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GLOBAL ATTRACTORS FOR DISCRETE DISPERSE DYNAMICAL SYSTEMS

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Abstract. In this paper we discuss the asymptotic behavior of trajectories of discrete disperse dynamical systems generated by set-valued mappings

1. Discrete disperse dynamical systems

Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. A discrete-time dynamical system is described by a space of states and a transition operator which can be set-valued. Usually in the dynamical systems theory a transition operator is single-valued. In the present paper we consider a class of dynamical systems introduced in [3] and studied in [4, 5, 7, 8] with a compact metric space of states and a set-valued transition operator. Such dynamical systems describe economical models [1, 2, 6].

Let (X, ρ) be a compact metric space and let $a : X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in X \times X : \ y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \ x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we discuss convergence of trajectories of the dynamical system generated by the set-valued mapping a. Following [3, 4] this system is called a discrete disperse dynamical system.

First we define a trajectory of this system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Put

$$\Omega(a) = \{z \in X : \ \text{for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \text{ such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \epsilon\}. \tag{1.1}$$

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Clearly, $\Omega(a)$ is closed subset of (X,ρ). In the present paper the set $\Omega(a)$ will be called a global attractor of a. Note that in [3-5] $\Omega(a)$ was called a turnpike set of a. This terminology was motivated by mathematical economics [1, 2, 6].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x,E) = \inf\{\rho(x,y) : \ y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have $\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0$.

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

In this section we discuss uniform convergence of trajectories to the global attractor $\Omega(a)$.

The following useful result was obtained in [7].

Proposition 1.1. *Let $\epsilon > 0$. Then there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$*

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T(\epsilon)\} \le \epsilon.$$

The following theorem established in [7] provides necessary and sufficient conditions for uniform convergence of trajectories to the global attractor.

Theorem 1.2. *The following properties are equivalent:*

(1) *For each $\epsilon > 0$ there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ and each integer $t \ge T(\epsilon)$ we have $\rho(x_t, \Omega(a)) \le \epsilon$.*

(2) *If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ for all integers t, then $\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$.*

(3) *For each $\epsilon > 0$ there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \le \delta$ the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ holds for all integers $t \ge 0$.*

The following two theorems established in [7] show that convergence of trajectories to the global attractor holds even in the presence of computational errors.

Theorem 1.3. *Let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \le \delta$ and for each integer $t \ge 0$ the following inequality holds:*

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T(\epsilon)\} \le \epsilon.$$

Theorem 1.4. *Assume that property (2) from Theorem 1.2 holds. Then for each $\epsilon > 0$ there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying*

$$\rho(x_{t+1}, a(x_t)) \le \delta \text{ for all integers } t \ge 0$$

the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ holds for each integer $t \ge T(\epsilon)$.

2. Examples

Denote by $\Pi(X)$ the set of all nonempty closed subsets of (X, ρ). For each $A, B \in \Pi(X)$ set

$$H(A, B) = \max\{\sup_{x\in A} \rho(x, B), \sup_{y\in B} \rho(y, A)\}. \tag{2.1}$$

It is known that the space $(\Pi(X), H)$ is a complete metric space.

Example 1. Let $a : X \to X$ satisfy $\rho(a(x), a(y)) \leq \rho(x, y)$ for each $x, y \in X$. Since the mapping a is single-valued it is not difficult to see that $a(\Omega(a)) \subset \Omega(a)$ and property (3) from Theorem 1.2 holds.

Example 2. Let $a : X \to X$ satisfy the following condition:

(C1) for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that for each pair $x, y \in X$ satisfying $\rho(x, y) \leq \delta$ we have $\rho(a^n x, a^n y) \leq \epsilon$ for all natural numbers n.

It was shown in [7] that the property (3) from Theorem 1.2 holds.

Example 3. Let $a : X \to 2^X \setminus \emptyset$ have a closed graph. Assume that

$$H(a(x), a(y)) \leq c\rho(x, y) \text{ for all } x, y \in X$$

with a constant $c \in (0, 1)$. It was shown in [7] that the property (3) from Theorem 1.2 holds.

3. Spaces of set-valued mappings

In this section we consider classes of discrete disperse dynamical systems whose global attractors are a singleton.

Denote by $\mathcal{A}$ the set of all mappings $a : X \to \Pi(X)$ with closed graphs. For each $a_1, a_2 \in \mathcal{A}$ set

$$d_{\mathcal{A}}(a_1, a_2) = \sup\{H(a_1(x), a_2(x)) : x \in X\}.$$

It is clear that the metric space $(\mathcal{A}, d_{\mathcal{A}})$ is complete.

Denote by $\mathcal{A}_c$ the set of all continuous mappings $a : X \to \Pi(X)$ which belong to $\mathcal{A}$, by $\mathcal{A}_f$ the set of all $a \in \mathcal{A}$ such that $a(x)$ is a singleton for each $x \in X$ and set $\mathcal{A}_{fc} = \mathcal{A}_f \cap \mathcal{A}_c$. Clearly $\mathcal{A}_f$, $\mathcal{A}_c$ and $\mathcal{A}_{fc}$ are closed subsets of $(\mathcal{A}, d_{\mathcal{A}})$.

Let $\mathcal{M}$ be one of the following spaces: $\mathcal{A}$; $\mathcal{A}_c$; $\mathcal{A}_f$; $\mathcal{A}_{fc}$. The space $\mathcal{M}$ is equipped with the metric $d_{\mathcal{A}}$.

Denote by $\mathcal{M}_{reg}$ the set of all $a \in \mathcal{M}$ such that $\Omega(a)$ is a singleton and that properties (1-3) from Theorem 1.2 hold.

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $(\mathcal{M}, d_{\mathcal{A}})$. The following result established in [7] shows that most elements of $\bar{\mathcal{M}}_{reg}$ (in the sense of Baire category) belong to $\mathcal{M}_{reg}$.

Theorem 3.1. *The set $\mathcal{M}_{reg}$ contains a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$.*

4. Stable points and Lyapunov functions

Let (X, ρ) be a compact metric space and let $\Pi(X)$ be the collection of all closed nonempty subsets of (X, ρ) equipped with the Hausdorff metric H defined by (2.1).

We consider a continuous mapping $a : X \to \Pi(X)$. Denote by $\mathcal{L}_a$ the set of all continuous functions $s : X \to R^1$ such that

$$s(x) \geq 0 \text{ for al } x \in X,$$

$$s(y) \leq s(x) \text{ for all } x \in X \text{ and all } y \in a(x).$$

A function $s \in \mathcal{L}_a$ is a Lyupunov function for the dynamical system generated by the mapping a.

For each $s \in \mathcal{L}_a$ set

$$(s \circ a)(x) = \max\{s(y) : \ y \in a(x)\}, \ x \in X$$

and

$$W_s = \{x \in X : \ s(x) = (s \circ a)(x)\}.$$

The following result was established in [4, Proposition 16.1].

Proposition 4.1. *For any $s \in \mathcal{L}_a$ the set W_s is nonempty and closed.*

Set

$$W_a = \cap_{s \in \mathcal{L}_a} W_s.$$

The following two theorems were established in [4, Theorems 16.1 and 16.2].

Theorem 4.2. *The set W is a collection of all $x \in X$ for which there exists a trajectory $\{x_t\}_{t=0}^{\infty} \subset X$ of a such that $x_0 = x$ and for all $s \in \mathcal{L}_a$,*

$$s(x_0) = s(x_i), \ i = 1, 2, \ldots$$

Theorem 4.3. *For each trajectory $\{x_t\}_{t=0}^{\infty}$ of a,*

$$\lim_{t \to \infty} \rho(x_t, W) = 0.$$

A point $x \in X$ is called stable if for any $\epsilon > 0$ there exists a trajectory $\{x_n\}_{n=0}^{\infty} \subset X$ such that $x_0 = x$ and $\liminf_{n \to \infty} \rho(x_n, x) \leq \epsilon$.

Denote by Π_a the set of all stable points and consider the set $\Omega(a)$ introduced in Section 1.

It is not difficult to see that

$$\Pi_a \subset \Omega(a) \subset W_a.$$

The following theorem is the main result of [5].

Theorem 4.4. *Assume that $a : X \to \Pi(X)$ satisfies*

$$H(a(x), a(y)) \leq \rho(x, y)$$

for all $x, y \in X$. Then $\Pi_a = W_a$.

5. Generalizations of Theorems 1.2-1.4

In this section we discuss a class of dynamical systems described by a set-valued transition operator $a : K \to 2^X \setminus \{\emptyset\}$, where K is a closed subset of a compact metric space X. In the previous sections we considered the case with $K = X$.

Let (X, ρ) be a compact metric space, K be a nonempty closed subset of (X, ρ) and let $a : K \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in K \times X : \ y \in a(x)\}$$

is a closed subset of $X \times X$.

For each $x \in X$ and each nonempty closed set $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : \ y \in E\}.$$

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $\{x_i\}_{t=0}^{\infty} \subset K$ and if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

A sequence $\{x_t\}_{t=0}^{T} \subset X$, where T is a natural number, is called a trajectory of a (or just a trajectory if the mapping a is understood) if $\{x_t\}_{t=0}^{T-1} \subset K$ and if $x_{t+1} \in a(x_t)$ for all integers $t \in [0, T-1]$.

In this section we use the following assumption introduced in [8]:

(A) For each integer $n \geq 1$ there exists a sequence $\{x_t\}_{t=0}^{n} \subset X$ such that for each integer t satisfying $0 \leq t \leq n-1$ there is $(y_t, z_t) \in \mathrm{graph}(a)$ such that $\rho(x_t, y_t) \leq 1/n$ and $\rho(x_{t+1}, z_t) \leq 1/n$.

The following result was established in [8].

Proposition 5.1. *Assume that (A) holds. Then there exists a trajectory $\{x_t\}_{t=0}^{\infty}$ of a.*

In this section we assume that (A) holds.

Put

$$\Omega(a) = \{z \in X : \ \text{for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \text{ such that } \liminf_{t \to \infty} \rho(z, x_t) \leq \epsilon\}.$$

Clearly, $\Omega(a)$ is a nonempty closed subset of K. It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have

$$\lim_{t \to \infty} \rho(x_t, \Omega(a)) = 0.$$

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t \to \infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) = B$.

In this section we discuss uniform convergence of trajectories to the global attractor $\Omega(a)$.

The following results were established in [8].

Theorem 5.2. *Let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number T such that if $\{x_t\}_{t=0}^{T} \subset X$ and if $\{(y_t, z_t)\}_{t=0}^{T-1} \subset \mathrm{graph}(a)$ satisfy*

$$\rho(x_t, y_t) \leq \delta, \ \rho(x_{t+1}, z_t) \leq \delta$$

for all $t = 0, \dots, T-1$, then

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T\} \leq \epsilon.$$

Theorem 5.3. *The following properties are equivalent:*

(1) *If a sequence* $\{x_t\}_{t=-\infty}^{\infty} \subset K$ *satisfies* $x_{t+1} \in a(x_t)$ *for all integers* t, *then* $\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$.

(2) *For each* $\epsilon > 0$ *there is a natural number* L *such that for each trajectory* $\{x_t\}_{t=0}^{\infty}$,

$$\rho(x_t, \Omega(a)) \le \epsilon \text{ for all integers } t \ge L.$$

(3) *Let* $\epsilon > 0$. *Then there exist* $\delta > 0$ *and a natural number* L *such that for each integer* $T > 2L$ *and each sequence* $\{x_t\}_{t=0}^{T} \subset X$ *which satisfies for all* $t = 0, \dots, T-1$

$$\inf\{\rho(x_t, y) + \rho(x_{t+1}, z) : (y, z) \in \text{ graph}(a)\} \le \delta$$

the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ *holds for all integers* $t = L, \dots, T-L$.

Theorem 5.4. *Assume that the property (1) from Theorem 5.3 holds and let* $\epsilon > 0$. *Then there exist* $\delta > 0$ *and a natural number* L *such that for each integer* $T > L$ *and each sequence* $\{x_t\}_{t=0}^{T} \subset X$ *which satisfies*

$$\rho(x_0, \Omega(a)) < \delta$$

and

$$\inf\{\rho(x_t, y) + \rho(x_{t+1}, z) : \ (y, z) \in \text{ graph}(a)\} \le \delta$$

for $t = 0, \dots, T-1$, *the following inequality holds:*

$$\rho(x_t, \Omega(a)) \le \epsilon, \ t = 0, \dots, T-L.$$

6. Extensions of Theorem 3.1

In this section we consider classes of discrete disperse dynamical systems whose global attractors are a singleton.

Let (Y, ρ) be a compact metric space. Denote by $\Pi(Y)$ the set of all nonempty closed subsets of Y.

For each $x \in Y$ and each $A \in \Pi(Y)$ put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

For each $A, B \in \Pi(Y)$ set

$$H_Y(A, B) = \max\{\sup_{x \in A} \rho(x, B), \ \sup_{y \in B} \rho(y, A)\}.$$

It is known that the space $(\Pi(Y), H_Y)$ is a complete metric space.

Let (X, ρ) be a compact metric space. For each $(x_1, x_2), (y_1, y_2) \in X \times X$ set

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2).$$

Then $(X \times X, \rho_1)$ is a compact metric space. We consider the complete metric space $(\Pi(X \times X), H_{X \times X})$.

It is clear that any set-valued mapping defined on a subset of X with values in $\Pi(X)$ is identified with its graph.

For each $S \in \Pi(X \times X)$ set

$$K_S = \{x \in X : \ \text{there is } y \in X \text{ such that } (x, y) \in S\},$$

$$a_S(x) = \{y \in X : \ (x, y) \in S\}, \ x \in K_S.$$

Denote by $\mathcal{M}$ the set of all $S \in \Pi(X \times X)$ such that a_S possesses a trajectory $\{x_t\}_{t=0}^{\infty}$. Denote by $\bar{\mathcal{M}}$ the closure of $\mathcal{M}$ in the space $(\Pi(X \times X), H_{X \times X})$.

Let K be a nonempty closed subset of X. Denote by $\Pi_K(X \times X)$ the set of all $S \in \Pi(X \times X)$ such that $K_S = K$. Clearly $\Pi_K(X \times X)$ is a closed subset of $(\Pi(X \times X), H_{X \times X})$. Set

$$\mathcal{M}_K = \mathcal{M} \cap \Pi_K(X \times X).$$

Denote by $\bar{\mathcal{M}}_K$ the closure of $\mathcal{M}_K$ in $(\Pi(X \times X), H_{X \times X})$.

We equip the spaces $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_K$ with the metric $H_{X \times X}$.

In [8] it was established the following result which shows that most elements of $\bar{\mathcal{M}}$ (respectively $\bar{\mathcal{M}}_K$) (in the sense of Baire category) belong to $\mathcal{M}$ (respectively, $\mathcal{M}_K$).

Theorem 6.1. *The set $\mathcal{M}$ (respectively, $\mathcal{M}_K$) contains a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}$ (respectively, $\bar{\mathcal{M}}_K$).*

Denote by $\mathcal{M}_r$ the set of all $S \in \Pi(X \times X)$ such that a_S possesses a trajectory $\{x_t\}_{t=0}^{\infty}$, $\Omega(a_S)$ is a singleton and the property (1) from Theorem 5.3 holds with $a = a_S$.

Set

$$\mathcal{M}_{r,K} = \mathcal{M}_r \cap \Pi_K(X \times X).$$

Denote by $\bar{\mathcal{M}}_r$ (respectively, $\bar{\mathcal{M}}_{r,K}$) the closure of $\mathcal{M}_r$ (respectively, $\mathcal{M}_{r,K}$) in the metric space $(\Pi(X \times X), H_{X \times X})$.

We equip the spaces $\bar{\mathcal{M}}_r$ and $\bar{\mathcal{M}}_{r,K}$ with the metric $H_{X \times X}$.

In [8] it was established the following result which shows that most elements of $\bar{\mathcal{M}}_r$ (respectively $\bar{\mathcal{M}}_{r,K}$) (in the sense of Baire category) belong to $\mathcal{M}_r$ (respectively, $\mathcal{M}_{r,K}$).

Theorem 6.2. *The set $\mathcal{M}_r$ (respectively, $\mathcal{M}_{r,K}$) contains a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_r$ (respectively, $\bar{\mathcal{M}}_{r,K}$).*

References

[1] Z. Dzalilov, A. M. Rubinov and P. E. Kloeden, *Lyapunov sequences and a turnpike theorem without convexity*, Set-Valued Analysis **6** (1998), 277–302.

[2] V. L. Makarov and A. M. Rubinov, *Mathematical Theory of Economic Dynamics and Equilibria*, Nauka, Moscow (1973), English trans.: Springer-Verlag, New York, 1977.

[3] A. M. Rubinov, *Turnpike sets in discrete disperse dynamical systems*, Sib. Math. J. **21** (1980), 136–146.

[4] A. M. Rubinov, *Multivalued Mappings and their Applications in Economic Mathematical Problems*, Nauka, Leningrad, 1980.

[5] A. J. Zaslavski, *Turnpike sets of continuous transformations in compact metric spaces*, Sib. Math. J. **23** (1982), 198–203.

[6] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, New York, 2006.

[7] A. J. Zaslavski, *Uniform convergence to global attractors for discrete disperse dynamical systems*, Nonlinear Dynamics and System Theory **7** (2007), 315–325.

[8] A. J. Zaslavski, *Convergence of trajectories of discrete disperse dynamical systems*, Preprint (2007).

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