial conditions (2.3) is the cheap control problem.

In the sequel of this paper, we concentrate on the following case:

$$H_2 = 0, \quad H_4 = 0, \quad G_2(\tau) \equiv 0, \quad G_4(\tau) \equiv 0. \tag{2.9}$$

By the control transformation

$$u(t) = (1/\varepsilon)v(t), \tag{2.10}$$

where v is a new control, this cheap control problem becomes

$$dx(t)/dt = A_1x(t) + A_2y(t) + H_1x(t-h) + \int_{-h}^0 G_1(\tau)x(t+\tau)d\tau, \tag{2.11}$$

(2.12)

$$\varepsilon dy(t)/dt = \varepsilon\left\{A_3x(t) + A_4y(t) + H_3x(t-h) + \int_{-h}^0 G_3(\tau)x(t+\tau)d\tau\right\} + v(t),$$

$$J(v) \triangleq \int_0^{+\infty} \Big[x'(t)D_x x(t) + y'(t)D_y y(t) + v'(t)Mv(t)\Big]dt \to \min_v. \tag{2.13}$$

It should be noted that the system (2.11)-(2.12) is singularly perturbed [15]. The state variable $x(\cdot)$ is slow, and the one $y(\cdot)$ is fast. It is seen that in this system, the slow state variable is with a delay, while the fast state variable contains no any delay. It should be also noted that, due to the lack of a delay in the fast state variable, the initial conditions (2.3) for the system (2.11)-(2.12) become as follows:

$$x(\tau) = \varphi_x(\tau), \quad \tau \in [-h, 0); \quad x(0) = \varphi_{0x}, \quad y(0) = \varphi_{0y}, \tag{2.14}$$

where $\varphi_x(\tau) \in L^2[-h, 0; E^{n-r}]$, $\varphi_{0x} \in E^{n-r}$ and $\varphi_{0y} \in E^r$ are given.

In the sequel, we deal with the problem of minimizing the cost functional $J(v)$ along trajectories of the system (2.11)-(2.12) with the initial conditions (2.14). This problem is called the original optimal control problem (OOCP). It is clear that once an optimal (suboptimal) control of the OOCP is obtained, the respective optimal (suboptimal) control of the problem (2.3),(2.6)-(2.7),(2.8),(2.9) is obtained directly by using the equation (2.10).

2.2. **Control optimality conditions.**

Definition 2.1. For a given ε, the system (2.11)-(2.12) is said to be L^2-stabilizable if for each triplet

$$\mathcal{T} \triangleq \Big(\varphi_x(\cdot), \varphi_{0x}, \varphi_{0y}\Big) \in L^2[-h, 0; E^{n-r}] \times E^{n-r} \times E^r, \tag{2.15}$$

there exists a control function $v(t)$, $\Big(v(\cdot)\in L^2[0,+\infty;E^r]\Big)$, such that the solution $z(t)=\mathrm{col}\Big(x(t),y(t)\Big)$ of the system (2.11)-(2.12) with the initial conditions (2.14) satisfies the inclusion $z(\cdot)\in L^2[0,+\infty;E^n]$.

Let introduce the following block-matrices of the dimension $n\times n$

$$S(\varepsilon)=\begin{pmatrix}0 & 0\\ 0 & \varepsilon^{-2}M^{-1}\end{pmatrix},\qquad D=\begin{pmatrix}D_x & 0\\ 0 & D_y\end{pmatrix}. \tag{2.16}$$

Using (2.16) and the results of [4], let us write down the set of Riccati-type algebraic, ordinary differential and partial differential equations for the matrices P, $Q(\tau)$ and $R(\tau,\rho)$ associated with the OOCP. This set has the form

$$PA+A^{'}P-PS(\varepsilon)P+Q(0)+Q^{'}(0)+D=0, \tag{2.17}$$

$$dQ(\tau)/d\tau=\Big(A-S(\varepsilon)P\Big)^{'}Q(\tau)+PG(\tau)+R(0,\tau), \tag{2.18}$$

$$(\partial/\partial\tau+\partial/\partial\rho)R(\tau,\rho)=G^{'}(\tau)Q(\rho)+Q^{'}(\tau)G(\rho)-Q^{'}(\tau)S(\varepsilon)Q(\rho). \tag{2.19}$$

The matrices $Q(\tau)$ and $R(\tau,\rho)$ satisfy the boundary conditions

$$Q(-h)=PH,\quad R(-h,\tau)=H^{'}Q(\tau),\quad R(\tau,-h)=Q^{'}(\tau)H. \tag{2.20}$$

The set of equations (2.17)-(2.20) is considered in the domain

$$\mathcal{D}=\{(\tau,\rho):-h\le\tau\le 0,\ -h\le\rho\le 0\}. \tag{2.21}$$

It is seen that the matrix-valued functions $Q(\tau)$ and $R(\tau,\rho)$ are present in the set (2.17)-(2.19) with deviating arguments. The problem (2.17)-(2.20) is, in general, of a high dimension. Moreover, due to the expression for $S(\varepsilon)$ (see (2.16)), this problem is ill-posed for $\varepsilon\to+0$.

Let, for some $\varepsilon>0$, the triplet $\{P(\varepsilon),Q(\tau,\varepsilon),R(\tau,\rho,\varepsilon)\}$ be a solution of (2.17)-(2.20) in the domain $\mathcal{D}$. Consider the linear bounded operator $\mathcal{F}_\varepsilon:\mathcal{M}[-h,0;n;n]\to\mathcal{M}[-h,0;n;n]$ given by the equation

$$\mathcal{F}_\varepsilon[f(\cdot)]$$

$$=\left(P(\varepsilon)f_E+\int_{-h}^{0}Q(\rho,\varepsilon)f_L(\rho)d\rho,\ Q^{'}(\tau,\varepsilon)f_E+\int_{-h}^{0}R(\tau,\rho,\varepsilon)f_L(\rho)d\rho\right), \tag{2.22}$$

where $f(\cdot)=(f_E,f_L(\cdot))$, $f_E\in E^n$, $f_L(\cdot)\in L^2[-h,0;E^n]$.

By virtue of [4] (Theorems 5.8, 5.9, 6.1), one directly has the following lemma.

Lemma 2.2. *Let, for a given $\varepsilon>0$, the system (2.11)-(2.12) is L^2-stabilizable. Then, for this ε, there exist a solution $\{P(\varepsilon),Q(\tau,\varepsilon),R(\tau,\rho,\varepsilon)\}$ of (2.17)-(2.20) such that the operator $\mathcal{F}_\varepsilon$ is self-adjoint and positive. Moreover:*

(a) *such a solution is unique;*
(b) *the matrix $P(\varepsilon)$ is positive definite;*
(c) *the OOCP has the unique optimal state-feedback control*

$$v_\varepsilon^*[z(\cdot)](t)=-\varepsilon^{-1}M^{-1}B'\left[P(\varepsilon)z(t)+\int_{-h}^{0}Q(\tau,\varepsilon)z(t+\tau)d\tau\right],\quad z=col(x,y); \tag{2.23}$$

(d) *the closed-loop system (2.11)-(2.12),(2.23) is L^2-stable, i.e., for any given $\varphi_x(\cdot) \in L^2[-h, 0; E^{n-r}]$, $\varphi_{0x} \in E^{n-r}$ and $\varphi_{0y} \in E^r$, the solution $z(t, \varepsilon) = col\Big(x(t, \varepsilon), y(t, \varepsilon)\Big)$ of (2.11)-(2.12) with $v(t) = v^*_\varepsilon[z(\cdot)](t)$ and the initial conditions (2.14) satisfies the inclusion $z(t, \varepsilon) \in L^2[0, +\infty; E^n]$.*

2.3. **Objectives of the paper.** Our objectives in this paper are the following:
(i) to construct and justify an asymptotic solution of the set (2.17)-(2.20);
(ii) to derive ε-free sufficient conditions for the existence and uniqueness of solution to the OOCP uniformly valid for all sufficiently small $\varepsilon > 0$;
(iii) to obtain an asymptotically suboptimal (as $\varepsilon \to +0$) state-feedback control for the OOCP.

3. Zero-order asymptotic solution of (2.17)-(2.20)

3.1. **Transformation of (2.17)-(2.20).** In order to remove the singularities at $\varepsilon = 0$ from the right-hand sides of the equations (2.17)-(2.19), we represent the solution $\{P(\varepsilon), Q(\tau, \varepsilon), R(\tau, \rho, \varepsilon)\}$ of (2.17)-(2.20) in the block form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P_2'(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \qquad Q(\tau, \varepsilon) = \begin{pmatrix} Q_1(\tau, \varepsilon) & Q_2(\tau, \varepsilon) \\ \varepsilon Q_3(\tau, \varepsilon) & \varepsilon Q_4(\tau, \varepsilon) \end{pmatrix}, \tag{3.1}$$

$$R(\tau, \rho, \varepsilon) = \begin{pmatrix} R_1(\tau, \rho, \varepsilon) & R_2(\tau, \rho, \varepsilon) \\ R_2'(\rho, \tau, \varepsilon) & R_3(\tau, \rho, \varepsilon) \end{pmatrix}, \tag{3.2}$$

where $P_j(\varepsilon),\ R_j(\tau, \rho, \varepsilon),\ (j = 1, 2, 3)$ are matrices of the dimensions $(n-r) \times (n-r),\ (n-r) \times r,\ r \times r$, respectively; $Q_i(\tau, \varepsilon),\ (i = 1, ..., 4)$ are matrices of the dimensions $(n-r) \times (n-r),\ (n-r) \times r,\ r \times (n-r),\ r \times r$, respectively.

Note that, subject to some symmetry assumptions on $P_l(\varepsilon)$ and $R_l(\tau, \rho, \varepsilon),\ (l = 1, 3)$, the form (3.1)-(3.2) provides the operator $\mathcal{F}_\varepsilon$ to be self-adjoint.

By substituting (3.1)-(3.2), as well as the block representations for the matrices A, H, $G(\tau)$, $S(\varepsilon)$ and D (see (2.4)-(2.5),(2.16)) into (2.17)-(2.20), and using (2.9), the system (2.17)-(2.20) becomes as follows (in this system of equations, for simplicity, we omit the designation of the dependence of the unknown matrices on ε):

$$P_1 A_1 + A_1' P_1 + \varepsilon P_2 A_3 + \varepsilon A_3' P_2' - P_2 M^{-1} P_2' + Q_1(0) + Q_1'(0) + D_x = 0, \tag{3.3}$$

$$P_1 A_2 + \varepsilon P_2 A_4 + \varepsilon A_1' P_2 + \varepsilon A_3' P_3 - P_2 M^{-1} P_3 + Q_2(0) + \varepsilon Q_3'(0) = 0, \tag{3.4}$$

$$\varepsilon P_2' A_2 + \varepsilon A_2' P_2 + \varepsilon P_3 A_4 + \varepsilon A_4' P_3 - P_3 M^{-1} P_3 + \varepsilon Q_4(0) + \varepsilon Q_4'(0) + D_y = 0, \tag{3.5}$$

$$\begin{aligned} dQ_1(\tau)/d\tau = {} & A_1' Q_1(\tau) + \varepsilon A_3' Q_3(\tau) - P_2 M^{-1} Q_3(\tau) \\ & + P_1 G_1(\tau) + \varepsilon P_2 G_3(\tau) + R_1(0, \tau), \end{aligned} \tag{3.6}$$

$$dQ_2(\tau)/d\tau = A_1' Q_2(\tau) + \varepsilon A_3' Q_4(\tau) - P_2 M^{-1} Q_4(\tau) + R_2(0, \tau), \tag{3.7}$$

$$\begin{aligned} \varepsilon dQ_3(\tau)/d\tau = {} & A_2' Q_1(\tau) + \varepsilon A_4' Q_3(\tau) - P_3 M^{-1} Q_3(\tau) \\ & + \varepsilon P_2' G_1(\tau) + \varepsilon P_3 G_3(\tau) + R_2'(\tau, 0), \end{aligned} \tag{3.8}$$

$$\varepsilon dQ_4(\tau)/d\tau = A_2' Q_2(\tau) + \varepsilon A_4' Q_4(\tau) - P_3 M^{-1} Q_4(\tau) + R_3(0, \tau), \tag{3.9}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_1(\tau,\rho) = G_1^{'}(\tau)Q_1(\rho) + Q_1^{'}(\tau)G_1(\rho)$$

$$+ \varepsilon G_3^{'}(\tau)Q_3(\rho) + \varepsilon Q_3^{'}(\tau)G_3(\rho) - Q_3^{'}(\tau)M^{-1}Q_3(\rho), \tag{3.10}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_2(\tau,\rho) = G_1^{'}(\tau)Q_2(\rho) + \varepsilon G_3^{'}(\tau)Q_4(\rho) - Q_3^{'}(\tau)M^{-1}Q_4(\rho), \tag{3.11}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_3(\tau,\rho) = -Q_4^{'}(\tau)M^{-1}Q_4(\rho), \tag{3.12}$$

$$Q_1(-h) = P_1H_1 + \varepsilon P_2H_3, \qquad Q_2(-h) = 0, \tag{3.13}$$

$$Q_3(-h) = P_2^{'}H_1 + P_3H_3, \qquad Q_4(-h) = 0, \tag{3.14}$$

$$R_1(-h,\tau) = H_1^{'}Q_1(\tau) + \varepsilon H_3^{'}Q_3(\tau), \quad R_1(\tau,-h) = Q_1^{'}(\tau)H_1 + \varepsilon Q_3^{'}(\tau)H_3, \tag{3.15}$$

$$R_2(-h,\tau) = H_1^{'}Q_2(\tau) + \varepsilon H_3^{'}Q_4(\tau), \qquad R_2(\tau,-h) = 0, \tag{3.16}$$

$$R_3(-h,\tau) = R_3(\tau,-h) = 0. \tag{3.17}$$

It is verified directly that we can set

$$Q_2(\tau) \equiv 0, \quad Q_4(\tau) \equiv 0, \quad R_2(\tau,\rho) \equiv 0, \quad R_3(\tau,\rho) \equiv 0, \quad (\tau,\rho) \in \mathcal{D} \tag{3.18}$$

without a formal contradiction with the system (3.3)-(3.17). In the sequel, we seek the solution of this system satisfying the condition (3.18).

By substitution (3.18) into (3.3)-(3.17), the latter is reduced to the system

$$P_1A_1 + A_1^{'}P_1 + \varepsilon P_2A_3 + \varepsilon A_3^{'}P_2^{'} - P_2M^{-1}P_2^{'} + Q_1(0) + Q_1^{'}(0) + D_x = 0, \tag{3.19}$$

$$P_1A_2 + \varepsilon P_2A_4 + \varepsilon A_1^{'}P_2 + \varepsilon A_3^{'}P_3 - P_2M^{-1}P_3 + \varepsilon Q_3^{'}(0) = 0, \tag{3.20}$$

$$\varepsilon P_2^{'}A_2 + \varepsilon A_2^{'}P_2 + \varepsilon P_3A_4 + \varepsilon A_4^{'}P_3 - P_3M^{-1}P_3 + D_y = 0, \tag{3.21}$$

$$dQ_1(\tau)/d\tau = A_1^{'}Q_1(\tau) + \varepsilon A_3^{'}Q_3(\tau) - P_2M^{-1}Q_3(\tau)$$

$$+ P_1G_1(\tau) + \varepsilon P_2G_3(\tau) + R_1(0,\tau), \tag{3.22}$$

$$\varepsilon dQ_3(\tau)/d\tau = A_2^{'}Q_1(\tau) + \varepsilon A_4^{'}Q_3(\tau) - P_3M^{-1}Q_3(\tau) + \varepsilon P_2^{'}G_1(\tau) + \varepsilon P_3G_3(\tau), \tag{3.23}$$

$$(\partial/\partial\tau + \partial/\partial\rho)R_1(\tau,\rho) = G_1^{'}(\tau)Q_1(\rho) + Q_1^{'}(\tau)G_1(\rho)$$

$$+ \varepsilon G_3^{'}(\tau)Q_3(\rho) + \varepsilon Q_3^{'}(\tau)G_3(\rho) - Q_3^{'}(\tau)M^{-1}Q_3(\rho), \tag{3.24}$$

$$Q_1(-h) = P_1H_1 + \varepsilon P_2H_3, \tag{3.25}$$

$$Q_3(-h) = P_2^{'}H_1 + P_3H_3, \tag{3.26}$$

$$R_1(-h,\tau) = H_1^{'}Q_1(\tau) + \varepsilon H_3^{'}Q_3(\tau), \quad R_1(\tau,-h) = Q_1^{'}(\tau)H_1 + \varepsilon Q_3^{'}(\tau)H_3. \tag{3.27}$$

The system (3.19)-(3.27) represents a singularly perturbed boundary-value problem for a hybrid set of equations, which contains matrix algebraic, and ordinary and partial differential equations of Riccati type. Moreover, the unknown matrices $Q_1(\tau)$, $Q_3(\tau)$ and $R_1(\tau,\rho)$ are with deviating arguments in this set. This problem is considered in the domain $\mathcal{D}$ with a non-smooth boundary. In order to construct the asymptotic solution of this problem, we adapt the idea of the boundary function method [25] (a short explanation of the boundary function method is presented in Appendix A, Section 11).

3.2. **Formal asymptotic solution of (3.19)-(3.27).** We seek the zero-order asymptotic solution of the problem (3.19)-(3.27) in the form

$$\{\bar{P}_{j0}, Q_{l0}(\tau,\varepsilon), R_{10}(\tau,\rho,\varepsilon)\}, \quad j=1,2,3, \quad l=1,3, \tag{3.28}$$

where the matrices $\bar{P}_{j0}$ are independent of ε, while the matrices $Q_{l0}(\tau,\varepsilon)$ and $R_{10}(\tau,\rho,\varepsilon)$ have the form

$$Q_{l0}(\tau,\varepsilon) = \bar{Q}_{l0}(\tau) + Q^{\tau}_{l0}(\eta), \quad l=1,3, \quad \eta = (\tau+h)/\varepsilon, \tag{3.29}$$

$$R_{10}(\tau,\rho,\varepsilon) = \bar{R}_{10}(\tau,\rho) + R^{\tau}_{10}(\eta,\rho) + R^{\rho}_{10}(\tau,\zeta) + R^{\tau,\rho}_{10}(\eta,\zeta), \quad \zeta = (\rho+h)/\varepsilon. \tag{3.30}$$

Here the terms with the bar are so called outer solution, the terms with the superscript "τ" are the boundary layer correction in a neighborhood of the boundary $\tau = -h$, the term with the superscript "ρ" is the boundary layer correction in a neighborhood of the boundary $\rho = -h$, and the term with the superscript "τ,ρ" is the boundary layer correction in a neighborhood of the corner point $(\tau = -h, \rho = -h)$. Equations and conditions for the asymptotic solution are obtained by substituting (3.28),(3.29) and (3.30) into (3.19)-(3.27) and equating coefficients for the same power of ε on both sides of the resulting equations, separately for the outer solution and for the boundary layer corrections of each type.

3.3. **Obtaining $Q^{\tau}_{10}(\eta)$.** Let us substitute $\bar{P}_{k0}$, $(k=1,2)$, (3.29) and (3.30) into (3.22) instead of P_k, $(k=1,2)$, $Q_l(\tau)$, $(l=1,3)$ and $R_1(\tau,\rho)$, respectively. After such a substitution, let us equate the coefficients of ε^{-1}, depending on η, on both sides of the resulting equation. Thus, we obtain the following equation for $Q^{\tau}_{10}(\eta)$:

$$dQ^{\tau}_{10}(\eta)/d\eta = 0, \quad \eta \geq 0. \tag{3.31}$$

Due to the boundary function method [25] (see also Appendix A, Section 11), we require that $Q^{\tau}_{10}(\eta) \to 0$ for $\eta \to +\infty$. Using this requirement, one directly has from (3.31)

$$Q^{\tau}_{10}(\eta) = 0 \quad \forall \eta \geq 0. \tag{3.32}$$

3.4. **Obtaining $R^{\tau}_{10}(\eta,\rho)$, $R^{\rho}_{10}(\tau,\zeta)$, $R^{\tau,\rho}_{10}(\eta,\zeta)$.** In order to obtain $R^{\tau}_{10}(\eta,\rho)$, $R^{\rho}_{10}(\tau,\zeta)$, $R^{\tau,\rho}_{10}(\eta,\zeta)$, let us substitute (3.29) and (3.30) into (3.24) instead of $Q_l(\tau)$, $(l=1,3)$ and $R_1(\tau,\rho)$, respectively. Then, let us equate the coefficients of ε^{-1}, separately depending on (η,ρ), (τ,ζ) and (η,ζ), on both sides of the resulting equation. Thus, the following equations are obtained for $R^{\tau}_{10}(\eta,\rho)$, $R^{\rho}_{10}(\tau,\zeta)$, $R^{\tau,\rho}_{10}(\eta,\zeta)$:

$$\partial R^{\tau}_{10}(\eta,\rho)/\partial\eta = 0, \quad \eta \geq 0, \tag{3.33}$$

$$\partial R^{\rho}_{10}(\tau,\zeta)/\partial\zeta = 0, \quad \zeta \geq 0, \tag{3.34}$$

$$(\partial/\partial\eta + \partial/\partial\zeta) R^{\tau,\rho}_{10}(\eta,\zeta) = 0, \quad \eta \geq 0, \quad \zeta \geq 0. \tag{3.35}$$

Based on the boundary function method [25] (see also Appendix A, Section 11), we require that

$$\lim_{\eta\to+\infty} R^{\tau}_{10}(\eta,\rho) = 0, \quad \rho \in [-h,0], \tag{3.36}$$

(3.37) $$\lim_{\zeta\to+\infty} R^{\rho}_{10}(\tau,\zeta)=0,\qquad \tau\in[-h,0],$$

(3.38) $$\lim_{\eta+\zeta\to+\infty} R^{\tau,\rho}_{10}(\eta,\zeta)=0.$$

The equations (3.33)-(3.35) subject to the conditions (3.36)-(3.38) yield the unique solutions

(3.39) $$R^{\tau}_{10}(\eta,\rho)=0 \quad \forall(\eta,\rho)\in[0,+\infty)\times[-h,0],$$

(3.40) $$R^{\rho}_{10}(\tau,\zeta)=0 \quad \forall(\tau,\zeta)\in[-h,0]\times[0,+\infty),$$

(3.41) $$R^{\tau,\rho}_{10}(\eta,\zeta)=0 \quad \forall(\eta,\zeta)\in[0,+\infty)\times[0,+\infty).$$

3.5. **Obtaining the outer solution.**

3.5.1. *Equations and Conditions for the Outer Solution.* Equations and conditions for the outer solution are obtained by substituting (3.28) into the system (3.19)-(3.27) instead of $\{P_j, Q_l(\tau), R_1(\tau,\rho)\}$, $(j=1,2,3;\ l=1,3)$ and equating those coefficients for ε^0, which are the outer solution terms, on both sides of the resulting equations. Thus, we have in the domain $\mathcal{D}$

(3.42) $$\bar P_{10}A_1+A_1'\bar P_{10}-\bar P_{20}M^{-1}\bar P_{20}'+\bar Q_{10}(0)+\bar Q_{10}'(0)+D_x=0,$$

(3.43) $$\bar P_{10}A_2-\bar P_{20}M^{-1}\bar P_{30}=0,$$

(3.44) $$-\bar P_{30}M^{-1}\bar P_{30}+D_y=0,$$

(3.45) $$d\bar Q_{10}(\tau)/d\tau=A_1'\bar Q_{10}(\tau)-\bar P_{20}M^{-1}\bar Q_{30}(\tau)+\bar P_{10}G_1(\tau)+\bar R_{10}(0,\tau),$$

(3.46) $$A_2'\bar Q_{10}(\tau)-\bar P_{30}M^{-1}\bar Q_{30}(\tau)=0,$$

(3.47) $$(\partial/\partial\tau+\partial/\partial\rho)\bar R_{10}(\tau,\rho)=G_1'(\tau)\bar Q_{10}(\rho)+\bar Q_{10}'(\tau)G_1(\rho)-\bar Q_{30}'(\tau)M^{-1}\bar Q_{30}(\rho),$$

(3.48) $$\bar Q_{10}(-h)=\bar P_{10}H_1,$$

(3.49) $$\bar R_{10}(-h,\tau)=H_1'\bar Q_{10}(\tau),\quad \bar R_{10}(\tau,-h)=\bar Q_{10}'(\tau)H_1.$$

The equation (3.44) has the following unique symmetric positive definite solution [27]

(3.50) $$\bar P_{30}=M^{1/2}\Big(M^{-1/2}D_yM^{-1/2}\Big)^{1/2}M^{1/2},$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of respective symmetric positive definite matrix, the one "-1/2" denotes the square root of respective inverse matrix.

The equations (3.43) and (3.46) yield, respectively,

(3.51) $$\bar P_{20}=\bar P_{10}A_2\alpha^{-1},$$

and

(3.52) $$\bar Q_{30}(\tau)=(\alpha')^{-1}A_2'\bar Q_{10}(\tau).$$

where

$$\alpha \triangleq M^{-1}\bar{P}_{30} = M^{-1/2}\Big(M^{-1/2}D_y M^{-1/2}\Big)^{1/2} M^{1/2}. \tag{3.53}$$

Since D_y is positive definite, all eigenvalues of α are real positive.

Eliminating $\bar{P}_{20}$ and $\bar{Q}_{30}(\tau)$ from the equations (3.42),(3.45) and (3.47) by using (3.50) and (3.51)-(3.52), we obtain the set of equations

$$\bar{P}_{10}A_1 + A_1^{'}\bar{P}_{10} + \bar{Q}_{10}(0) + \bar{Q}_{10}^{'}(0) + D_x - \bar{P}_{10}A_2D_y^{-1}A_2^{'}\bar{P}_{10} = 0, \tag{3.54}$$

$$d\bar{Q}_{10}(\tau)/d\tau = A_1^{'}\bar{Q}_{10}(\tau) + \bar{P}_{10}G_1(\tau) + \bar{R}_{10}(0,\tau) - \bar{P}_{10}A_2D_y^{-1}A_2^{'}\bar{Q}_{10}(\tau), \tag{3.55}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\bar{R}_{10}(\tau,\rho) = G_1^{'}(\tau)\bar{Q}_{10}(\rho) + \bar{Q}_{10}^{'}(\tau)G_1(\rho)$$
$$-\bar{Q}_{10}^{'}(\tau)A_2D_y^{-1}A_2^{'}\bar{Q}_{10}(\rho). \tag{3.56}$$

Thus, in order to obtain the outer solution, one has to solve the system (3.54)-(3.56) with the boundary conditions (3.48)-(3.49).

3.5.2. *Reduced Optimal Control Problem and Solution of the Problem (3.48)-(3.49), (3.54)-(3.56).* Setting formally $\varepsilon = 0$ in the OOCP, one obtains the following problem, after a simple rearrangement and a redenoting x, y and J by $\bar{x}$, $\bar{y}$ and $\bar{J}$, respectively,

$$d\bar{x}(t)/dt = A_1\bar{x}(t) + H_1\bar{x}(t-h) + \int_{-h}^{0} G_1(\tau)\bar{x}(t+\tau)d\tau + A_2\bar{y}(t), \tag{3.57}$$

$$\bar{J} \triangleq \int_0^{+\infty}\Big[\bar{x}^{'}(t)D_x\bar{x}(t) + \bar{y}^{'}(t)D_y\bar{y}(t)\Big]dt \to \min. \tag{3.58}$$

$$\bar{x}(\tau) = \varphi_x(\tau), \quad \tau \in [-h, 0); \quad \bar{x}_0 = \varphi_{0x}. \tag{3.59}$$

Since the variable $\bar{y}(t)$ does not satisfy any equation for $t \in [0, +\infty)$, the cost functional $\bar{J}$ can be minimized only by a proper choice of $\bar{y}(t)$, $t \in [0, +\infty)$. This means that the variable $\bar{y}(t)$ is a control variable in the problem (3.57)-(3.59). This optimal control problem is called the reduced optimal control problem (ROCP) associated with the OOCP.

Definition 3.1. The system (3.57) is said to be L^2-stabilizable if for each pair

$$\mathcal{P} \triangleq \Big(\varphi_x(\cdot), \varphi_{0x}\Big) \in L^2[-h, 0; E^{n-r}] \times E^{n-r}, \tag{3.60}$$

there exists a control function $\bar{y}(t)$, $\Big(\bar{y}(\cdot) \in L^2[0, +\infty; E^r]\Big)$, such that the solution $\bar{x}(t)$ of the system (3.57) with the initial conditions (3.59) satisfies the inclusion $\bar{x}(\cdot) \in L^2[0, +\infty; E^{n-r}]$.

In the sequel, we assume:

A1. The system (3.57) is L^2-stabilizable, i.e. (see [26]),

$$\mathrm{rank}\Big(W_1(\lambda) - \lambda I_{n-r},\ A_2\Big) = n - r \quad \forall\lambda:\ \mathrm{Re}\lambda \geq 0,$$

where

$$W_1(\lambda) = A_1 + H_1 \exp(-\lambda h) + \int_{-h}^{0} G_1(\tau) \exp(\lambda \tau) d\tau.$$

Let the problem (3.48)-(3.49), (3.54)-(3.56) have a solution

$$\bar{\mathcal{S}} \overset{\triangle}{=} \left\{ \bar{P}_{10}, \bar{Q}_{10}(\tau), \bar{R}_{10}(\tau, \rho) \right\}$$

in the domain $\mathcal{D}$. Based on this solution, let construct the linear bounded operator $\bar{\mathcal{F}} : \mathcal{M}[-h, 0; n-r; n-r] \to \mathcal{M}[-h, 0; n-r; n-r]$ given by the equation

(3.61)

$$\bar{\mathcal{F}}[g(\cdot)] = \left(\bar{P}_{10} g_E + \int_{-h}^{0} \bar{Q}_{10}(\rho) g_L(\rho) d\rho,\ \bar{Q}_{10}^{'}(\tau) g_E + \int_{-h}^{0} \bar{R}_{10}(\tau, \rho) g_L(\rho) d\rho \right),$$

where $g(\cdot) = (g_E, g_L(\cdot)), g_E \in E^{n-r}, g_L(\cdot) \in L^2[-h, 0; E^{n-r}]$.

Based on results of [4] (Theorems 5.8, 5.9, 6.1), one directly obtains the following lemma.

Lemma 3.2. *Under the assumption A1, the optimal feedback control of the ROCP exists, is unique and has the form*

$$\bar{y}^*[\bar{x}(t), \bar{x}_h(t)] = -D_y^{-1} A_2^{'} \left[\bar{P}_{10} \bar{x}(t) + \int_{-h}^{0} \bar{Q}_{10}(\tau) \bar{x}(t+\tau) d\tau \right], \tag{3.62}$$

where $t \geq 0$; $\bar{x}_h(t) = \{\bar{x}(t+\tau) \ \ \forall \tau \in [-h, 0)\}$; the matrices $\bar{P}_{10}$ and $\bar{Q}_{10}(\tau)$ are the respective components of the unique solution $\bar{\mathcal{S}}$ to the problem (3.48)-(3.49),(3.54)-(3.56), meeting the conditions:

(i) *$\bar{P}_{10}$ is a symmetric positive-definite matrix;*
(ii) *$\bar{R}_{10}^{'}(\tau, \rho) = \bar{R}_{10}(\rho, \tau)$;*
(iii) *the operator $\bar{\mathcal{F}}$ is self-adjoint and positive.*

Moreover, the system (3.57) with the optimal control $y_s^*[\cdot]$ is L^2-stable, i.e., for any initial conditions (3.59), its solution $\bar{x}^*(t)$ belongs to $L^2[0, +\infty; E^{n-r}]$.

3.6. **Obtaining $Q_3^{\tau}(\eta)$.** Let us substitute $\bar{P}_{k0}$, $(k = 2, 3)$ and (3.29) into (3.23) instead of P_k, $(k = 2, 3)$ and $Q_l(\tau)$, $(l = 1, 3)$, respectively. After such a substitution, let us equate the coefficients of ε^0, depending on η, on both sides of the resulting equation. Using (3.32), (3.39)-(3.41) and (3.53), we obtain the following equation for $Q_{30}^{\tau}(\eta)$:

$$dQ_{30}^{\tau}(\eta)/d\eta = -\alpha^{'} Q_{30}^{\tau}(\eta), \quad \eta \geq 0. \tag{3.63}$$

The condition for $Q_{30}^{\tau}(\eta)$ is obtained by substituting $\bar{P}_{k0}$, $(k = 2, 3)$ and (3.29) into (3.26) instead of P_k, $(k = 2, 3)$ and $Q_3(\tau)$, respectively, and equating the coefficients of ε^0 on both sides of the resulting equation. Thus, we obtain

$$Q_{30}^{\tau}(0) = \bar{P}_{20}^{'} H_1 + \bar{P}_{30} H_3 - \bar{Q}_{30}(-h). \tag{3.64}$$

Let us transform this condition. First, using the equation (3.48) and the equations (3.51)-(3.52), one has

$$\bar{Q}_{30}(-h) = \bar{P}_{20}^{'} H_1. \tag{3.65}$$

Now, the equations (3.64) and (3.65) yield

$$Q_{30}^{\tau}(0) = \bar{P}_{30}H_3, . \tag{3.66}$$

Solving the initial-value problem (3.63),(3.66), we obtain

$$Q_{30}^{\tau}(\eta) = \bar{P}_{30}H_3 \exp(-\alpha^{'}\eta), \quad \eta \geq 0. \tag{3.67}$$

Since all eigenvalues of the matrix α are real positive, the equation (3.67) leads to the inequality

$$\left\|Q_{30}^{\tau}(\eta)\right\| \leq a \exp(-\beta\eta), \quad \eta \geq 0, \tag{3.68}$$

where $a > 0$ and $\beta > 0$ are some constants.

3.7. **Justification of the asymptotic solution.**

Theorem 3.3. *Under the assumption A1, there exists a positive number ε_1^* such that, for all $\varepsilon \in (0, \varepsilon_1^*]$, the problem (3.19)-(3.27) has a solution $\{P_j(\varepsilon), Q_l(\tau,\varepsilon), R_1(\tau,\rho,\varepsilon),\ j = 1,2,3,\ l = 1,3\}$ in the domain $\mathcal{D}$. For all $(\tau,\rho,\varepsilon) \in \mathcal{D} \times (0, \varepsilon_1^*]$, this solution satisfies the conditions*

$$P_l^{'}(\varepsilon) = P_l(\varepsilon), \quad l = 1,3; \qquad R_1^{'}(\tau,\rho,\varepsilon) = R_1(\rho,\tau,\varepsilon), \tag{3.69}$$

and the inequalities

$$\left\|P_j(\varepsilon) - \bar{P}_{j0}\right\| \leq a\varepsilon, \qquad \left\|R_1(\tau,\rho,\varepsilon) - \bar{R}_{10}(\tau,\rho)\right\| \leq a\varepsilon, \quad j = 1,2,3, \tag{3.70}$$

$$\left\|Q_1(\tau,\varepsilon) - \bar{Q}_{10}(\tau)\right\| \leq a\varepsilon, \qquad \left\|Q_3(\tau,\varepsilon) - Q_{30}(\tau,\varepsilon)\right\| \leq a\varepsilon, \tag{3.71}$$

where $a > 0$ is some constant independent of ε.

The proof of the theorem is presented in Section 5.

As a direct consequence of Theorem 3.3, we have the following corollary.

Corollary 3.4. *Under the assumption A1, for all $\varepsilon \in (0, \varepsilon_1^*]$, the problem (3.3)-(3.17) has a solution $\{P_j(\varepsilon), Q_i(\tau,\varepsilon), R_j(\tau,\rho,\varepsilon),\ j = 1,2,3,\ i = 1,...,4\}$. The components $Q_k(\tau,\varepsilon)$, $(k = 2,4)$ and $R_l(\tau,\rho,\varepsilon)$, $(l = 2,3)$ of this solution satisfy (3.18). The other components of this solution constitute the solution of the problem (3.19)-(3.27) mentioned in Theorem 3.3.*

3.8. **Operators based on the asymptotic solution.** Let us consider the following $n \times n$ block matrices

$$\bar{P}_0(\varepsilon) = \begin{pmatrix} \bar{P}_{10} & \varepsilon\bar{P}_{20} \\ \varepsilon\bar{P}_{20}^{'} & \varepsilon\bar{P}_{30} \end{pmatrix}, \qquad Q_0(\tau,\varepsilon) = \begin{pmatrix} \bar{Q}_{10}(\tau) & 0 \\ \varepsilon Q_{30}(\tau,\varepsilon) & 0 \end{pmatrix}, \tag{3.72}$$

$$\bar{Q}_0(\tau,\varepsilon) = \begin{pmatrix} \bar{Q}_{10}(\tau) & 0 \\ \varepsilon\bar{Q}_{30}(\tau) & 0 \end{pmatrix}, \quad \bar{R}_0(\tau,\rho) = \begin{pmatrix} \bar{R}_{10}(\tau,\rho) & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.73}$$

For a given $\varepsilon > 0$, consider two linear bounded operators $\mathcal{F}_{0,\varepsilon} : \mathcal{M}[-h,0;n;n] \to \mathcal{M}[-h,0;n;n]$ and $\bar{\mathcal{F}}_{0,\varepsilon} : \mathcal{M}[-h,0;n;n] \to \mathcal{M}[-h,0;n;n]$ given by the following equations

$$\mathcal{F}_{0,\varepsilon}[f(\cdot)]$$

$$= \left(\bar{P}_0(\varepsilon) f_E + \int_{-h}^{0} Q_0(\rho, \varepsilon) f_L(\rho) d\rho, \; Q_0'(\tau, \varepsilon) f_E + \int_{-h}^{0} \bar{R}_0(\tau, \rho) f_L(\rho) d\rho \right), \tag{3.74}$$

$$\bar{\mathcal{F}}_{0,\varepsilon}[f(\cdot)]$$

$$= \left(\bar{P}_0(\varepsilon) f_E + \int_{-h}^{0} \bar{Q}_0(\rho, \varepsilon) f_L(\rho) d\rho, \; \bar{Q}_0'(\tau, \varepsilon) f_E + \int_{-h}^{0} \bar{R}_0(\tau, \rho) f_L(\rho) d\rho \right), \tag{3.75}$$

where $f(\cdot) = (f_E, f_L(\cdot)), f_E \in E^n, f_L(\cdot) \in L^2[-h, 0; E^n]$.

By virtue of the equation (3.50) and Lemma 3.2, the operators $\mathcal{F}_{0,\varepsilon}$ and $\bar{\mathcal{F}}_{0,\varepsilon}$ are self-adjoint.

Let us denote by $\mathcal{F}_\varepsilon$ the linear bounded operator given by (2.22), where $\{P(\varepsilon), Q(\tau, \varepsilon), R(\tau, \rho, \varepsilon)\}$, $(\tau, \rho) \in \mathcal{D}$ is the solution of the set (2.17)-(2.20) having the block form (3.1)-(3.2) and satisfying (3.18),(3.69),(3.70)-(3.71).

Lemma 3.5. *Let the assumption A1 be satisfied. Let the operator $\bar{\mathcal{F}}$, given by (3.61), be uniformly positive. Then, there exists a number $\varepsilon_2^* > 0$, such that the operators $\mathcal{F}_{0,\varepsilon}$, $\bar{\mathcal{F}}_{0,\varepsilon}$ and $\mathcal{F}_\varepsilon$ are positive for all $\varepsilon \in (0, \varepsilon_2^*]$.*

The proof of the lemma is presented in Section 6.

4. ε-free conditions for the existence and uniqueness of solution to the OOCP

Lemma 4.1. *Let the assumption A1 be valid. Then, for all $\varepsilon > 0$, the system (2.11)-(2.12) is L^2-stabilizable.*

Proof. Due to results of [26], for any $\varepsilon > 0$, the system (2.11)-(2.12) is L^2-stabilizable if and only if the following condition is satisfied

$$\text{rank} \begin{pmatrix} W_1(\lambda) - \lambda I_{n-r} & A_2 & 0 \\ W_3(\lambda) & A_4 - \lambda I_r & \varepsilon^{-1} I_r \end{pmatrix} = n \quad \forall \lambda : \text{Re}\lambda \geq 0, \tag{4.1}$$

where

$$W_3(\lambda) = A_3 + H_3 \exp(-\lambda h) + \int_{-h}^{0} G_3(\tau) \exp(\lambda \tau) d\tau.$$

The validity of the equation (4.1) for any positive ε directly follows from the assumption A1, which proves the lemma. □

Lemmas 2.2, 3.5, 4.1, Theorem 3.3, Corollary 3.3 and the results of [4] (Theorems 5.7, 5.9, 6.1) directly yield the following theorem.

Theorem 4.2. *Let the assumption A1 be satisfied. Let the operator $\bar{\mathcal{F}}$, given by (3.61), be uniformly positive. Then, there exists a number $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:*

- (i) *the set of Riccati-type equations (2.17)-(2.20) has the unique solution $\{P(\varepsilon), Q(\tau, \varepsilon), R(\tau, \rho, \varepsilon)\}$, $(\tau, \rho) \in \mathcal{D}$ providing the operator $\mathcal{F}_\varepsilon$ to be self-adjoint and positive, and the matrix $P(\varepsilon)$ to be positive definite;*
- (ii) *this solution has the block form (3.1)-(3.2) and satisfies the conditions (3.18) and the inequalities (3.70)-(3.71);*

(iii) *the OOCP has the unique optimal state-feedback control (2.23), and this control provides the L^2-stability for the closed-loop system (2.11)-(2.12),(2.23);*
(iv) *the optimal value of $J(v)$ in the OOCP has the form*

$$J_\varepsilon^* = \varphi_0^{'} P(\varepsilon)\varphi_0 + 2\left(\varphi_{0x}^{'}\int_{-h}^{0} Q_1(\tau,\varepsilon)\varphi_x(\tau)d\tau + \varepsilon\varphi_{0y}^{'}\int_{-h}^{0} Q_3(\tau,\varepsilon)\varphi_x(\tau)d\tau\right)$$

$$+ \int_{-h}^{0}\int_{-h}^{0}\varphi_x^{'}(\tau)R_1(\tau,\rho,\varepsilon)\varphi_x(\rho)d\tau d\rho, \qquad \varphi_0 = col(\varphi_{0x},\varphi_{0y}). \tag{4.2}$$

5. Proof of Theorem 3.3

The proof is based on the following auxiliary results.

5.1. **Auxiliary Lemmas.** Consider the system

$$dw(t)/dt = \tilde{A}(\varepsilon)w(t) + \tilde{H}(\varepsilon)w(t-h) + \int_{-h}^{0}\tilde{G}(\tau,\varepsilon)w(t+\tau)d\tau, \quad t \geq 0, \tag{5.1}$$

where $w \in E^n$ and the $n\times n$-matrices $\tilde{A}(\varepsilon)$, $\tilde{H}(\varepsilon)$ and $\tilde{G}(\tau,\varepsilon)$ have the block form

$$\tilde{A}(\varepsilon) = \begin{pmatrix} \tilde{A}_1(\varepsilon) & \tilde{A}_2(\varepsilon) \\ \varepsilon^{-1}\tilde{A}_3(\varepsilon) & \varepsilon^{-1}\tilde{A}_4(\varepsilon)\end{pmatrix}, \qquad \tilde{H}(\varepsilon) = \begin{pmatrix} \tilde{H}_1(\varepsilon) & 0 \\ \varepsilon^{-1}\tilde{H}_3(\varepsilon) & 0\end{pmatrix}, \tag{5.2}$$

$$\tilde{G}(\tau,\varepsilon) = \begin{pmatrix} \tilde{G}_{11}(\tau,\varepsilon) + \tilde{G}_{12}(\eta,\varepsilon) & 0 \\ \varepsilon^{-1}[\tilde{G}_{31}(\tau,\varepsilon) + \tilde{G}_{32}(\eta,\varepsilon)] & 0\end{pmatrix}, \quad \eta = (\tau+h)/\varepsilon, \tag{5.3}$$

the blocks $\tilde{A}_1(\varepsilon)$, $\tilde{H}_1(\varepsilon)$, $\tilde{G}_{11}(\tau,\varepsilon)$ and $\tilde{G}_{12}(\eta,\varepsilon)$ are of the dimension $(n-r)\times(n-r)$, and the blocks $\tilde{A}_3(\varepsilon)$, $\tilde{H}_3(\varepsilon)$, $\tilde{G}_{31}(\tau,\varepsilon)$ and $\tilde{G}_{32}(\eta,\varepsilon)$ are of the dimension $r\times(n-r)$.

We assume that:

A2. There exists a constant $\tilde{\varepsilon}_1 > 0$, such that:
(a) $\tilde{A}_i(\varepsilon)$ and $\tilde{H}_l(\varepsilon)$, $(i = 1,...,4;\ l = 1,3)$ satisfy the Lipshitz condition with respect to $\varepsilon \in [0,\tilde{\varepsilon}_1]$;
(b) $\tilde{G}_{l1}(\tau,\varepsilon)$, $(l = 1,3)$ are piece-wise continuous with respect to $\tau \in [-h,0]$ for each $\varepsilon \in [0,\tilde{\varepsilon}_1]$;
(c) $\tilde{G}_{l1}(\tau,\varepsilon)$, $(l = 1,3)$ satisfy the Lipshitz condition with respect to $\varepsilon \in [0,\tilde{\varepsilon}_1]$ uniformly in $\tau \in [-h,0]$;
(d) $\tilde{G}_{l2}(\eta,\varepsilon)$, $(l = 1,3)$ are piece-wise continuous with respect to $\eta \in [0,h/\varepsilon]$ for each $\varepsilon \in (0,\tilde{\varepsilon}_1]$;
(e) $\tilde{G}_{l2}(\eta,\varepsilon)$, $(l = 1,3)$ satisfy the inequality

$$\|G_{l2}(\eta,\varepsilon)\| \leq a\exp(-\beta\eta) \quad \forall(\eta,\varepsilon)\in[0,+\infty)\times[0,\tilde{\varepsilon}_1], \tag{5.4}$$

where a and β are some positive constants independent of ε.

A3. The matrix $\tilde{A}_4(0)$ is a Hurwitz one.

A4. The reduced-order subsystem associated with (5.1)

$$d\bar{w}_1(t)/dt = \Gamma\bar{w}_1(t) + \Theta\bar{w}_1(t-h) + \int_{-h}^{0}\Omega(\tau)\bar{w}_1(t+\tau)d\tau, \quad t \geq 0, \tag{5.5}$$

where $\bar{w}_1(t) \in E^{n-r}$, and

$$\Gamma = \tilde{A}_1(0) - \tilde{A}_2(0)\tilde{A}_4^{-1}(0)\tilde{A}_3(0), \quad \Theta = \tilde{H}_1(0) - \tilde{A}_2(0)\tilde{A}_4^{-1}(0)\tilde{H}_3(0), \tag{5.6}$$

$$\Omega(\tau) = \tilde{G}_{11}(\tau, 0) - \tilde{A}_2(0)\tilde{A}_4^{-1}(0)\tilde{G}_{31}(\tau, 0), \tag{5.7}$$

is asymptotically stable, i.e., all roots λ of the equation

$$\det\left[\Gamma + \exp(-\lambda h)\Theta + \int_{-h}^{0} \exp(\lambda\tau)\Omega(\tau)d\tau - I_{n-r}\right] = 0 \tag{5.8}$$

have negative real parts.

Let $\Psi(t, \varepsilon)$ be the fundamental matrix of the system (5.1), i.e., it satisfies this system and the initial conditions

$$\Psi(t, \varepsilon) = 0, \quad t < 0; \qquad \Psi(0, \varepsilon) = I_n. \tag{5.9}$$

Lemma 5.1. *Let $\Psi_1(t, \varepsilon), \Psi_2(t, \varepsilon), \Psi_3(t, \varepsilon)$ and $\Psi_4(t, \varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of the matrix $\Psi(t, \varepsilon)$ of the dimensions $(n-r) \times (n-r), (n-r) \times r, r \times (n-r)$ and $r \times r$, respectively. Under the assumptions A2-A4, there exists a constant $\tilde{\varepsilon}_2 > 0$ ($\tilde{\varepsilon}_2 \le \tilde{\varepsilon}_1$), such that for all $t \ge 0$ and $\varepsilon \in (0, \tilde{\varepsilon}_2]$, the following inequalities are satisfied:*

$$\|\Psi_l(t, \varepsilon)\| \le a \exp(-\nu t), \quad (l = 1, 3), \qquad \|\Psi_2(t, \varepsilon)\| \le a\varepsilon \exp(-\nu t), \tag{5.10}$$

$$\|\Psi_4(t, \varepsilon)\| \le a \exp(-\nu t)[\varepsilon + \exp(-\beta t/\varepsilon)], \tag{5.11}$$

where $a > 0, \nu > 0$ and $\beta > 0$ are some constants independent of ε.

Proof. The lemma is a direct extension of results of [5] (Theorem 2.1) in the case where the slow state variable has a single point-wise delay and a distributed delay, while the fast state variable has no delays. The inequalities (5.10)-(5.11) are proved in the same way as the similar inequalities in [5]. □

Consider the particular case of the system (5.1) with the coefficients

$$\tilde{A}(\varepsilon) = A - S(\varepsilon)\bar{P}_0(\varepsilon), \quad \tilde{H}(\varepsilon) = H, \quad \tilde{G}(\tau, \varepsilon) = G(\tau) - S(\varepsilon)Q_0(\tau, \varepsilon), \tag{5.12}$$

where A, H and $G(\tau)$ are defined in (2.4)-(2.5),(2.9); $\bar{P}_0(\varepsilon)$ and $Q_0(\tau, \varepsilon)$ are defined in (3.72).

Let $\Lambda(t, \varepsilon)$ be the fundamental matrix of the system (5.1) with the matrices of coefficients given by (5.12). Let $\Lambda_1(t, \varepsilon), \Lambda_2(t, \varepsilon), \Lambda_3(t, \varepsilon)$ and $\Lambda_4(t, \varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of the matrix $\Lambda(t, \varepsilon)$ of the dimensions $(n-r) \times (n-r), (n-r) \times r, r \times (n-r)$ and $r \times r$, respectively.

Lemma 5.2. *Under the assumption A1, there exists a constant $\tilde{\varepsilon}_3 > 0$, such that for all $t \ge 0$ and $\varepsilon \in (0, \tilde{\varepsilon}_3]$, the following inequalities are satisfied:*

$$\|\Lambda_l(t, \varepsilon)\| \le a \exp(-\nu t), \ (l = 1, 3), \qquad \|\Lambda_2(t, \varepsilon)\| \le a\varepsilon \exp(-\nu t), \tag{5.13}$$

$$\|\Lambda_4(t, \varepsilon)\| \le a \exp(-\nu t)[\varepsilon + \exp(-\beta t/\varepsilon)], \tag{5.14}$$

where $a > 0, \nu > 0$ and $\beta > 0$ are some constants independent of ε.

Proof. First, let note that the matrices in (5.12) satisfy the assumption A2. Moreover, for the matrix $\tilde{A}(\varepsilon)$, given in (5.12), the block $\tilde{A}_4(\varepsilon)$ has the form $\tilde{A}_4(\varepsilon) = \varepsilon A_4 - \alpha$. The latter means the fulfilment of the assumption A3 for this matrix.

Now, let construct the reduced-order subsystem, associated with the system (5.1),(5.12), and show the asymptotic stability of this subsystem. By virtue of (5.2)-(5.3) and (5.12), one has

$$\tilde{A}_1(\varepsilon) = A_1, \ \ \tilde{A}_2(\varepsilon) = A_2, \ \ \tilde{A}_3(\varepsilon) = \varepsilon A_3 - M^{-1}\bar{P}_{20}', \ \ \tilde{A}_4(\varepsilon) = \varepsilon A_4 - M^{-1}\bar{P}_{30}, \tag{5.15}$$

$$\tilde{H}_l(\varepsilon) = H_l, \quad l = 1, 3, \tag{5.16}$$

$$\tilde{G}_{11}(\tau, \varepsilon) = G_1(\tau), \quad \tilde{G}_{12}(\eta, \varepsilon) = 0, \tag{5.17}$$

$$\tilde{G}_{31}(\tau, \varepsilon) = \varepsilon G_3(\tau) - M^{-1}\bar{Q}_{30}(\tau), \quad \tilde{G}_{32}(\eta, \varepsilon) = -M^{-1}Q_{30}^{\tau}(\eta). \tag{5.18}$$

By substituting (5.15)-(5.18) into (5.6)-(5.7), one obtains (after some rearrangements) the matrices Γ, Θ and $\Omega(\tau)$ of coefficients for the reduced-order subsystem (5.5) associated with the system (5.1),(5.12). Namely,

$$\Gamma = A_1 - A_2\bar{P}_{30}^{-1}\bar{P}_{20}', \quad \Theta = H_1, \quad \Omega(\tau) = G_1(\tau) - A_2\bar{P}_{30}^{-1}\bar{Q}_{30}(\tau). \tag{5.19}$$

Let transform equivalently the expressions for Γ and $\Omega(\tau)$. Substituting (3.51) and (3.52) into the expressions for Γ and $\Omega(\tau)$, respectively, and using (3.53), yield after some rearrangements

$$\Gamma = A_1 - A_2\bar{P}_{30}^{-1}M\bar{P}_{30}^{-1}A_2'\bar{P}_{10}, \quad \Omega(\tau) = G_1(\tau) - A_2\bar{P}_{30}^{-1}M\bar{P}_{30}^{-1}A_2'\bar{Q}_{10}(\tau). \tag{5.20}$$

From the equation (3.44), one directly has

$$\bar{P}_{30}^{-1}M\bar{P}_{30}^{-1} = D_y^{-1}. \tag{5.21}$$

Finally, substituting (5.21) into (5.20), we obtain

$$\Gamma = A_1 - A_2D_y^{-1}A_2'\bar{P}_{10}, \quad \Omega(\tau) = G_1(\tau) - A_2D_y^{-1}A_2'\bar{Q}_{10}(\tau). \tag{5.22}$$

Now, let us consider the system (3.57) with the control $\bar{y}(t) = \bar{y}^*[\bar{x}(t), \bar{x}_h(t)]$, where $\bar{y}^*[\cdot]$ is given by (3.62). Substituting (3.62) into (3.57) instead of $\bar{y}(t)$, one obtains the closed-loop system

$$d\bar{x}(t)/dt = (A_1 - A_2D_y^{-1}A_2'\bar{P}_{10})\bar{x}(t) + H_1\bar{x}(t-h)$$
$$+ \int_{-h}^{0} \Big[G_1(\tau) - A_2D_y^{-1}A_2'\bar{Q}_{10}(\tau)\Big]\bar{x}(t+\tau)d\tau. \tag{5.23}$$

Due to Lemma 3.2, this system is L^2-stable.

Consider the characteristic equation of the system (5.23)

$$\det\Big[A_1 - A_2D_y^{-1}A_2'\bar{P}_{10} + \exp(-\lambda h)H_1$$
$$+ \int_{-h}^{0}\exp(\lambda\tau)\Big(G_1(\tau) - A_2D_y^{-1}A_2'\bar{Q}_{10}(\tau)\Big)d\tau - I_{n-r}\Big] = 0. \tag{5.24}$$

Using L^2-stability of (5.23) and Proposition B.1 from Appendix B, Section 12 (for more details, see Theorem 5.3 in [4]), we obtain that all roots λ of (5.24) have negative real part. The latter means that the system (5.23) is asymptotically stable in the sense mentioned in the assumption A4.

Comparing the reduced-order subsystem (5.5),(5.19) with the system (5.23), and using the equivalent expressions for Γ and $\Omega(\tau)$ (see the equation (5.22)), one can conclude that these systems coincide with each other. Thus, the reduced-order subsystem (5.5),(5.19), associated with the system (5.1),(5.12), is asymptotically stable, i.e., this system satisfies the assumption A4.

Thus, the system (5.1),(5.12) satisfies the assumptions A2-A4. The latter, along with Lemma 5.1, directly yields the statement of the lemma. □

5.2. **Main part of the Theorem's Proof.** Let us make the following transformation of variables in the problem (3.19)-(3.27)

$$P_j(\varepsilon) = \bar{P}_{j0} + \delta_{Pj}(\varepsilon), \quad j = 1,2,3, \tag{5.25}$$

$$Q_1(\tau,\varepsilon) = \bar{Q}_{10}(\tau) + \varepsilon\bar{P}_{20}H_3 + \delta_{Q1}(\tau,\varepsilon), \quad Q_3(\tau,\varepsilon) = Q_{30}(\tau,\varepsilon) + \delta_{Q3}(\tau,\varepsilon), \tag{5.26}$$

$$R_1(\tau,\rho,\varepsilon) = \bar{R}_{10}(\tau,\rho) + \varepsilon\Big[Q_{30}^{'}(\tau,\varepsilon)H_3 + H_3^{'}Q_{30}(\rho,\varepsilon) - H_3^{'}\bar{P}_{30}H_3\Big] + \delta_{R1}(\tau,\rho,\varepsilon), \tag{5.27}$$

where $\delta_{Pj}(\varepsilon)$, $(j = 1,2,3)$, $\delta_{Ql}(\tau,\varepsilon)$, $(l = 1,3)$ and $\delta_{R1}(\tau,\rho,\varepsilon)$ are new matrix-valued variables of corresponding dimensions.

Let us introduce in the consideration the following $n \times n$-matrices

$$\delta_P(\varepsilon) = \begin{pmatrix} \delta_{P1}(\varepsilon) & \varepsilon\delta_{P2}(\varepsilon) \\ \varepsilon\delta_{P2}^{'}(\varepsilon) & \varepsilon\delta_{P3}(\varepsilon) \end{pmatrix}, \quad \delta_Q(\tau,\varepsilon) = \begin{pmatrix} \delta_{Q1}(\tau,\varepsilon) & 0 \\ \varepsilon\delta_{Q3}(\tau,\varepsilon) & 0 \end{pmatrix}, \tag{5.28}$$

$$\delta_R(\tau,\rho,\varepsilon) = \begin{pmatrix} \delta_{R1}(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.29}$$

Substituting (5.25)-(5.27) into the problem (3.19)-(3.27) and using (5.28)-(5.29), as well as the equation (3.29) for $l = 3$, the set (3.42)-(3.49) and the set (3.63)-(3.64), one obtains the following problem for the new matrix-valued variables $\delta_P(\varepsilon)$, $\delta_Q(\tau,\varepsilon)$ and $\delta_R(\tau,\rho,\varepsilon)$ in the domain $\mathcal{D}$

$$\delta_P(\varepsilon)\tilde{A}(\varepsilon) + \tilde{A}^{'}(\varepsilon)\delta_P(\varepsilon) + \delta_Q(0,\varepsilon) + \delta_Q^{'}(0,\varepsilon) + D_P(\varepsilon) - \delta_P(\varepsilon)S(\varepsilon)\delta_P(\varepsilon) = 0, \tag{5.30}$$

$$d\delta_Q(\tau,\varepsilon)/d\tau = \tilde{A}^{'}(\varepsilon)\delta_Q(\tau,\varepsilon) + \delta_P(\varepsilon)\tilde{G}(\tau,\varepsilon)$$
$$+ \delta_R(0,\tau,\varepsilon) + D_Q(\tau,\varepsilon) - \delta_P(\varepsilon)S(\varepsilon)\delta_Q(\tau,\varepsilon), \tag{5.31}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\delta_R(\tau,\rho,\varepsilon) = \tilde{G}^{'}(\tau,\varepsilon)\delta_Q(\rho,\varepsilon) + \delta_Q^{'}(\tau,\varepsilon)\tilde{G}(\rho,\varepsilon)$$
$$+ D_R(\tau,\rho,\varepsilon) - \delta_Q^{'}(\tau,\varepsilon)S(\varepsilon)\delta_Q(\rho,\varepsilon), \tag{5.32}$$

$$\delta_Q(-h,\varepsilon) = \delta_P(\varepsilon)H, \tag{5.33}$$

$$\delta_R(-h,\tau,\varepsilon) = H^{'}\delta_Q(\tau,\varepsilon), \quad \delta_R(\tau,-h,\varepsilon) = \delta_Q^{'}(\tau,\varepsilon)H, \tag{5.34}$$

where the matrices $\tilde{A}(\varepsilon)$ and $\tilde{G}(\tau,\varepsilon)$ are given in (5.12).

Matrices $D_P(\varepsilon)$, $D_Q(\tau,\varepsilon)$ and $D_R(\tau,\rho,\varepsilon)$ are expressed in a known form by the matrices $\bar{P}_0(\varepsilon)$ and $Q_0(\tau,\varepsilon)$. These matrices are represented in the block form as follows:

$$D_P(\varepsilon) = \begin{pmatrix} D_{P,1}(\varepsilon) & D_{P,2}(\varepsilon) \\ D_{P,2}^{'}(\varepsilon) & D_{P,3}(\varepsilon) \end{pmatrix}, \quad D_Q(\tau,\varepsilon) = \begin{pmatrix} D_{Q,1}(\tau,\varepsilon) & 0 \\ D_{Q,3}(\tau,\varepsilon) & 0 \end{pmatrix}, \tag{5.35}$$

$$D_R(\tau,\rho,\varepsilon) = \begin{pmatrix} D_{R,1}(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.36}$$

The dimensions of the blocks in (5.35)-(5.36) are the same as in (5.28)-(5.29), and

$$D'_{P,l}(\varepsilon) = D_{P,l}(\varepsilon), \quad l = 1,3; \quad D'_{R,1}(\tau,\rho,\varepsilon) = D_{R,1}(\rho,\tau,\varepsilon). \tag{5.37}$$

Moreover, by application of Lemma 3.2, the equations (3.52) and (3.67), and the inequality (3.68), it can be shown the existence of a constant $\tilde{\varepsilon}_4 > 0$ such that $D_{Q,l}(\tau,\varepsilon)$, $(l = 1,3)$ and $D_{R,1}(\tau,\rho,\varepsilon)$ are continuous in $\tau \in [-h,0]$ and in $(\tau,\rho) \in \mathcal{D}$, respectively, for any $\varepsilon \in (0,\tilde{\varepsilon}_4]$, and the following inequalities are satisfied for all $\varepsilon \in (0,\tilde{\varepsilon}_4]$:

$$\|D_{P,j}(\varepsilon)\| \le a\varepsilon, \quad j = 1,2,3, \tag{5.38}$$

$$\|D_{Q,1}(\tau,\varepsilon)\| \le a\{\varepsilon + \exp[-\beta(\tau+h)/\varepsilon]\}, \quad \|D_{Q,3}(\tau,\varepsilon)\| \le a\varepsilon, \tag{5.39}$$

$$\|D_{R,1}(\tau,\rho,\varepsilon)\| \le a\{\varepsilon + \exp[-\beta(\tau+h)/\varepsilon] + \exp[-\beta(\rho+h)/\varepsilon]\}, \tag{5.40}$$

where $(\tau,\rho) \in \mathcal{D}$; $a > 0$ and $\beta > 0$ are some constants independent of ε.

Let us denote

$$\Delta_P[\delta_P](\varepsilon) \stackrel{\triangle}{=} D_P(\varepsilon) - \delta_P(\varepsilon)S(\varepsilon)\delta_P(\varepsilon), \tag{5.41}$$

$$\Delta_Q[\delta_P,\delta_Q](\tau,\varepsilon) \stackrel{\triangle}{=} D_Q(\tau,\varepsilon) - \delta_P(\varepsilon)S(\varepsilon)\delta_Q(\tau,\varepsilon), \tag{5.42}$$

$$\Delta_R[\delta_Q](\tau,\rho,\varepsilon) \stackrel{\triangle}{=} D_R(\tau,\rho,\varepsilon) - \delta'_Q(\tau,\varepsilon)S(\epsilon)\delta_Q(\rho,\varepsilon), \tag{5.43}$$

$$\tilde{\Lambda}(t,\tau,\varepsilon) \stackrel{\triangle}{=} \Lambda(t-\tau-h,\varepsilon)H + \int_{-\tau}^{h} \Lambda(t-\tau-\rho,\varepsilon)\tilde{G}(-\rho,\varepsilon)d\rho. \tag{5.44}$$

By virtue of (2.5),(2.9),(2.16) and (5.12), the matrix $\tilde{\Lambda}(t,\tau,\varepsilon)$ has the block form

$$\tilde{\Lambda}(t,\tau,\varepsilon) = \begin{pmatrix} \tilde{\Lambda}_1(t,\tau,\varepsilon) & 0 \\ \tilde{\Lambda}_3(t,\tau,\varepsilon) & 0 \end{pmatrix}. \tag{5.45}$$

Moreover, using Lemma 5.2 yields the following inequalities for all $t \ge 0$, $\tau \in [-h,0]$ and $\varepsilon \in (0,\tilde{\varepsilon}_3]$:

$$\|\tilde{\Lambda}_l(t,\tau,\varepsilon)\| \le a\exp(-\nu t), \quad l = 1,3, \tag{5.46}$$

where $a > 0$ and $\nu > 0$ are some constants independent of ε.

Let estimate the matrices $\delta_P(\varepsilon)$, $\delta_Q(\tau,\varepsilon)$ and $\delta_R(\tau,\rho,\varepsilon)$. For this purpose, we will transform equivalently the problem (5.30)-(5.34). This transformation is based on some results of [4], namely, on Theorem 6.1 and its proof (see the equations (6.3),(6.6),(6.10)-(6.11) and (B.4),(B.6),(B.18) in [4]). Using these results allows us to transform the problem (5.30)-(5.34) to an equivalent set of integral equations. Thus, by using the notations (5.41)-(5.44) and the above mentioned results of [4], as well as the fact that $\Lambda(t,\varepsilon)$ is the fundamental matrix of the system (5.1) with the coefficients (5.12), we can rewrite the problem (5.30)-(5.34) in the following equivalent integral form (similarly to [9], [10])

$$\delta_P(\varepsilon) = \int_0^{+\infty} \Big[\Lambda'(t,\varepsilon)\Delta_P[\delta_P](\varepsilon)\Lambda(t,\varepsilon)$$

$$+\int_{-h}^{0}\Lambda^{'}(t,\varepsilon)\Delta_Q[\delta_P,\delta_Q](\tau,\varepsilon)\Lambda(t+\tau,\varepsilon)d\tau$$

$$+\int_{-h}^{0}\Lambda^{'}(t+\tau,\varepsilon)\Delta_Q^{'}[\delta_P,\delta_Q](\tau,\varepsilon)\Lambda(t,\varepsilon)d\tau$$

$$+\int_{-h}^{0}\int_{-h}^{0}\Lambda^{'}(t+\tau,\varepsilon)\Delta_R[\delta_Q](\tau,\rho,\varepsilon)\Lambda(t+\rho,\varepsilon)d\rho d\tau\Big]dt, \tag{5.47}$$

$$\delta_Q(\tau,\varepsilon)=\int_{0}^{+\infty}\Big[\Lambda^{'}(t,\varepsilon)\Delta_P[\delta_P](\varepsilon)\tilde{\Lambda}(t,\tau,\varepsilon)$$

$$+\int_{-h}^{0}\Lambda^{'}(t,\varepsilon)\Delta_Q[\delta_P,\delta_Q](\rho,\varepsilon)\tilde{\Lambda}(t+\rho,\tau,\varepsilon)d\rho$$

$$+\int_{-h}^{0}\Lambda^{'}(t+\rho,\varepsilon)\Delta_Q^{'}[\delta_P,\delta_Q](\rho,\varepsilon)\tilde{\Lambda}(t,\tau,\varepsilon)d\rho$$

$$+\int_{-h}^{0}\int_{-h}^{0}\Lambda^{'}(t+\sigma,\varepsilon)\Delta_R[\delta_Q](\sigma,\rho,\varepsilon)\tilde{\Lambda}(t+\rho,\tau,\varepsilon)d\sigma d\rho\Big]dt$$

$$+\int_{0}^{\tau+h}\Big[\Lambda^{'}(t,\varepsilon)\Delta_Q[\delta_P,\delta_Q](\tau-t,\varepsilon)$$

$$+\int_{-h}^{0}\Lambda^{'}(t+\rho,\varepsilon)\Delta_R[\delta_Q](\rho,\tau-t,\varepsilon)d\rho\Big]dt, \tag{5.48}$$

$$\delta_R(\tau,\rho,\varepsilon)=\int_{0}^{+\infty}\Big[\tilde{\Lambda}^{'}(t,\tau,\varepsilon)\Delta_P[\delta_P](\varepsilon)\tilde{\Lambda}(t,\rho,\varepsilon)$$

$$+\int_{-h}^{0}\tilde{\Lambda}^{'}(t,\tau,\varepsilon)\Delta_Q[\delta_P,\delta_Q](\sigma,\varepsilon)\tilde{\Lambda}(t+\sigma,\rho,\varepsilon)d\sigma$$

$$+\int_{-h}^{0}\tilde{\Lambda}^{'}(t+\sigma,\tau,\varepsilon)\Delta_Q^{'}[\delta_P,\delta_Q](\sigma,\varepsilon)\tilde{\Lambda}(t,\rho,\varepsilon)d\sigma$$

$$+\int_{-h}^{0}\int_{-h}^{0}\tilde{\Lambda}^{'}(t+\sigma,\tau,\varepsilon)\Delta_R[\delta_Q](\sigma,\sigma_1,\varepsilon)\tilde{\Lambda}(t+\sigma_1,\rho,\varepsilon)d\sigma d\sigma_1\Big]dt$$

$$+\int_{0}^{\rho+h}\Big[\tilde{\Lambda}^{'}(t,\tau,\varepsilon)\Delta_Q[\delta_P,\delta_Q](\rho-t,\varepsilon)$$

$$+\int_{-h}^{0}\tilde{\Lambda}^{'}(t+\sigma,\tau,\varepsilon)\Delta_R[\delta_Q](\sigma,\rho-t,\varepsilon)d\sigma\Big]dt$$

$$+\int_{0}^{\tau+h}\Big[\Delta_Q^{'}[\delta_P,\delta_Q](\tau-t,\varepsilon)\tilde{\Lambda}(t,\rho,\varepsilon)$$

$$+\int_{-h}^{0}\Delta_R[\delta_Q](\sigma,\tau-t,\varepsilon)\tilde{\Lambda}(t+\sigma,\rho,\varepsilon)d\sigma\Big]dt$$

$$+\int_{0}^{\min(\tau+h,\rho+h)}\Delta_R[\delta_Q](\tau-t,\rho-t,\varepsilon)dt. \tag{5.49}$$

It is verified directly that

$$0\le\min(\tau+h,\rho+h)\le h,\qquad(\tau,\rho)\in\mathcal{D}. \tag{5.50}$$

Now, applying the procedure of successive approximations to the set (5.47)-(5.49) with zero initial approximation for δ_P, δ_Q and δ_R, and taking into account Lemma 5.2, the equations (5.35)-(5.36),(5.37), (5.41)-(5.44) and the inequalities (5.38)-(5.40), (5.46) and (5.50), one directly obtains the existence of solution $\{\delta_P(\varepsilon), \delta_Q(\tau,\varepsilon), \delta_R(\tau,\rho,\varepsilon)\}$ of the system (5.47)-(5.49) (and, consequently, of the problem (5.30)-(5.34)), having the block form (5.28)-(5.29), and satisfying the following conditions and inequalities for all $(\tau,\rho) \in \mathcal{D}$ and $\varepsilon \in (0, \tilde{\varepsilon}_5]$, where $0 < \tilde{\varepsilon}_5 \le \min(\tilde{\varepsilon}_3, \tilde{\varepsilon}_4)$ is some constant,

$$\delta'_{Pl}(\varepsilon) = \delta_{Pl}(\varepsilon), \quad l = 1,3; \qquad \delta'_{R1}(\tau,\rho,\varepsilon) = \delta_{R1}(\rho,\tau,\varepsilon), \tag{5.51}$$

$$\|\delta_{Pj}(\varepsilon)\| \le a\varepsilon, \;\; j = 1,2,3; \;\; \|\delta_{Ql}(\tau,\varepsilon)\| \le a\varepsilon, \;\; l = 1,3; \;\; \|\delta_{R1}(\tau,\rho,\varepsilon)\| \le a\varepsilon, \tag{5.52}$$

where $a > 0$ is some constant independent of ε.

The conditions (5.51) and the inequalities (5.52), along with the equations (5.25)-(5.27), directly yield the statements of the theorem. □

6. Proof of Lemma 3.5

Let begin with the proof of the positiveness of the operator $\mathcal{F}_{0,\varepsilon}$. Remind that, for a given $\varepsilon > 0$, the operator $\mathcal{F}_{0,\varepsilon}$ is positive if

$$\Big(\mathcal{F}_{0,\varepsilon}[f(\cdot)], f(\cdot)\Big)_{\mathcal{M}} \ge 0 \quad \forall f(\cdot) \in \mathcal{M}[-h,0;n;n]. \tag{6.1}$$

The condition (6.1), along with the equation (3.74), yields

$$\gamma(\varepsilon) \overset{\triangle}{=} \Big(\mathcal{F}_{0,\varepsilon}[f(\cdot)], f(\cdot)\Big)_{\mathcal{M}} = f'_E \bar{P}_0(\varepsilon) f_E + 2 f'_E \int_{-h}^{0} Q_0(\rho,\varepsilon) f_L(\rho) d\rho$$

$$+ \int_{-h}^{0}\int_{-h}^{0} f'_L(\tau) \bar{R}_0(\tau,\rho) f_L(\rho) d\tau d\rho \ge 0 \quad \forall f_E \in E^n, \quad f_L(\cdot) \in L^2[-h,0;E^n]. \tag{6.2}$$

Let $f_{E,(n-r)}$ and $f_{E,r}$ be the upper and lower blocks of the vector f_E of the dimensions $(n-r)$ and r, respectively. Similarly, let, for any $\tau \in [-h,0]$, $f_{L,(n-r)}(\tau)$ and $f_{L,r}(\tau)$ be the upper and lower blocks of the vector $f_L(\tau)$ of the dimensions $(n-r)$ and r, respectively. Then, by using the equation (3.61) and the block form of the matrices $\bar{P}_0(\varepsilon)$, $Q_0(\tau,\varepsilon)$ and $\bar{R}_0(\tau,\rho)$ (see (3.72)-(3.73)), the expression for $\gamma(\varepsilon)$ can be rewritten as follows

$$\gamma(\varepsilon) = \gamma_1 + \varepsilon\gamma_2 + \varepsilon\gamma_3(\varepsilon), \tag{6.3}$$

where

$$\gamma_1 = \Big(\bar{\mathcal{F}}[g_f(\cdot)], g_f(\cdot)\Big)_{\mathcal{M}}, \tag{6.4}$$

$$g_f(\cdot) \overset{\triangle}{=} (f_{E,(n-r)}, f_{L,(n-r)}(\cdot)) \in \mathcal{M}[-h,0;n-r;n-r],$$

$$\gamma_2 = f'_{E,r} \bar{P}_{30} f_{E,r}, \tag{6.5}$$

$$\gamma_3(\varepsilon) = 2 f'_{E,(n-r)} \bar{P}_{20} f_{E,r} + 2 f'_{E,r} \int_{-h}^{0} Q_{30}(\rho,\varepsilon) f_{L,(n-r)}(\rho) d\rho. \tag{6.6}$$

Since, due to the assumption of the lemma, the operator $\bar{\mathcal{F}}$ is uniformly positive, then there exists a positive number μ_1 such that

$$\gamma_1 \geq 2\mu_1\Big(\|f_{E,(n-r)}\|^2 + \|f_{L,(n-r)}(\cdot)\|_{L^2}^2\Big)$$

$$\geq \mu_1\Big(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\Big)^2$$

$$\forall f_{E,(n-r)} \in E^n, f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}]. \tag{6.7}$$

Using the positive definiteness of the matrix $\bar{P}_{30}$, one obtains the inequality

$$\gamma_2 \geq \mu_2\|f_{E,r}\|^2 \quad \forall f_{E,r} \in E^r, \tag{6.8}$$

where $\mu_2 > 0$ is some constant.

Moreover, using the expression for $Q_{30}(\tau,\varepsilon)$ (see (3.29)) and the inequality (3.68) yields the following estimate of $\gamma_3(\varepsilon)$ for all $\varepsilon > 0$:

$$|\gamma_3(\varepsilon)| \leq 2\mu_3\|f_{E,r}\|\Big(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\Big)$$

$$\forall f_{E,r} \in E^r, f_{E,(n-r)} \in E^{n-r}, f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}], \tag{6.9}$$

where $\mu_3 > 0$ is some constant independent of ε.

Using the equation (6.3) and the inequalities (6.7)-(6.9), we directly obtain for all $\varepsilon > 0$

$$\gamma(\varepsilon) \geq \mu_1\Big(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\Big)^2 + \varepsilon\mu_2\|f_{E,r}\|^2$$

$$- 2\varepsilon\mu_3\|f_{E,r}\|\Big(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\Big)$$

$$\forall f_{E,(n-r)} \in E^{n-r},\ f_{E,r} \in E^r,\ f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}]. \tag{6.10}$$

A simple equivalent transformation of the right-hand side of the inequality (6.10) yields the inequality

$$\gamma(\varepsilon) \geq \Big(\sqrt{\mu_1}(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}) - \sqrt{\varepsilon\mu_2}\|f_{E,r}\|\Big)^2$$

$$+ 2\sqrt{\varepsilon}(\sqrt{\mu_1\mu_2} - \sqrt{\varepsilon}\mu_3)\|f_{E,r}\|\Big(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\Big)$$

$$\forall f_{E,(n-r)} \in E^{n-r},\ f_{E,r} \in E^r,\ f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}]. \tag{6.11}$$

The latter implies that, for all $\varepsilon \in (0, \mu_1\mu_2/\mu_3^2]$,

$$\gamma(\varepsilon) \geq 0 \quad \forall f_{E,(n-r)} \in E^{n-r},\ f_{E,r} \in E^r,\ f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}]. \tag{6.12}$$

The inequality (6.12), along with the equation (6.2), directly yields the inequality (6.1), which completes the proof of the positiveness of the operator $\mathcal{F}_{0,\varepsilon}$. The positiveness of the operators $\bar{\mathcal{F}}_{0,\varepsilon}$ and $\mathcal{F}_\varepsilon$ is proved similarly. Note that in such a proof for $\mathcal{F}_\varepsilon$, Theorem 3.3 is used. □

7. Suboptimal feedback controls of the OOCP

7.1. **ε-free state-feedback control.** Consider the following state-feedback control for the OOCP

$$v = \bar{v}_0[z(\cdot)](t) = -\varepsilon^{-1}M^{-1}B'\left[\bar{P}_0(\varepsilon)z(t) + \int_{-h}^{0}\bar{Q}_0(\tau,\varepsilon)z(t+\tau)d\tau\right], \tag{7.1}$$

where $z = \mathrm{col}(x, y)$.

This control is obtained from the OOCP optimal state-feedback control (2.23) by replacing there the matrices $P(\varepsilon)$ and $Q(\tau,\varepsilon)$ by the ones $\bar{P}_0(\varepsilon)$ and $\bar{Q}_0(\tau,\varepsilon)$, respectively.

Substituting the block form of the state variable z and of the matrices B, $\bar{P}_0(\varepsilon)$ and $\bar{Q}_0(\tau,\varepsilon)$ (see (2.2), (3.72) and (3.73)) into (7.1) yields after a simple rearrangement

$$v = \bar{v}_0[z(\cdot)](t) = -M^{-1}\left(\bar{P}_{20}'x(t) + \bar{P}_{30}y(t) + \int_{-h}^{0}\bar{Q}_{30}(\tau)x(t+\tau)d\tau\right). \tag{7.2}$$

It is seen that the state-feedback control $v = \bar{v}_0[z(\cdot)](t)$ is independent of ε.

Substituting $v = \bar{v}_0[z(\cdot)](t)$ into the system (2.11)-(2.12), one obtains the system

$$dx(t)/dt = A_1x(t) + A_2y(t) + H_1x(t-h) + \int_{-h}^{0}G_1(\tau)x(t+\tau)d\tau, \tag{7.3}$$

$$\varepsilon dy(t)/dt = \left(\varepsilon A_3 - M^{-1}\bar{P}_{20}'\right)x(t) + \left(\varepsilon A_4 - M^{-1}\bar{P}_{30}\right)y(t)$$

$$+\varepsilon H_3x(t-h) + \int_{-h}^{0}\left(\varepsilon G_3(\tau) - M^{-1}\bar{Q}_{30}(\tau)\right)x(t+\tau)d\tau. \tag{7.4}$$

Lemma 7.1. *Under the assumption A1, there exists a positive constant $\bar{\varepsilon}_1$, such that for all $\varepsilon \in (0, \bar{\varepsilon}_1]$ the system (7.3)-(7.4) is L^2-stable.*

Proof. Let, for a given $\varepsilon > 0$, $\bar{\Lambda}(t,\varepsilon)$ be the fundamental matrix solution of the system (7.3)-(7.4). Let $\bar{\Lambda}_1(t,\varepsilon)$, $\bar{\Lambda}_2(t,\varepsilon)$, $\bar{\Lambda}_3(t,\varepsilon)$ and $\bar{\Lambda}_4(t,\varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of $\bar{\Lambda}(t,\varepsilon)$ of the dimensions $(n-r)\times(n-r)$, $(n-r)\times r$, $r\times(n-r)$ and $r\times r$, respectively. Then, similarly to the proof of Lemma 5.2, one obtains the existence of a constant $\bar{\varepsilon}_1 > 0$, such that the following inequalities are satisfied for all $t \geq 0$ and $\varepsilon \in (0, \bar{\varepsilon}_1]$:

$$\|\bar{\Lambda}_l(t,\varepsilon)\| \leq a\exp(-\nu t),\ (l = 1, 3), \quad \|\bar{\Lambda}_2(t,\varepsilon)\| \leq a\varepsilon\exp(-\nu t), \tag{7.5}$$

$$\|\bar{\Lambda}_4(t,\varepsilon)\| \leq a\exp(-\nu t)[\varepsilon + \exp(-\beta t/\varepsilon)], \tag{7.6}$$

where $a > 0, \nu > 0$ and $\beta > 0$ are some constants independent of ε.

Using the variation-of-constant formula [12] and the block form of the fundamental matrix solution $\bar{\Lambda}(t,\varepsilon)$ of the system (7.3)-(7.4), one can write down the solution of this system with the initial conditions (2.14) as follows

$$x(t,\varepsilon) = \bar{\Lambda}_1(t,\varepsilon)\varphi_{0x} + \bar{\Lambda}_2(t,\varepsilon)\varphi_{0y} + \int_{-h}^{0}\hat{\Lambda}_1(t,\tau,\varepsilon)\varphi_x(\tau)d\tau, \quad t \geq 0, \tag{7.7}$$

$$y(t,\varepsilon) = \bar{\Lambda}_3(t,\varepsilon)\varphi_{0x} + \bar{\Lambda}_4(t,\varepsilon)\varphi_{0y} + \int_{-h}^{0}\hat{\Lambda}_3(t,\tau,\varepsilon)\varphi_x(\tau)d\tau, \quad t \geq 0, \tag{7.8}$$

where

$$\hat{\Lambda}_1(t,\tau,\varepsilon) = \bar{\Lambda}_1(t-\tau-h,\varepsilon)H_1 + \bar{\Lambda}_2(t-\tau-h,\varepsilon)H_3 + \int_{-\tau}^{h}\Big[\bar{\Lambda}_1(t-\tau-\rho,\varepsilon)G_1(-\rho) + \bar{\Lambda}_2(t-\tau-\rho,\varepsilon)\Big(G_3(-\rho) - \varepsilon^{-1}M^{-1}\bar{Q}_{30}(-\rho)\Big)\Big]d\rho, \tag{7.9}$$

$$\hat{\Lambda}_3(t,\tau,\varepsilon) = \bar{\Lambda}_3(t-\tau-h,\varepsilon)H_1 + \bar{\Lambda}_4(t-\tau-h,\varepsilon)H_3 + \int_{-\tau}^{h}\Big[\bar{\Lambda}_3(t-\tau-\rho,\varepsilon)G_1(-\rho) + \bar{\Lambda}_4(t-\tau-\rho,\varepsilon)\Big(G_3(-\rho) - \varepsilon^{-1}M^{-1}\bar{Q}_{30}(-\rho)\Big)\Big]d\rho. \tag{7.10}$$

By virtue of (7.5)-(7.6), we obtain the following inequalities for all $\varepsilon \in (0,\bar{\varepsilon}_1]$

$$\|\hat{\Lambda}_l(t,\tau,\varepsilon)\| \le a\exp(-\nu t), \quad t \ge 0,\ \tau \in [-h,0], \quad l = 1,3, \tag{7.11}$$

where $a > 0$ and $\nu > 0$ are some constants independent of ε.

Now, using the equations (7.7)-(7.8), the inequalities (7.5)-(7.6) and (7.11), and the Cauchy inequality, we directly obtain the following estimates of $x(t,\varepsilon)$ and $y(t,\varepsilon)$ for all $\varepsilon \in (0,\bar{\varepsilon}_1]$

$$\|x(t,\varepsilon)\| \le a\exp(-\nu t)\Big(\|\varphi_{0x}\| + \|\varphi_{0y}\| + \|\varphi_x\|_{L^2}\Big), \quad t \ge 0, \tag{7.12}$$

$$\|y(t,\varepsilon)\| \le a\exp(-\nu t)\Big(\|\varphi_{0x}\| + \|\varphi_{0y}\| + \|\varphi_x\|_{L^2}\Big), \quad t \ge 0, \tag{7.13}$$

where $a > 0$ and $\nu > 0$ are some constants independent of ε.

The inequalities (7.12)-(7.13) imply the L^2-stability of the system (7.3)-(7.4) for all $\varepsilon \in (0,\bar{\varepsilon}_1]$. Thus, the lemma is proved. □

Consider the following system of algebraic, ordinary differential and partial differential equations with respect to $n \times n$-matrices $\tilde{P}$, $\tilde{Q}(\tau)$ and $\tilde{R}(\tau,\rho)$

$$\tilde{P}\tilde{A}(\varepsilon) + \tilde{A}'(\varepsilon)\tilde{P} + \tilde{Q}(0) + \tilde{Q}'(0) + D + \bar{P}_0(\varepsilon)S(\varepsilon)\bar{P}_0(\varepsilon) = 0, \tag{7.14}$$

$$d\tilde{Q}(\tau)/d\tau = \tilde{A}'(\varepsilon)\tilde{Q}(\tau) + \tilde{P}\hat{G}(\tau,\varepsilon) + \tilde{R}(0,\tau) + \bar{P}_0(\varepsilon)S(\varepsilon)\bar{Q}_0(\tau,\varepsilon), \tag{7.15}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\tilde{R}(\tau,\rho) = \hat{G}'(\tau,\varepsilon)\tilde{Q}(\rho) + \tilde{Q}'(\tau)\hat{G}(\rho,\varepsilon) + \bar{Q}_0'(\tau,\varepsilon)S(\varepsilon)\bar{Q}_0(\rho,\varepsilon), \tag{7.16}$$

where the matrices $S(\varepsilon)$ and D are given by (2.16), the matrix $\tilde{A}(\varepsilon)$ is given in (5.12), and the matrix $\hat{G}(\tau,\varepsilon)$ has the form

$$\hat{G}(\tau,\varepsilon) = G(\tau) - S(\varepsilon)\bar{Q}_0(\tau,\varepsilon), \tag{7.17}$$

the matrix $G(\tau)$ is given in (2.5),(2.9).

The system (7.14)-(7.16) is considered subject to the boundary conditions

$$\tilde{Q}(-h) = \tilde{P}H, \quad \tilde{R}(-h,\tau) = H'\tilde{Q}(\tau), \quad \tilde{R}(\tau,-h) = \tilde{Q}'(\tau)H, \tag{7.18}$$

where the matrix H is given in (2.5),(2.9).

Let $\bar{J}_{0\varepsilon}$ be the value of the cost functional $J(v)$ (see (2.13)) obtained by employing the state-feedback control (7.2) in the system (2.11)-(2.12) subject to the initial conditions (2.14).

Lemma 7.2. *Let the assumption A1 is satisfied. Then, for all* $\varepsilon \in (0, \bar{\varepsilon}_1]$*:*

(a) *the problem (7.14)-(7.16),(7.18) has the unique solution* $\{\tilde{P}(\varepsilon), \tilde{Q}(\tau,\varepsilon), \tilde{R}(\tau,\rho,\varepsilon)\}$ *in the domain* $\mathcal{D}$*;*

(b) *the operator* $\tilde{\mathcal{F}}_\varepsilon : \mathcal{M}[-h,0;n;n] \to \mathcal{M}[-h,0;n;n]$*, given by the equation*

$$\tilde{\mathcal{F}}_\varepsilon[f(\cdot)]$$

$$= \left(\tilde{P}(\varepsilon)f_E + \int_{-h}^{0} \tilde{Q}(\rho,\varepsilon)f_L(\rho)d\rho,\ \tilde{Q}^{'}(\tau,\varepsilon)f_E + \int_{-h}^{0} \tilde{R}(\tau,\rho,\varepsilon)f_L(\rho)d\rho\right), \tag{7.19}$$

where $f(\cdot) = (f_E, f_L(\cdot)), f_E \in E^n, f_L(\cdot) \in L^2[-h,0;E^n]$*, is self-adjoint and positive;*

(c) *the matrix* $\tilde{P}(\varepsilon)$ *is positive definite.*

Proof. The statements of the lemma are direct consequences of Lemma 7.1 and the results of [4] (Theorem 5.3, Proposition 5.4, Theorem 5.5). □

Lemma 7.3. *Let the assumption A1 is satisfied. Then, there exists a positive number* $\bar{\varepsilon}_2$ $(\bar{\varepsilon}_2 \le \bar{\varepsilon}_1)$*, such that for all* $\varepsilon \in (0, \bar{\varepsilon}_2]$*:*

(a) *the solution* $\{\tilde{P}(\varepsilon), \tilde{Q}(\tau,\varepsilon), \tilde{R}(\tau,\rho,\varepsilon)\}$, $(\tau,\rho) \in \mathcal{D}$ *of the problem (7.14)-(7.16),(7.18) has the block form*

$$\tilde{P}(\varepsilon) = \begin{pmatrix} \tilde{P}_1(\varepsilon) & \varepsilon\tilde{P}_2(\varepsilon) \\ \varepsilon\tilde{P}_2^{'}(\varepsilon) & \varepsilon\tilde{P}_3(\varepsilon) \end{pmatrix}, \qquad \tilde{Q}(\tau,\varepsilon) = \begin{pmatrix} \tilde{Q}_1(\tau,\varepsilon) & 0 \\ \varepsilon\tilde{Q}_3(\tau,\varepsilon) & 0 \end{pmatrix}, \tag{7.20}$$

$$\tilde{R}(\tau,\rho,\varepsilon) = \begin{pmatrix} \tilde{R}_1(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.21}$$

where $\tilde{P}_j(\varepsilon)$, $(j = 1,2,3)$ *are matrices of the dimensions* $(n-r)\times(n-r)$, $(n-r)\times r$, $r \times r$*, respectively;* $\tilde{Q}_l(\tau,\varepsilon)$, $(l = 1,3)$ *are matrices of the dimensions* $(n-r)\times(n-r)$, $r\times(n-r)$*, respectively;* $\tilde{R}_1(\tau,\rho,\varepsilon)$ *is a matrix of the dimension* $(n-r)\times(n-r)$*;*

(b) *these matrices satisfy the inequalities*

$$\left\|\tilde{P}_j(\varepsilon) - \bar{P}_{j0}\right\| \le a\varepsilon, \qquad j = 1,2,3, \tag{7.22}$$

$$\left\|\tilde{Q}_1(\tau,\varepsilon) - \bar{Q}_{10}(\tau)\right\| \le a\varepsilon, \qquad \left\|\tilde{Q}_3(\tau,\varepsilon) - Q_{30}(\tau,\varepsilon)\right\| \le a\varepsilon, \qquad \tau \in [-h,0], \tag{7.23}$$

$$\left\|\tilde{R}_1(\tau,\rho,\varepsilon) - \bar{R}_{10}(\tau,\rho)\right\| \le a\varepsilon, \qquad (\tau,\rho) \in \mathcal{D}, \tag{7.24}$$

where the matrices $\bar{P}_{j0}$, $(j = 1,2,3)$, $\bar{Q}_{10}(\tau)$, $Q_{30}(\tau,\varepsilon)$ *and* $\bar{R}_{10}(\tau,\rho)$ *are the same as in Theorem 3.3;* $a > 0$ *is some constant independent of* ε*;*

(c) *the value* $\bar{J}_{0\varepsilon}$ *can be expressed as*

$$\bar{J}_{0\varepsilon} = \varphi_0^{'}\tilde{P}(\varepsilon)\varphi_0 + 2\left(\varphi_{0x}^{'}\int_{-h}^{0}\tilde{Q}_1(\tau,\varepsilon)\varphi_x(\tau)d\tau + \varepsilon\varphi_{0y}^{'}\int_{-h}^{0}\tilde{Q}_3(\tau,\varepsilon)\varphi_x(\tau)d\tau\right)$$

$$+\int_{-h}^{0}\int_{-h}^{0}\varphi_x^{'}(\tau)\tilde{R}_1(\tau,\rho,\varepsilon)\varphi_x(\rho)d\tau d\rho, \qquad \varphi_0 = col(\varphi_{0x},\varphi_{0y}). \tag{7.25}$$

Proof. The statements (a) and (b) of the lemma are obtained similarly to Theorem 3.3 and Corollary 3.3. The statement (c) follows directly from Lemma 7.1, the results of [4] (Theorem 5.3, Proposition 5.4 and Theorem 5.5) and the statement (a) of the lemma. □

Based on Lemma 7.3 (statements (a) and (b)), the following lemma is proved very similarly to Lemma 3.5.

Lemma 7.4. *Let the assumption A1 be satisfied. Let the operator* $\bar{\mathcal{F}}$*, given by (3.61), be uniformly positive. Then, there exist positive numbers* ν_1*,* ν_2 *and* ν_3*, such that, for all* $\varepsilon \in (0, \nu_1\nu_2/\nu_3^2]$ *and* $f(\cdot) \in \mathcal{M}[-h,0;n;n]$*, the following inequality is satisfied*

$$\left(\tilde{\mathcal{F}}_\varepsilon[f(\cdot)], f(\cdot)\right)_{\mathcal{M}} \geq \left(\sqrt{\nu_1}(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}) - \sqrt{\varepsilon\nu_2}\|f_{E,r}\|\right)^2$$

$$+\, 2\sqrt{\varepsilon}(\sqrt{\nu_1\nu_2} - \sqrt{\varepsilon}\nu_3)\|f_{E,r}\|\left(\|f_{E,(n-r)}\| + \|f_{L,(n-r)}(\cdot)\|_{L^2}\right) \geq 0,$$

$$f_{E,(n-r)} \in E^{n-r},\ f_{E,r} \in E^r,\ f_{L,(n-r)}(\cdot) \in L^2[-h,0;E^{n-r}], \tag{7.26}$$

implying that the operator $\tilde{\mathcal{F}}_\varepsilon$ *is positive for all* $\varepsilon \in (0, \nu_1\nu_2/\nu_3^2]$*.*

Theorem 7.5. *Let the assumption A1 be satisfied. Let the operator* $\bar{\mathcal{F}}$*, given by (3.61), be uniformly positive. Then, there exists a positive number* $\bar{\varepsilon}_1^*$*, such that the following inequality is satisfied for all* $\varepsilon \in (0, \bar{\varepsilon}_1^*]$*:*

$$0 \leq \bar{J}_{0\varepsilon} - J_\varepsilon^* \leq a\varepsilon^{3/2}\left((\|\varphi_0\|)^2 + (\|\varphi_x(\cdot)\|_{L^2})^2\right), \tag{7.27}$$

where J_ε^* *is the optimal value of the functional* $J(v)$ *in the OOCP;* $a > 0$ *is some constant independent of* ε*.*

Moreover, if $\varphi_x(\cdot) \in L^\infty[-h,0;E^{n-r}]$, then there exists a positive number $\bar{\varepsilon}_2^*$, such that the following inequality is satisfied for all $\varepsilon \in (0, \bar{\varepsilon}_2^*]$:

$$0 \leq \bar{J}_{0\varepsilon} - J_\varepsilon^* \leq a\varepsilon^2\left((\|\varphi_0\|)^2 + (\|\varphi_x(\cdot)\|_\infty)^2\right), \tag{7.28}$$

with some positive constant a independent of ε.

The proof of the theorem is presented in Section 8.

7.2. **ε-dependent state-feedback control.** In this subsection, we consider another state-feedback control for the OOCP. Namely,

$$v = v_{0\varepsilon}[z(\cdot)](t) = -\varepsilon^{-1}M^{-1}B'\left[\bar{P}_0(\varepsilon)z(t) + \int_{-h}^{0} Q_0(\tau,\varepsilon)z(t+\tau)d\tau\right], \tag{7.29}$$

where $z = \mathrm{col}(x, y)$.

This control is obtained from the OOCP optimal state-feedback control (2.23) by replacing there the matrices $P(\varepsilon)$ and $Q(\tau,\varepsilon)$ by the ones $\bar{P}_0(\varepsilon)$ and $Q_0(\tau,\varepsilon)$, respectively.

Substituting the block form of the state variable z and of the matrices B, $\bar{P}_0(\varepsilon)$ and $Q_0(\tau,\varepsilon)$ (see (2.2) and (3.72)) into (7.29) yields after a simple rearrangement

$$v = v_{0\varepsilon}[z(\cdot)](t) = -M^{-1}\left(\bar{P}_{20}'x(t) + \bar{P}_{30}y(t) + \int_{-h}^{0} Q_{30}(\tau,\varepsilon)x(t+\tau)d\tau\right). \tag{7.30}$$

It is seen that the state-feedback control $v = v_{0\varepsilon}[z(\cdot)](t)$ depends on ε.

Let $J_{0\varepsilon}$ be the value of the cost functional $J(v)$ (see (2.13)) obtained by employing the state-feedback control (7.30) in the system (2.11)-(2.12) subject to the initial conditions (2.14).

Theorem 7.6. *Let the assumption A1 be satisfied. Let the operator $\bar{\mathcal{F}}$, given by (3.61), be uniformly positive. Then, there exists a positive number ε_0^*, such that the following inequality is satisfied for all $\varepsilon \in (0, \varepsilon_0^*]$:*

$$0 \le J_{0\varepsilon} - J_\varepsilon^* \le a\varepsilon^2\Big((\|\varphi_0\|)^2 + (\|\varphi_x(\cdot)\|_{L^2})^2\Big), \tag{7.31}$$

where J_ε^* is the optimal value of the functional $J(v)$ in the OOCP; $a > 0$ is some constant independent of ε.

Proof. The theorem is proved similarly to Theorem 7.5. □

8. Proof of Theorem 7.5

Consider the matrices

$$\tilde{\gamma}_P(\varepsilon) \stackrel{\triangle}{=} \tilde{P}(\varepsilon) - P(\varepsilon), \qquad \tilde{\gamma}_Q(\tau,\varepsilon) \stackrel{\triangle}{=} \tilde{Q}(\tau,\varepsilon) - Q(\tau,\varepsilon), \tag{8.1}$$

$$\tilde{\gamma}_R(\tau,\rho,\varepsilon) \stackrel{\triangle}{=} \tilde{R}(\tau,\rho,\varepsilon) - R(\tau,\rho,\varepsilon), \tag{8.2}$$

where $\{\tilde{P}(\varepsilon), \tilde{Q}(\tau,\varepsilon), \tilde{R}(\tau,\rho,\varepsilon)\}$ is the unique solution of the problem (7.14)-(7.16), (7.18); $\{P(\varepsilon), Q(\tau,\varepsilon), R(\tau,\rho,\varepsilon)\}$ is the solution of the problem (2.17)-(2.20) mentioned in Theorem 4.2.

By using the sets of equations (7.14)-(7.16),(7.18) and (2.17)-(2.20), one can show that the matrices $\tilde{\gamma}_P(\varepsilon)$, $\tilde{\gamma}_Q(\tau,\varepsilon)$ and $\tilde{\gamma}_R(\tau,\rho,\varepsilon)$ satisfy the following problem

$$\tilde{\gamma}_P(\varepsilon)\tilde{A}(\varepsilon) + \tilde{A}'(\varepsilon)\tilde{\gamma}_P(\varepsilon) + \tilde{\gamma}_Q(0,\varepsilon) + \tilde{\gamma}_Q'(0,\varepsilon) + \tilde{D}_P(\varepsilon) = 0, \tag{8.3}$$

$$d\tilde{\gamma}_Q(\tau,\varepsilon)/d\tau = \tilde{A}'(\varepsilon)\tilde{\gamma}_Q(\tau,\varepsilon) + \tilde{\gamma}_P(\varepsilon)\hat{G}(\tau,\varepsilon) + \tilde{\gamma}_R(0,\tau,\varepsilon) + \tilde{D}_Q(\tau,\varepsilon), \tag{8.4}$$

$$(\partial/\partial\tau + \partial/\partial\rho)\tilde{\gamma}_R(\tau,\rho,\varepsilon) = \hat{G}'(\tau,\varepsilon)\tilde{\gamma}_Q(\rho,\varepsilon) + \tilde{\gamma}_Q'(\tau,\varepsilon)\hat{G}(\rho,\varepsilon) + \tilde{D}_R(\tau,\rho,\varepsilon), \tag{8.5}$$

$$\tilde{\gamma}_Q(-h,\varepsilon) = \tilde{\gamma}_P(\varepsilon)H, \quad \tilde{\gamma}_R(-h,\tau,\varepsilon) = H'\tilde{\gamma}_Q(\tau,\varepsilon), \quad \tilde{\gamma}_R(\tau,-h,\varepsilon) = \tilde{\gamma}_Q'(\tau,\varepsilon)H, \tag{8.6}$$

where

$$\tilde{D}_P(\varepsilon) = [P(\varepsilon) - \bar{P}_0(\varepsilon)]S(\varepsilon)[P(\varepsilon) - \bar{P}_0(\varepsilon)], \tag{8.7}$$

$$\tilde{D}_Q(\tau,\varepsilon) = [P(\varepsilon) - \bar{P}_0(\varepsilon)]S(\varepsilon)[Q(\tau,\varepsilon) - \bar{Q}_0(\tau,\varepsilon)], \tag{8.8}$$

$$\tilde{D}_R(\tau,\rho,\varepsilon) = [Q(\tau,\varepsilon) - \bar{Q}_0(\tau,\varepsilon]' S(\varepsilon)[Q(\rho,\varepsilon) - \bar{Q}_0(\rho,\varepsilon)]. \tag{8.9}$$

Using Theorem 4.2 and the block form of the matrices $\bar{P}_0(\varepsilon)$ and $\bar{Q}_0(\tau,\varepsilon)$ (see (3.72)-(3.73)) yields the following block form of the matrices $\tilde{D}_Q(\tau,\varepsilon)$ and $\tilde{D}_R(\tau,\rho,\varepsilon)$

$$\tilde{D}_Q(\tau,\varepsilon) = \begin{pmatrix} \tilde{D}_{Q1}(\tau,\varepsilon) & 0 \\ \tilde{D}_{Q3}(\tau,\varepsilon) & 0 \end{pmatrix}, \quad \tilde{D}_R(\tau,\rho,\varepsilon) = \begin{pmatrix} \tilde{D}_{R1}(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}, \tag{8.10}$$

where the matrices $\tilde{D}_{Q1}(\tau,\varepsilon)$ and $\tilde{D}_{Q3}(\tau,\varepsilon)$ have the dimensions $(n-r)\times(n-r)$ and $r\times(n-r)$, respectively; the matrix $\tilde{D}_{R1}(\tau,\rho,\varepsilon)$ has the dimension $(n-r)\times(n-r)$. Moreover, by using the inequalities (3.70)-(3.71), one obtains the following estimates for the matrices $\tilde{D}_P(\varepsilon)$, $\tilde{D}_{Ql}(\tau,\varepsilon)$, $(l=1,3)$ and $\tilde{D}_{R1}(\tau,\rho,\varepsilon)$ for all $(\tau,\rho)\in\mathcal{D}$ and $\varepsilon\in(0,\varepsilon_1^*]$:

$$\|\tilde{D}_P(\varepsilon)\| \le a\varepsilon^2, \qquad \|\tilde{D}_{Ql}(\tau,\varepsilon)\| \le a\varepsilon[\varepsilon + \exp(-\beta(\tau+h)/\varepsilon)], \quad l=1,3, \tag{8.11}$$

$$\|\tilde{D}_{R1}(\tau,\rho,\varepsilon)\| \le a[\varepsilon^2 + \varepsilon\exp(-\beta(\tau+h)/\varepsilon)$$
$$+ \varepsilon\exp(-\beta(\rho+h)/\varepsilon) + \exp(-\beta(\tau+\rho+2h)/\varepsilon)], \tag{8.12}$$

where $a>0$ and $\beta>0$ are some constants independent of ε.

Similarly to Lemmas 7.2 and 7.3, it is obtained that, for all $\varepsilon\in(0,\min(\varepsilon_1^*,\bar{\varepsilon}_2)]$, the problem (8.3)-(8.6) has the unique solution $\{\tilde{\gamma}_P(\varepsilon),\tilde{\gamma}_Q(\tau,\varepsilon),\tilde{\gamma}_Q(\tau,\rho,\varepsilon)\}$ in the domain $\mathcal{D}$, and the components $\tilde{\gamma}_Q(\tau,\varepsilon)$ and $\tilde{\gamma}_R(\tau,\rho,\varepsilon)$ of this solution have the block form

$$\tilde{\gamma}_Q(\tau,\varepsilon) = \begin{pmatrix} \tilde{\gamma}_{Q1}(\tau,\varepsilon) & 0 \\ \varepsilon\tilde{\gamma}_{Q3}(\tau,\varepsilon) & 0 \end{pmatrix}, \quad \tilde{\gamma}_R(\tau,\rho,\varepsilon) = \begin{pmatrix} \tilde{\gamma}_{R1}(\tau,\rho,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}, \tag{8.13}$$

where the matrices $\tilde{\gamma}_{Q1}(\tau,\varepsilon)$ and $\tilde{\gamma}_{Q3}(\tau,\varepsilon)$ have the dimensions $(n-r)\times(n-r)$ and $r\times(n-r)$, respectively; the matrix $\tilde{\gamma}_{R1}(\tau,\rho,\varepsilon)$ has the dimension $(n-r)\times(n-r)$.

Now, rewriting the system (8.3)-(8.6) in the equivalent integral form (similarly to the proof of Theorem 3.3), and using the inequalities (7.5)-(7.6) and (8.11)-(8.12), one directly obtains the following inequalities for all $(\tau,\rho)\in\mathcal{D}$ and $\varepsilon\in(0,\bar{\varepsilon}_1^*]$ with some $0<\bar{\varepsilon}_1^*\le\min(\varepsilon_1^*,\bar{\varepsilon}_2)$:

$$\|\tilde{\gamma}_P(\varepsilon)\| \le a\varepsilon^2, \qquad \|\tilde{\gamma}_{Ql}(\tau,\varepsilon)\| \le a\varepsilon^2, \quad l=1,3, \tag{8.14}$$

$$\|\tilde{\gamma}_{R1}(\tau,\rho,\varepsilon)\| \le a\varepsilon[\varepsilon + \exp(-\beta|\tau-\rho|/\varepsilon)], \tag{8.15}$$

where $a>0$ and $\beta>0$ some constants independent of ε.

Using (4.2) and (7.25), we obtain

$$\bar{J}_{0\varepsilon} - J_\varepsilon^* = \varphi_0'\tilde{\gamma}_P(\varepsilon)\varphi_0$$
$$+ 2\left(\varphi_{0x}'\int_{-h}^{0}\tilde{\gamma}_{Q1}(\tau,\varepsilon)\varphi_x(\tau)d\tau + \varepsilon\varphi_{0y}'\int_{-h}^{0}\tilde{\gamma}_{Q3}(\tau,\varepsilon)\varphi_x(\tau)d\tau\right)$$

$$+\int_{-h}^{0}\int_{-h}^{0}\varphi_x'(\tau)\tilde{\gamma}_{R1}(\tau,\rho,\varepsilon)\varphi_x(\rho)d\tau d\rho, \qquad \varphi_0=\mathrm{col}(\varphi_{0x},\varphi_{0y}). \tag{8.16}$$

Using this equation, as well as the inequalities (8.14)-(8.15) and the Cauchy inequality, directly yields the inequality (7.27) stated in the theorem. The inequality (7.28) follows from (8.14)-(8.16) and the inequality $\|\varphi_x(\tau)\| \le \|\varphi_x(\cdot)\|_\infty$ for almost all $\tau \in [-h, 0]$. Thus, the theorem is proved. □

9. Direct method of suboptimal solution of the OOCP

In this section, we propose another method of constructing a suboptimal state-feedback control for the OOCP. This method is not based on the asymptotic solution of the set of Riccati-type matrix equations arising in the control optimality conditions for the OOCP, but it is based on an asymptotic decomposition of the OOCP into two much simpler ε-free subproblems, the slow and fast ones.

9.1. **Slow subproblem.** The slow subproblem is obtained from the OOCP by setting there formally $\varepsilon = 0$ and redenoting x, y, v and J by x_s, y_s, v_s and J_s, respectively. Thus, one obtains

$$dx_s(t)/dt = A_1x_s(t) + A_2y_s(t) + H_1x_s(t-h) + \int_{-h}^{0} G_1(\tau)x_s(t+\tau)d\tau, \tag{9.1}$$

$$v_s(t) = 0, \qquad t \in [0,+\infty), \tag{9.2}$$

$$J_s \triangleq \int_0^{+\infty}\left[x_s'(t)D_xx_s(t) + y_s'(t)D_yy_s(t)\right]dt \to \min, \tag{9.3}$$

$$x_s(\tau) = \varphi_x(\tau), \quad \tau \in [-h,0); \qquad x(0) = \varphi_{x0}. \tag{9.4}$$

It is seen that the slow subproblem consists of the equation (9.2) and the problem of minimizing the cost functional J_s along trajectories of the system (9.1) with the initial conditions (9.4). Since the variable $y_s(t)$ does not satisfy any equation, the cost functional J_s can be minimized only by a proper choice of $y_s(t)$, i.e., in the problem (9.1),(9.3),(9.4), the variable $y_s(t)$ is a control variable.

Comparing the problem (9.1),(9.3),(9.4) with the ROCP introduced in Section 3.5.2, one can see that these problems coincide with each other. Thus, due to Lemma 3.1, the optimal state-feedback control of the problem (9.1),(9.3),(9.4) has the form

$$y_s^*[x_s(t), x_{sh}(t)] = -D_y^{-1}A_2'\left[\bar{P}_{10}x_s(t) + \int_{-h}^{0}\bar{Q}_{10}(\tau)x_s(t+\tau)d\tau\right], \tag{9.5}$$

where $t \ge 0$; $x_{sh}(t) = \{x_s(t+\tau) \;\; \forall\tau \in [-h,0)\}$; the matrices $\bar{P}_{10}$ and $\bar{Q}_{10}(\tau)$ are the respective components of the unique solution $\bar{\mathcal{S}}$ to the problem (3.48)-(3.49),(3.54)-(3.56), satisfying the conditions mentioned in Lemma 3.2.

9.2. **Fast subproblem.** The fast subproblem is obtained in the following way: (I) the slow variable $x(\cdot)$ is removed from the equation (2.12) and the performance index (2.13) of the OOCP; (II) the transformation of variables $t = \varepsilon\xi, y(\varepsilon\xi) = y_f(\xi), v(\varepsilon\xi) = v_f(\xi), J(v(\varepsilon\xi)) = \varepsilon J_f(v_f(\xi))$ is made in the resulting problem, where ξ, y_f, v_f and J_f are new independent variable, state, control and cost functional, respectively. As a result, one obtains the problem

$$dy_f(\xi)/d\xi = \varepsilon A_4 y_f(\xi) + v_f(\xi), \tag{9.6}$$

$$J_f(v_f) \stackrel{\triangle}{=} \int_0^{+\infty} \Big[y_f'(\xi) D_y y_f(\xi) + v_f'(\xi) M v_f(\xi)\Big] d\xi \to \min_{v_f}. \tag{9.7}$$

Now, neglecting formally the term with the multiplier ε in (9.6) yields the fast subproblem consisting of the system

$$dy_f(\xi)/d\xi = v_f(\xi) \tag{9.8}$$

and the performance index (9.7).

Due to results of [14], we obtain the following lemma.

Lemma 9.1. *The fast subproblem (9.7)-(9.8) with a given initial value $y_f(0)$ of the state variable has the unique optimal state-feedback control*

$$v_f^*[y_f(\xi)] = -M^{-1} P_f y_f(\xi), \tag{9.9}$$

where the $r \times r$-matrix P_f is the unique symmetric positive definite solution of the algebraic Riccati equation

$$P_f M^{-1} P_f - D_y = 0. \tag{9.10}$$

Moreover, the optimal trajectory $y_f(\xi)$ of the fast subproblem satisfies the inequality

$$\|y_f(\xi)\| \le a \exp(-\beta\xi) \|y_f(0)\|, \qquad \xi \ge 0, \tag{9.11}$$

where $a > 0$ and $\beta > 0$ are some constants.

Comparing the equation (9.10) with the equation (3.44), one can see that these equations coincide with each other, implying that

$$P_f = \bar{P}_{30}. \tag{9.12}$$

9.3. **Composite control for the OOCP.** In this subsection, based on the control $v_s(t)$ of the slow subproblem, the optimal state-feedback control $y_s^*[x_s(t), x_{sh}(t)]$ of the problem (9.1),(9.3),(9.4) and the optimal control $v_f^*[y_f(\xi)]$ of the fast subproblem, we construct a composite state-feedback control for the OOCP. Then, we show an asymptotic suboptimality (for all sufficiently small $\varepsilon > 0$) of this composite control.

The composite control is obtained in the form

$$v_c[x(t), y(t), x_h(t)] = v_s(t) + v_f^*[\tilde{y}(t/\varepsilon)], \tag{9.13}$$

where $\tilde{y}(t/\varepsilon)$ is defined as follows

$$\tilde{y}(t/\varepsilon) \stackrel{\triangle}{=} y(t) - y_s^*[x(t), x_h(t)]. \tag{9.14}$$

Substituting (9.2) and (9.9) into (9.13), and using (9.12) and (9.14), yield after some rearrangement

$$v_c[x(t), y(t), x_h(t)] = -M^{-1}\bar{P}_{30}\Big\{y(t)$$

$$+D_y^{-1}A_2'\Big[\bar{P}_{10}x(t) + \int_{-h}^{0}\bar{Q}_{10}(\tau)x(t+\tau)d\tau\Big]\Big\}. \tag{9.15}$$

By virtue of the equations (3.51),(3.52) and (5.21), the expression (9.15) can be transformed equivalently as follows

$$v_c[x(t), y(t), x_h(t)] = -M^{-1}\left(\bar{P}_{20}'x(t) + \bar{P}_{30}y(t) + \int_{-h}^{0}\bar{Q}_{30}(\tau)x(t+\tau)d\tau\right). \tag{9.16}$$

Comparing the expression (9.16) for the composite control with the expression (7.2) for the ε-free suboptimal control of the OOCP, we obtain that these controls coincide with each other. Thus, the statements of Theorem 7.5 also are valid for the composite control $v_c[x(t), y(t), x_h(t)]$, meaning its suboptimality in the OOCP for all sufficiently small $\varepsilon > 0$.

10. Conclusions

In this paper, the infinite horizon linear-quadratic optimal control problem for a system with point-wise and distributed delays was considered. It is assumed that the system consists of two modes. One of them is controlled directly, while the other is controlled through the first one. Moreover, the case where the state variable of the mode, controlled directly, has no delays is treated. The control cost in the cost functional is assumed to be small with respect to the state cost, i.e., the considered problem is the cheap control problem. By a simple control transformation, this problem was converted to the optimal control problem for a system with a small multiplier $\varepsilon > 0$ for a part of the derivatives (a singularly perturbed system). In this singularly perturbed system, the slow state variable has delays, while the fast state variable has not. For the transformed control problem, considered in the sequel as an original one, two methods of suboptimal solution were proposed. The first method is based on the asymptotic solution of the set of Riccati-type matrix equations associated with the control optimality conditions. This method yields two suboptimal state-feedback controls, ε-free and ε-dependent ones. It was established that, in the case of a square-integrable initial function for the slow state variable, the ε-free suboptimal state-feedback control provides the corresponding value of the cost functional to be within an $O(\varepsilon^{3/2})$-vicinity of the optimal value. In the case of an essentially bounded initial function for the slow state variable, the value of the cost functional, corresponding to this control, belongs to $O(\varepsilon^2)$-vicinity of the optimal value. The ε-dependent state-feedback control provides the corresponding value of the cost functional to be within an $O(\varepsilon^2)$-vicinity of the optimal value for any square-integrable initial function of the slow state variable.

The second method is based on an asymptotic decomposition of the original optimal control problem into two much simpler ε-free subproblems, the slow and fast ones. For each of these subproblems the optimal state-feedback control was obtained. Then, using these controls, the suboptimal state-feedback composite control for the original problem was designed. It was shown that this composite control coincides with the ε-free state-feedback control obtained by the first method.

11. Appendix A: boundary function method

In this section, a short explanation of the boundary function method is given. This method is applied for asymptotic solution of various classes of singularly perturbed differential and difference equations (see e.g. [25] and references therein). Below, we describe an application of this method to an asymptotic solution of initial value problem for a set of linear ordinary differential equations with a small multiplier for a part of the derivatives.

Consider the following set of equations

$$dx/dt = A_1 x + A_2 y + f_1(t), \qquad t \in [0, T], \tag{A.1}$$

$$\varepsilon dy/dt = A_3 x + A_4 y + f_2(t), \qquad t \in [0, T], \tag{A.2}$$

where $x \in E^n$, $y \in E^m$; A_i, $(i = 1, ..., 4)$ are given constant matrices of corresponding dimensions; the $m \times m$-matrix A_4 is a Hurwitz one; $f_j(t)$, $(j = 1, 2)$ are given vectors of corresponding dimensions; the vector-valued functions $f_1(t)$ and $f_2(t)$ are continuously differentiable for $t \in [0, T]$; $\varepsilon > 0$ is a small parameter.

The system (A.1)-(A.2) is subject the following initial conditions

$$x(0) = x^0, \qquad y(0) = y^0, \tag{A.3}$$

where $x^0 \in E^n$ and $y^0 \in E^m$ are given.

Note that the state variables x and y are called the slow and fast ones.

We look for the zero-order asymptotic solution of the problem (A.1)-(A.3) in the form

$$x_0(t, \varepsilon) = \bar{x}_0(t) + x_0^t(\xi), \qquad y_0(t, \varepsilon) = \bar{y}_0(t) + y_0^t(\xi), \tag{A.4}$$

where

$$\xi = t/\varepsilon. \tag{A.5}$$

The terms $\bar{x}_0(t)$ and $\bar{y}_0(t)$ constitute the outer part of the asymptotic solution, or simply the outer solution. The terms $x_0^t(\xi)$ and $y_0^t(\xi)$ constitute the boundary layer correction of the asymptotic solution in a right-hand neighborhood of $t = 0$, or simply the boundary layer correction.

Substituting $x_0(t, \varepsilon)$ and $y_0(t, \varepsilon)$ into (A.1)-(A.2) instead of x and y, respectively, one obtains after some rearrangement

$$d\bar{x}_0(t)/dt + \varepsilon^{-1} dx_0^t(\xi)/d\xi = A_1 \bar{x}_0(t) + A_2 \bar{y}_0(t) + f_1(t) + A_1 x_0^t(\xi) + A_2 y_0^t(\xi), \tag{A.6}$$

$$\varepsilon d\bar{y}_0(t)/dt + dy_0^t(\xi)/d\xi = A_3 \bar{x}_0(t) + A_4 \bar{y}_0(t) + f_2(t) + A_3 x_0^t(\xi) + A_4 y_0^t(\xi), \tag{A.7}$$

Now, equating the coefficients for ε^0, which depend on t, on both sides of the equations (A.6) and (A.7) yields the following equations for the outer solution

$$d\bar{x}_0(t)/dt = A_1 \bar{x}_0(t) + A_2 \bar{y}_0(t) + f_1(t), \qquad t \in [0, T], \tag{A.8}$$

$$0 = A_3 \bar{x}_0(t) + A_4 \bar{y}_0(t) + f_2(t), \qquad t \in [0, T]. \tag{A.9}$$

The system (A.8)-(A.9) is differential-algebraic. However, since A_4 is a Hurwitz matrix, the algebraic equation (A.9) can be resolved with respect to $\bar{y}_0(t)$, yielding

$$\bar{y}_0(t) = -A_4^{-1}\Big(A_3\bar{x}_0(t) + f_2(t)\Big), \qquad t \in [0, T]. \tag{A.10}$$

Substituting (A.10) into (A.8) converts the latter to the differential equation with respect to $\bar{x}_0(t)$

$$d\bar{x}_0(t)/dt = \bar{A}_0\bar{x}_0(t) + \bar{f}_0(t), \qquad t \in [0, T], \tag{A.11}$$

where

$$\bar{A}_0 = A_1 - A_2A_4^{-1}A_3, \qquad \bar{f}_0(t) = f_1(t) - A_2A_4^{-1}f_2(t). \tag{A.12}$$

In order to obtain a single solution of the equation (A.11), one needs an initial condition. This condition will be obtained below.

Now, let proceed to the boundary correction. Equating the coefficients for ε^{-1}, which depend on ξ, on both sides of the equation (A.6) yields the following equation for the term $x_0^t(\xi)$ of the boundary layer correction

$$dx_0^t(\xi)/d\xi = 0, \qquad \xi \geq 0. \tag{A.13}$$

Similarly, equating the coefficients for ε^0, which depend on ξ, on both sides of the equation (A.7) yields the following equation for the term $y_0^t(\xi)$ of the boundary layer correction

$$dy_0^t(\xi)/d\xi = A_3x_0^t(\xi) + A_4y_0^t(\xi), \qquad \xi \geq 0. \tag{A.14}$$

In order to obtain a single solution of the set (A.13)-(A.14), we also need an additional condition. By such a condition, we use a reasonable requirement that the boundary layer correction is considerable only in some right-hand neighborhood of $\xi = 0$, and it tends to zero while $\xi \to +\infty$, i.e.,

$$\lim_{\xi \to +\infty} x_0^t(\xi) = 0, \tag{A.15}$$

$$\lim_{\xi \to +\infty} y_0^t(\xi) = 0. \tag{A.16}$$

Thus, solving the equation (A.13) subject to the condition (A.15) yields

$$x_0^t(\xi) = 0, \qquad \xi \geq 0. \tag{A.17}$$

Substituting (A.17) into (A.14), we obtain the differential equation with respect to $y_0^t(\xi)$

$$dy_0^t(\xi)/d\xi = A_4y_0^t(\xi), \qquad \xi \geq 0. \tag{A.18}$$

Thus, we have obtained one term $x_0^t(\xi)$ (see (A.17)) of the zero-order asymptotic solution (A.4) to the initial-value problem (A.1)-(A.3). For the terms $\bar{x}_0(t)$ and $y_0^t(\xi)$, we have two differential equations (A.11) and (A.18), respectively. For the term $\bar{y}_0(t)$, we have the explicit expression (A.10) by $\bar{x}_0(t)$.

In order to complete the obtaining the zero-order asymptotic solution (A.4), we need conditions for the differential equations (A.11) and (A.18). It should be noted, that although the condition (A.15) helped us to distinguish the unique solution of the equation (A.13), the similar condition (A.16) cannot help us in such a task for the equation (A.18) because, due to A_4 is a Hurwitz matrix, all solutions of this equation satisfy (A.16).

Substituting $x_0(t,\varepsilon)$ and $y_0(t,\varepsilon)$, given by (A.4), into (A.3) instead of $x(\cdot)$ and $y(\cdot)$, respectively, one has

$$\bar{x}_0(0) + x_0^t(0) = x^0, \tag{A.19}$$

$$\bar{y}_0(0) + y_0^t(0) = y^0, \tag{A.20}$$

Now, the equations (A.17) and (A.19) yield the initial condition for the differential equation (A.11)

$$\bar{x}_0(0) = x^0, \tag{A.21}$$

while the equation (A.20), along with (A.10) and (A.21), yield the initial condition for the differential equation (A.18)

$$y_0^t(0) = y^0 + A_4^{-1}\Big(A_3x^0 + f_2(0)\Big). \tag{A.22}$$

Solving the differential equation (A.11) subject to the initial condition (A.21), we obtain

$$\bar{x}_0(t) = \exp\left(\bar{A}_0 t\right)x^0 + \int_0^t \exp\left(\bar{A}_0(t-s)\right)\bar{f}_0(s)ds, \qquad t \in [0,T]. \tag{A.23}$$

Solving the differential equation (A.18) subject to the initial condition (A.22), one has

$$y_0^t(\xi) = \Big[y^0 + A_4^{-1}\Big(A_3x^0 + f_2(0)\Big)\Big]\exp\left(A_4\xi\right), \qquad \xi \geq 0. \tag{A.24}$$

Thus, we have completed the formal constructing the zero-order asymptotic solution (A.4) of the initial-value problem (A.1)-(A.3).

As a direct consequence of results of [25] (Chapter 2), we obtain the following proposition, which justifies the zero-order asymptotic solution (A.4).

Proposition A.1.‘*There exists a positive number ε^*, such that for all $\varepsilon \in (0,\varepsilon^*]$ the solution col$\Big(x(t,\varepsilon), y(t,\varepsilon)\Big)$ of the initial-value problem (A.1)-(A.3) satisfies the inequalities*

$$\|x(t,\varepsilon) - x_0(t,\varepsilon)\| \leq a\varepsilon, \qquad t \in [0,T], \tag{A.25}$$

$$\|y(t,\varepsilon) - y_0(t,\varepsilon)\| \leq a\varepsilon, \qquad t \in [0,T], \tag{A.26}$$

where $a > 0$ is some constant independent of ε.

12. Appendix B: Criterion of L^2-stability of a linear system sith delays

Consider the system

$$dz(t)/dt = Az(t) + Hz(t-h) + \int_{-h}^0 G(\tau)z(t+\tau)d\tau, \quad t \geq 0, \tag{B.1}$$

where $z(t) \in E^n$; $h > 0$ is a given constant time delay; A, H and $G(\tau)$ are given time-invariant matrices of corresponding dimensions; the matrix-valued function $G(\tau)$ is piece-wise continuous for $\tau \in [-h, 0]$.

Along with the system (B.1), consider its characteristic equation

$$\det\Big[A + \exp(-\lambda h)H + \int_{-h}^{0} \exp(\lambda\tau)G(\tau)d\tau - I_n\Big] = 0. \tag{B.2}$$

The following proposition is formulated and proved in [4] (Theorem 5.3).

Proposition B.1. *The system (B.1) is L^2-stable if and only if all roots λ of its characteristic equation (B.2) have negative real parts.*

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