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SINGULAR INFINITE HORIZON CALCULUS OF VARIATIONS. APPLICATIONS TO FISHERIES MANAGEMENT

ELADIO OCAÑA ANAYA, PIERRE CARTIGNY, AND PATRICE LOISEL

Abstract. We consider a calculus of variations problem in infinite horizon linear with respect to the velocities. In our case the admissible curves stay in a bounded interval and we prove that the MRAP (Most Rapid Approach Pathes) from any initial conditions to the solutions of the (algebraic) Euler-Lagrange equation are optimal. We use a result on the uniqueness of the solution of a Hamilton-Jacobi equation. We propose some new and straightforward proofs. Particularly we show that boundary conditions, that are essential for the uniqueness, are satisfied under some assumptions that we detail. Finally we underline the limits for the applications (fisheries examples) of the established results.

1. Introduction

The aim of this paper is to study the solutions of an infinite horizon optimal control problem that is linear with respect to the control. Such a problem can be seen as equivalent to the following infinite horizon calculus of variations problem:

$$\int_{t=0}^{+\infty} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt\,,$$

where the state variable x is scalar and the velocities belong to a closed set.

For such a problem it is known that the Euler-lagrange equation is available as an optimal necessary condition [4]. While in general this equation is a differential one, in our case it is algebraic. Moreover from the autonomous character, the interior solutions are given by constant values. The question is thus to determine the optimal solutions from initial conditions that don't correspond to the solutions of Euler-Lagrange. Moreover, as we don't consider any concavity assumption, we have to solve the question of the optimality of these particular solutions.

Such a problem has been considered in different papers with different methods. Clark [6] considers the case (with finite horizon) when the Euler-lagrange equation possesses only one solution, $\bar{x}$. He proves that the Most Rapid Approach Paths, $MRAP(x_0, \bar{x})$ from any initial condition x_0 to $\bar{x}$, are optimal. This result is obtained by using the Green's theorem, a technique already presented in such a context by Miele [9].

Using the same approach, this question was considered in infinite horizon and in presence of a finite number of solutions for the Euler-Lagrange equation, by Sethi [12].

Hartl and Feichtinger [8] underlined that the sign condition considered by the preceding authors did not suffice in the infinite horizon framework. They prove

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that an other condition, specific to this context insures the optimality of the MRAP. This last condition plays the role of a transversality condition at infinity.

More recently in [10], an approach via the value function of the problem was proposed. The authors proved that the value function and the value of the objective along a MRAP are (viscosity) solutions of the same Hamilton-Jacobi equation. By using a result on the uniqueness of the solutions of such an equation [2], [3], the optimality of the MRAP can be deduced. When there is no constraint on the sate variable, i.e. $x \in R^1$, then such a result can be established in the class of the BUC (Bounded Uniformly Continuous) functions [10]. We underline that the behaviour at infinity of this class of functions can be interpreted as a transversality condition. Now in presence of constraints for the state variable, more precisely when the state variable belongs to an open bounded set, then a corresponding condition such as a particular value on the boundaries, has to be considered. In [11] such condition is not considered. In the present paper we visit again this question and we give a proof for the optimality of the MRAP taking into account the condition on the boundary. Our proofs are straightforward and we precise the role of the assumptions in order to establish the different properties of the functions we use.

The paper contains two main parts. In the first one (section 2), a general case will be studied. In the second one (section 3), an application to a fishery management problem will be given.

We shall use the following convension: $0 \cdot \infty = \infty \cdot 0 = 0$.

2. Statement of the problem and assumptions

Given a subset $[a, b] \subset R^1$ (with $a < b$) and an arbitrary element $x_0 \in [a, b]$, let us consider the following set, termed the admissible set,

$$\begin{aligned} \operatorname{Adm}(x_0) \;=\; \{\, & x : [0, \infty[\to [a, b], \text{ locally absolutly continuous, such that} \\ & x(0) = x_0 \text{ and } f^-(x(t)) \leq \dot{x}(t) \leq f^+(x(t)) \text{ almost every } t\,\} \end{aligned}$$

where f^- and f^+ are two given real functions defined on $[a, b]$.

The aim of this work is to study necessary and sufficient conditions in order that the most rapid approach path, defined below, is indeed the optimal solution of the following calculus of variations problem:

$$(VP) \qquad \sup_{x(\cdot) \in \operatorname{Adm}(x_0)} J[x(\cdot)]$$

with

$$J[x(\cdot)] = \int_{t=0}^{+\infty} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt\,,$$

where A and B are also two given real functions.

Let us associate with problem (VP) the function $V : [a, b] \to R^1$, termed the valued function, defined by

$$V(x_0) = \sup_{x(\cdot) \in \operatorname{Adm}(x_0)} \int_{t=0}^{+\infty} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt. \tag{2.1}$$

In the sequel we will use the following assumptions:

(H1) The function $A(\cdot)$ is continuously differentiable on $[a,b]$ and the function $B(\cdot)$ is continuous on (a,b).
(H2) The functions $f^-(\cdot)$ and $f^+(\cdot)$ are both Lipschitz continuous on $[a,b]$.
(H3) For all $x \in (a,b)$, one has $f^-(x) < 0 < f^+(x)$.
(H4) $f^+(a) > f^-(a) = 0$ and $f^-(b) < f^+(b) = 0$.

Remark 2.1. i) From assumptions $(H2)$-$(H4)$ the interval $[a,b]$ is an invariant subset of the differential inclusion $\dot{x}(t) \in [f^-(x(t)), f^+(x(t))]$:

- for any $x_0 \in (a,b)$, the solution paths $x(\cdot)$ of this differential inclusion with initial condition $x(0) = x_0$, are lower and upper bounded by $x^{\pm}(\cdot)$, respectively the solutions of the differential equations $\dot{x}(t) = f^{\pm}(x(t))$ with initial condition x_0 (assumption (H_2) ensures existence and uniqueness of solutions) . But $x^{\pm}(\cdot)$ are strictly bounded by a and b respectively. The constant path a (resp. b) is a stable equilibrium of $f^-(\cdot)$ (resp. $f^+(\cdot)$). It follows that any such paths are bounded and thereby defined on the whole interval $[0,\infty)$. In particular the admissible set $\mathrm{Adm}(x_0)$ is non empty.
- With same arguments we obtain that the sets $\mathrm{Adm}(a)$ and $\mathrm{Adm}(b)$ are non empty too. Moreover, $\mathrm{Adm}(a)$ contains the equilibrium curve $x(t) = a$ for all $t \geq 0$. Symetric properties stand for $\mathrm{Adm}(b)$.

ii) In some problem of fisheries management $f^+(a) = f^-(a) = 0$, this is the case for the Shaefer model [6]. In this case the set $\mathrm{Adm}(a)$ reduces then to the single constant curve a. The results presented in the first part don't cover this case. We will study it in the framework of the application to the fishery management in the second part of this work.

iii) For curves that don't approach the boundaries a nor b, $B(\cdot)$ remains bounded and then the improper integral in $J[\cdot]$ converges ($A(\cdot)$ is bounded on $[a,b]$ but $B(\cdot)$ can be unbounded near a or b). Later we will introduce an assumption that ensures the convergence of the integral on the whole interval $[a,b]$ in all cases.

2.1. **The turnpike optimality.** It is well known [4], that if an interior solution $x(\cdot)$ of problem (VP) exists, then it has to satisfy the Euler-Lagrange necessary condition. In our case, i.e. with a linearity with respect to the velocity, this condition becomes an algebraic equation:

$$C(x) := A'(x) + \delta B(x) = 0\,. \tag{2.1.1}$$

Denotes by $\bar{X}$ the set of zeroes on (a,b) of equation (2.1.1):

$$\bar{X} = \{\, \bar{x} \in (a,b) \,:\, C(\bar{x}) = 0 \,\}$$

and we will assume that it is not empty and possesses at most a finite number of elements.

Given $x_0 \in [a,b]$ and $\bar{x} \in \bar{X}$, the admissible *Most Rapid Approach Path* from x_0 to $\bar{x}$, denoted by $\mathrm{MRAP}(x_0,\bar{x})$, is (if exists) the path $x^*(\cdot) \in \mathrm{Adm}(x_0)$ satisfying the following condition:

$$|x^*(t) - \bar{x}| \leq |x(t) - \bar{x}| \quad \text{for all} \quad t \geq 0\,,$$

for every $x(\cdot) \in \mathrm{Adm}(x_0)$.

Remark 2.2. i) The fact that the definition of the set $\bar{X}$ is only considered on (a,b) is because the function $B(\cdot)$ is not defined on the extremal points of this interval.

ii) Under assumptions $(H2)$-$(H3)$, for any $x_0 \in (a,b)$ and any $\bar{x} \in \bar{X}$, there exists an unique MRAP$(x_0,\bar{x})$. Indeed, such a path is the unique solution of the following Cauchy problem

$$\dot{x}(t) = \begin{cases} f^+(x(t)) & \text{if} \quad x(t) < \bar{x}, \\ 0 & \text{if} \quad x(t) = \bar{x}, \\ f^-(x(t)) & \text{if} \quad x(t) > \bar{x} \end{cases}$$

for almost every t, with initial condition $x(0) = x_0$. This definition extends to $x_0 = a, b$ under the assumption $(H4)$. Moreover, for all $(x_0,\bar{x}) \in [a,b] \times \bar{X}$, the path MRAP$(x_0,\bar{x})$ belongs to Adm(x_0).

iii) We underline that the problem that we consider isn't concave, i.e. the integrand in the definition of $J[\cdot]$ isn't a concave function with respect to $(x,\dot{x})$. Thus, the Euler-Lagrange equation is a necessary optimality condition, but not a sufficient one.

2.2. **Continuity of the value function.** With an easy integration by parts in the definition of J we can associate to the problem (VP), the functional J_2 defined by

$$J_2[x(\cdot)] = \int_{t=0}^{+\infty} e^{-\delta t}\, C(x(t))\dot{x}(t)\, dt$$

We have for all $x(\cdot) \in \text{Adm}(x)$,

$$J[x(\cdot)] = \frac{A(x) + J_2[x(\cdot)]}{\delta}\,.$$

Then, we introduce the associated value function V_2 defined on $[a,b]$ by

$$V_2(x_0) = \sup_{x(\cdot)\in \text{Adm}(x_0)} J_2[x(\cdot)]\,. \tag{2.2.1}$$

Therefore

$$V(\cdot) = \frac{A(\cdot) + V_2(\cdot)}{\delta}\,. \tag{2.2.2}$$

In the sequel we will also use the following assumption:

(H5) $0 < C(a^+) := \lim_{x\to a^+} C(x)$ and $0 > C(b^-) := \lim_{x\to b^-} C(x)$.

In the following proposition we shall prove the continuity of function $V_2(\cdot)$ on the whole interval $[a,b]$. For that, we shall also use the following assumption, ensuring in particular the finite value on the whole interval $[a,b]$ of the function $V_2(\cdot)$:

(H6) The functions $C(\cdot)f^+(\cdot)$ and $C(\cdot)f^-(\cdot)$ are both Lipschitz continuous on (a,b).

Before, the following lemma is useful

Lemma 2.3. *Assume that assumptions $(H2)$, $(H4)$ and $(H5)$ hold. If $V_2(a)$ is finite, then for any $\epsilon > 0$, there exists $x_\epsilon(\cdot) \in Adm(a)$ such that for all $t > 0$, one has $x_\epsilon(t) \in (a,b)$ and*

$$V_2(a) < \int_{t=0}^{+\infty} e^{-\delta t} C(x_\epsilon(t))\dot{x}_\epsilon(t)dt + \epsilon\,.$$

Same facts with the point b.

Proof. We first observe that the assumption $(H4)$ guarantees the existence of paths $x(\cdot) \in \mathrm{Adm}(a)$ satisfying $x(t) \in (a,b)$ for all $t > 0$. Now, since $V_2(a)$ is finite, for any $\epsilon > 0$ there exists $x_\epsilon(\cdot)$ belonging to $\mathrm{Adm}(a)$ such that

$$V_2(a) < \int_{t=0}^{+\infty} e^{-\delta t} C(x_\epsilon(t))\dot{x}_\epsilon(t)dt + \epsilon\,.$$

It remains to prove that such a path can be choosen satisfying $x_\epsilon(t) \in (a,b)$ for all $t > 0$. For that, let us define

$$\bar{t} = \max\{\, t \geq 0 : x_\epsilon(t) = a\,\}.$$

- If $\bar{t} = \infty$, then considering $\tilde{x}(\cdot) = MRAP(a,\bar{x})$ with $\bar{x}$ being the lowest element in $\bar{X}$, we obtain that $\tilde{x}(t) \in (a,b)$ for all $t > 0$ and
$$V_2(a) < \int_{t=0}^{+\infty} e^{-\delta t} C(x_\epsilon(t))\dot{x}_\epsilon(t)dt + \epsilon < \int_{t=0}^{+\infty} e^{-\delta t} C(\tilde{x}(t))\dot{\tilde{x}}(t)dt + \epsilon\,.$$
Then, set $x_\epsilon(\cdot) = \tilde{x}(\cdot)$.
- If $0 < \bar{t} < \infty$, then taking $\delta > 0$ sufficiently small such that $\dot{x}_\epsilon(\bar{t}+t) > 0$ and $C(x_\epsilon(\bar{t}+t)) > 0$ for all $t \in (0,\delta]$, we define the path $\tilde{x}(\cdot)$ by
$$\tilde{x}(t) = \begin{cases} x_\epsilon(\bar{t}+t) & \text{if} \quad t \in [0,\delta], \\ x_\epsilon(\bar{t}+\delta) & \text{if} \quad t \in [\delta, \bar{t}+\delta], \\ x_\epsilon(t) & \text{if} \quad t \geq \bar{t}+\delta\,. \end{cases}$$
Then, $\tilde{x}(t) \in (a,b)$ for all $t > 0$ and
$$V_2(a) < \int_{t=0}^{+\infty} e^{-\delta t} C(x_\epsilon(t))\dot{x}_\epsilon(t)dt + \epsilon < \int_{t=0}^{+\infty} e^{-\delta t} C(\tilde{x}(t))\dot{\tilde{x}}(t)dt + \epsilon\,.$$
Again, set $x_\epsilon(\cdot) = \tilde{x}(\cdot)$.

The lemma follows. □

Proposition 2.4. *Under $(H2)$, $(H5)$ and $(H6)$ we have:*

(1) *If assumption $(H4)$ holds, then $V_2(\cdot)$ is continuous at points a and b.*
(2) *If assumption $(H3)$ holds, then $V_2(\cdot)$ is continuous on any point $x_0 \in (a,b)$.*

Proof. We first observe that the Lipschitz continuity on (a,b) of $C(\cdot)f^+(\cdot)$ and $C(\cdot)f^-(\cdot)$, ensures that $V_2(x_0)$ is finite for all $x_0 \in [a,b]$.

To simplify, we only prove the continuity of $V_2(\cdot)$ at point a. The proof on the other points uses the same technique described in this proof.

Take $z \in (a,b)$ and $\epsilon > 0$. We distinguish two cases:

i) If $V_2(a) \geq V_2(z)$: take $x_\epsilon(\cdot) \in \mathrm{Adm}(a)$ as Lemma 2.3. Then, a mesurable function $R^1_+ \ni t \to \lambda(t) \in (0,1)$ exists such that for all $t \geq 0$, one has

$$\dot{x}_\epsilon(t) = \lambda(t)f^-(x_\epsilon(t)) + (1-\lambda(t))f^+(x_\epsilon(t)) \quad \text{almost every } t\,.$$

Let $y(\cdot)$ be the unique solution path of the following Cauchy problem:

$$\begin{cases} \dot{y}(t) = \lambda(t)f^-(y(t)) + (1-\lambda(t))f^+(y(t)) & \text{for all} \quad t \geq 0 \\ y(0) = z\,. \end{cases}$$

Then,

$$\int_{t=0}^{+\infty} e^{-\delta t} C(y(t))\dot{y}(t)dt \leq V_2(z).$$

Thus,

$$V_2(a) - V_2(z) \leq \int_0^{+\infty} e^{-\delta t}|C(x_\epsilon)\dot{x}_\epsilon - C(y)\dot{y}|dt + \epsilon$$

$$= \int_0^{+\infty} e^{-\delta t}|\, C(x_\epsilon)[\lambda f^-(x_\epsilon) + (1-\lambda)f^+(x_\epsilon)] - C(y)[\lambda f^-(y) + (1-\lambda)f^+(y)]\,|dt + \epsilon$$

$$\leq \int_0^{+\infty} e^{-\delta t}|\,\lambda\,|\,|\,C(x_\epsilon)f^-(x_\epsilon) - C(y)f^-(y)\,| + |\,1-\lambda\,|\,|\,C(x_\epsilon)f^+(x_\epsilon) - C(y)f^+(y)\,|\,dt + \epsilon$$

$$\leq \int_0^{+\infty} e^{-\delta t}[\lambda\, k\,|x_\epsilon(t) - y(t)| + (1-\lambda)\, k\,|x_\epsilon(t) - y(t)|]dt + \epsilon$$

$$= \int_0^{+\infty} e^{-\delta t}\, k\,|x_\epsilon(t) - y(t)|dt + \epsilon$$

$k > 0$ stands for the maximum of the Lipschitz constant of $C(\cdot)f^+(\cdot)$ and $C(\cdot)f^-(\cdot)$),

$$= \int_0^{T} e^{-\delta t}\, k\,|x_\epsilon(t) - y(t)|dt + \int_T^{+\infty} e^{-\delta t}\, k\,|x_\epsilon(t) - y(t)|dt + \epsilon.$$

We choosed $T > 0$ satisfying $ke^{-\delta T}(b-a)/\delta \leq \epsilon$. By Gronwall lemma, from $|\dot{x_\epsilon}(t) - \dot{y}(t)| \leq M|x_\epsilon(t) - y(t)|$ we have $|x_\epsilon(t) - y(t)| \leq e^{Mt}|a - z|$ and thus, the preceding expression leads to

$$\leq \int_0^{T} e^{-\delta t}\, k\,|x_\epsilon(t) - y(t)|dt + 2\epsilon \leq 2\epsilon + k|a-z|\int_0^{T} e^{(M-\delta)t}dt.$$

For $|a - z| \to 0$, one obtains

$$V_2(a) - V_2(z) \leq 3\epsilon\,.$$

ii) If $V_2(a) \leq V_2(z)$: take $y_\epsilon(\cdot) \in \mathrm{Adm}(z)$ satisfying

- $V_2(z) < \int_{t=0}^{+\infty} e^{-\delta t} C(y_\epsilon(t))\dot{y}_\epsilon(t)dt + \epsilon$
- For $\lambda(t) \in [0,1]$ such that $\dot{y}_\epsilon(t) = \lambda(t)f^-(y_\epsilon(t)) + (1-\lambda(t))f^+(y_\epsilon(t))$, the solution path $x(\cdot)$ of system

$$\begin{cases} \dot{x}(t) = \lambda(t)f^-(x(t)) + (1-\lambda(t))f^+(x(t))\,,\ t \geq 0 \\ x(0) = a\,, \end{cases}$$

satisfies $x(t) \in \,]a,b]$ for all $t > 0$. By definition,

$$\int_{t=0}^{+\infty} e^{-\delta t} C(x(t))\dot{x}(t)dt \leq V_2(a).$$

Similar to the first case, we can conclude that for $|a - z| \to 0$, one obtains

$$V_2(z) - V_2(a) \leq 3\epsilon\,.$$

From items i) and ii), the continuity of V_2 at point a follows. □

Then, from relation (2.2.2) and from the fact that $A(\cdot)$ is continuous on $[a,b]$, the continuity of $V(\cdot)$ on this same interval follows using the same assumptions of Proposition 2.4.

2.3. **The value along the MRAPs.** We observe that $J_2[\cdot]$ is well defined along the MRAPs. Then, define the function $T : [a,b] \to R^1$ by

$$T(x_0) = \max_{\bar{x}\in\bar{X}} J[MRAP(x_0,\bar{x})]\,. \tag{2.3.1}$$

In this part we will prove the continuity of $T(\cdot)$ on $[a,b]$. For that, we introduce the function $T_2(\cdot)$ defined on $[a,b]$ by

$$T_2(\cdot) = \max_{\bar{x}\in\bar{X}} J_2[MRAP(\cdot,\bar{x})].$$

Then,

$$T(\cdot) = \frac{A(\cdot)+T_2(\cdot)}{\delta}\,.$$

Thus, if $T_2(\cdot)$ is continuous on $[a,b]$, from the continuity of $A(\cdot)$, we deduce the continuity of $T(\cdot)$ on the whole interval $[a,b]$.

The MRAPs denoted by $\tilde{x}(\cdot)$ are defined by one of the differential equations $\dot{y}(t) = f^+(y(t))$ or $\dot{y}(t) = f^-(y(t))$, depending of the position of the initial condition x to the particular solution of the Euler-Lagrange equation $\bar{x}_k \in \bar{X}$ that we consider. Let $\tau_k(x)$ be the first time such that $\tilde{x}(\cdot)$ crosses $\bar{x}_k$. Let $x \le x_k$ (analog result when $x \ge x_k$). As $f^+(\cdot) > 0$ on $[a,b)$, the function

$$[0,\tau_k(x)] \ni t \to y = \tilde{x}(t) \in [x,\bar{x}_k]$$

with $x \le \bar{x}_k$ is invertible. From $t = \int_x^y \frac{dz}{f^+(z)}$ we derive that

$$J_2[MRAP(x,\bar{x}_k)] = \int_x^{\bar{x}_k} C(\xi)e^{-\delta\int_x^\xi \frac{dz}{f^+(z)}}d\xi\ . \tag{2.3.2}$$

This proves that for each fixed $\bar{x}_k \in \bar{X}$, the function $J_2[MRAP(\cdot,\bar{x}_k)]$ is continuous on $[a,\bar{x}_k]$.

Similar argument shows the continuity of $J_2[MRAP(\cdot,\bar{x}_k)]$ on $[\bar{x}_k,b]$ by using in this case the function $f^-(\cdot)$ instead $f^+(\cdot)$.

In combining these two arguments, the continuity of $J_2[MRAP(\cdot,\bar{x}_k)]$ on $[a,b]$ follows.

Now we can deduce that the function $T_2(\cdot)$ is continuous on $[a,b]$, as the maximum of a finite number of continuous functions corresponding to the finite number of elements of $\bar{X}$.

Therefore, we have proved the following proposition:

Proposition 2.5. *Assume that assumptions $(H1)$-$(H5)$ hold, then $T(\cdot)$ is continuous on $[a,b]$.*

Remark 2.6. The preceding proposition extends the result obtained in [11], where the function $T_2(\cdot)$ was seen to be continuous only on (a,b).

2.4. **Behaviour of $V(\cdot)$ and $T(\cdot)$ on the boundary of $[a,b]$.** As it will be precise in the next section, in order to compare the two functions $T(\cdot)$ and $V(\cdot)$, it is necessary to know their values on the boundaries of $[a,b]$.

Hartl and Feichtinger in [8] gave two sufficient conditions in order to prove the optimality of the MRAPs for the problem (VP), in presence of a unique solution for the Euler-lagrange and when the state x belongs to R^1. The first one is a sign condition already introduced by Sethi [12], $C(\bar{x})(\bar{x}-x) \geq 0$ for all x. The second one (relation (17) in [8]) looks like a transversality condition. We prove that this second condition (given below (2.4.1)) is always satisfied in our framework scheme i.e. when the state belongs to the bounded set $[a,b]$.

Proposition 2.7. *Under assumptions $(H1)$-$(H5)$, for any $(x_0,\bar{x}) \in [a,b]\times\bar{X}$ and any $x(\cdot) \in Adm(x_0)$, one has*

$$\limsup_{t\to+\infty}\,[\,e^{-\delta t}\int_{x(t)}^{\bar{x}} B(\xi)d\xi\,] \geq 0\,. \tag{2.4.1}$$

Proof. Fix $(x_0,\bar{x}) \in [a,b]\times\bar{X}$ and $x(\cdot)\in \mathrm{Adm}(x_0)$. We will assume, without loss of generality, that the path $x(\cdot)$ remains below $\bar{x}$ (for the other situations we can adapt a technical analog to the following argument). It is clear that relation (2.4.1) holds if $\lim_{x\to a^+} B(x)$ is finite. Now, since $C(a^+)>0$, we can also assume that $\lim_{x\to a^+} B(x)=+\infty$. Then, an element $z\in (a,\bar{x})$ exists such that for all $\xi \in (a,z]$, one has $B(\xi)\geq 0$. Let us consider the following path:

$$y(t)=\begin{cases} z & \text{if} \quad x(t)\leq z\,,\\ x(t) & \text{if} \quad x(t)>z \end{cases}$$

and define, for all $t\geq 0$,

$$I_1(t)=\int_{y(t)}^{\bar{x}} B(\xi)d\xi \quad\text{and}\quad I_2(t)=\int_{x(t)}^{z} B(\xi)\chi_{(a,z]}(\xi)d\xi\,,$$

where, $\chi_{(a,z]}(\cdot)$ denotes the characteristic function of interval $(a,z]$:

$$\chi_{(a,z]}(\xi)=\begin{cases} 1 & \text{if}\ \ \xi\in(a,z],\\ 0 & \quad\text{otherwise}\,. \end{cases}$$

It follows that $I_1(\cdot)$ remains bounded on $[0,+\infty)$ and for any t in this interval, the quantity $I_2(t)$ is not negative. From this and from the fact that

$$\int_{x(t)}^{\bar{x}} B(\xi)d\xi = I_1(t)+I_2(t)\,,$$

the inequality in (2.4.1) follows. □

Now applying Theorem 7.1 of [12], we can conclude that the value $V(a)$ (resp. $V(b)$) coincides with the value $T(a)$ (resp. $T(b)$).

Proposition 2.8. *Under assumptions $(H1)$, $(H2)$ and $(H5)$, we have:*

i) If $f^+(a) > f^-(a)=0$, then $T(a)=V(a)$.
ii) If $f^-(b) < f^+(b)=0$, then $T(b)=V(b)$.

Proof. In order to apply Theorem 7.1 of [12], it is sufficient to ensure the existence of $MRAP(a, \bar{x})$ (resp. $MRAP(b, \bar{x})$), for any $\bar{x} \in \bar{X}$. But, this existence follows from the fact that $f^+(a) > f^-(a)$ (resp. $f^+(b) > f^-(b)$), as seen in Remark 2.1.1. ii). Thus, under the above assumptions, following the Sethi's proof we can apply Green's theorem associated to any pair of non-constant admissible paths belonging to Adm(a) (resp. Adm(b)). □

2.5. **Optimality of the MRAP.** In order to prove the optimality of the MRAPs, we will show that the two functions $V(\cdot)$ and $T(\cdot)$ coincide on $[a, b]$. To do that we use one of the well known results on the uniqueness of the solutions of an PDE. More precisely we use the theorem given in [2] (page 51), that gives such a result in the case of open bounded set. Thus we have to prove that

1) $T(\cdot)$ and $V(\cdot)$ are solutions of a same Hamilton-Jacobi equation $\delta U(x) + H(x, U'(x)) = 0$ on (a, b).
2) $T(\cdot)$ and $V(\cdot)$ are continuous on the whole interval $[a, b]$.
3) $T(\cdot)$ and $V(\cdot)$ take same values on the boundaries a, b.
4) $H(x, p)$ satisfies
$$|H(x, p) - H(y, p)| \leq F(|x - y|(1 + |p|))$$
where $F : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function with $F(0) = 0$, for all $x, y \in (a, b)$ and $p \in R^1$.

In the preceding sections we proved the items 2) and 3).

In order to prove item 1), we can compute straightforwardly, but as we have to establish this result on the open interval (a, b) only, we prefer use the result obtained in [11]. In this last paper, $V_2(\cdot)$ is proved to be a viscosity solution on (a, b) of the the following Hamilton-Jacobi equation:

$$\delta Z(x) + H(x, Z'(x)) = 0$$

where

$$H(x, Z'(x)) := -\max[(C(x) + Z'(x))f^-(x), (C(x) + Z'(x))f^+(x)]\,.$$

Moreover $T_2(\cdot)$ is proved to be a solution of the same equation if and only if $T_2(\cdot)$ is nonnegative. But in our case, $T_2(\cdot)$ is always nonnegative: denoting the solution of the Euler-Lagrange equation, $C(x) = 0$, by $\bar{x}_1 < \bar{x}_2 < ... < \bar{x}_n$, $C(\cdot)$ has a constant sign on each $(\bar{x}_i, \bar{x}_{i+1})$ and then by choosing $\dot{x}(t) = f^+(x(t))$ or $\dot{x}(t) = f^-(x(t))$ for $x(t) \in (\bar{x}_i, \bar{x}_{i+1})$ the value of the corresponding MRAP is positive and then $T_2(\cdot)$ is positive.

Then we deduce that $V(\cdot)$ is solution on (a, b) of the following Hamilton-Jacobi equation:

$$\delta Z(x) - A(x) - \max[(B(x) + Z'(x))f^-(x), (B(x) + Z'(x))f^+(x)] = 0$$

Moreover from the relation between $T(\cdot)$ and $T_2(\cdot)$, we derive that $T(\cdot)$ is solution of the same equation.

Finally, for the item 4), $H(\cdot)$ satisfy this condition is an easy consequence of the corresponding condition given in [11].

Now the proof is complete. We can conclude that $V(\cdot) = T(\cdot)$ on $[a, b]$ and that therefore the MRAPs are optimal solutions of the problem (VP).

Theorem 2.9. *Assume that assumptions* $(H1)$-$(H6)$ *hold, then the MRAPs are optimal solutions of the problem* (VP).

Remark 2.10. We have proved moreover that the problem (VP) has solutions, from the existence of the MRAPs.

3. Models for fisheries management: Shaefer's model

We begin with the most standard model of the management of a fishery introduced by Shaefer [6].

Let $x(t)$ stands for the stock of fishes at time t disponible in a particular zone. We assume that its evolution evolves with the following dynamic

$$\dot{x}(t) = f(x(t)) - qE(t)x(t), \ t \geq 0, \tag{3.1}$$

where $f(\cdot)$ is given by the logistic evolution $f(x) = rx(1 - \frac{x}{K})$, that corresponds to the natural growth of the population. The quantity qEx corresponds to the harvest, the function $E(\cdot)$ corresponds to the fishing effort, and satisfies $0 \leq E(t) \leq E_M$ for all $t \geq 0$. We consider only the significative values of the variable x: $x \in [0, K]$.

In the sequel we will assume that $qE_M = r > 0$. If the maximum available policy qE_M is followed, then the stock is drived to 0, a non acceptable situation, from a biological point of view. Thus generally $qE_M \leq r$ is considered in the literature.

Similar to the first part of this work, let us introduce the set of admissible paths at x_0, wich is given by

$$\begin{aligned} \mathrm{Adm}(x_0) \ = \ & \{x : [0, \infty[\to R^1, \text{ locally absolutly continuous, such that} \\ & x(0) = x_0 \text{ and } \exists\, E(\cdot) \text{ with } \dot{x} = f(x) - qEx \ \text{ almost every } t\,\} \end{aligned}$$

Now the harvest is sold on a market. Assume constant price p and cost c per unity of effort. Thus at each time the revenue is given by $(pqx(t) - c)E(t)$. A manager has to choose a policy of effort $E(\cdot)$, if any, in order to maximise the total actualised revenue:

$$\sup_{x(\cdot) \in \mathrm{Adm}(x_0)} J(x(\cdot)) \tag{P}$$

where

$$J(x(\cdot)) = \int_{t=0}^{+\infty} e^{-\delta t}(pqx(t) - c)E(t)dt\,. \tag{3.2}$$

The solution of this optimal control problem is well known: a unique constant solution exists, given for instance by the Pontryagin maximum principle. Depending of the initial stock $x(0)$, the optimal solutions consist to connect as quickly as possible this singular solution and to stay on it. This result is established by using the Green's theorem [6].

We revisit this problem with the approach proposed in the preceding part. As we will see this approach can't apply straight to this fishery problem.

Then, define the value function $V(\cdot)$ associate with problem (P):

$$V(x_0) = \sup_{x(\cdot)\in \mathrm{Adm}(x_0)} J(x(\cdot))\,. \tag{3.3}$$

3.1. Remark on the admissible set $\mathbf{Adm}(x_0)$.

i) For a given arbitrary control $E(\cdot)$, the point $x = 0$ is an equilibrium of system (3.1).

ii) By definition, $\mathrm{Adm}(0) = \{x(\cdot) \equiv 0\}$. Thus, for any admissible control $E(\cdot)$, the value of J at $x \equiv 0$ is

$$J(0) = -c \int_0^\infty e^{-\delta t} E(t)dt$$

and therefore taking $E \equiv 0$, one obtains $V(0) = 0$.

iii) Since the admissible functions $x(\cdot)$ and $E(\cdot)$ are bounded, then the functional J defined in (3.2) is always finite.

3.2. Continuity of the value function on $[0, K]$.

In this part we will prove the continuity property of the function $V(\cdot)$ on $[0, K]$, defined in (3.3).

Proposition 3.1. *The function $V(\cdot)$ is continuous on $[0, K]$.*

Proof. Let x, y be two arbitrary points of $[0, K]$. Then, for any $\epsilon > 0$ there exist functions $y_\epsilon(\cdot) \in \mathrm{Adm}(y)$ and $E_\epsilon(\cdot)$ (associated to $y_\epsilon(\cdot)$) such that

$$V(y) < \int_0^\infty e^{-\delta t}(pqy_\epsilon(t) - c)E_\epsilon(t)dt + \epsilon.$$

With this same ϵ, take $x_\epsilon(\cdot)$ the solution path of system (3.1) with $E = E_\epsilon$ and initial condition $x_\epsilon(0) = x$. Then,

$$V(x) \geq \int_0^\infty e^{-\delta t}(pqx_\epsilon(t) - c)E_\epsilon(t)dt.$$

Adding these two inequalities, one obtains

$$\begin{aligned} V(y) - V(x) \;&\leq\; \epsilon + \int_0^T e^{-\delta t}(pqE_\epsilon(t)(y_\epsilon(t) - x_\epsilon(t))dt \\ &\quad + \int_T^\infty e^{-\delta t}(pqE_\epsilon(t)(y_\epsilon(t) - x_\epsilon(t))dt \end{aligned}$$

where T is a real positive such that $(2pqKE_M\, e^{-\delta T})/\delta \leq \epsilon$. Then, (assuming without loss of generality that $V(x) \leq V(y)$), one obtains

$$\begin{aligned} |V(y) - V(x)| \;&\leq\; \epsilon + \int_0^T e^{-\delta t}pqE_M|y_\epsilon(t) - x_\epsilon(t)|dt + \int_T^\infty 2pqKE_M\, e^{-\delta t}dt \\ &=\; \epsilon + \frac{2pqKE_M}{\delta}\, e^{-\delta T} + \int_0^T e^{-\delta t}pqE_M|y_\epsilon(t) - x_\epsilon(t)|dt \\ &\leq\; 2\epsilon + \int_0^T e^{-\delta t}pqE_M|y_\epsilon(t) - x_\epsilon(t)|dt\,. \end{aligned}$$

To complete the proof, let us bound $|y_\epsilon(t) - x_\epsilon(t)|$ in term of $|x - y|$. Since x_ϵ and y_ϵ are both solutions of system (3.1) with $E = E_\epsilon$, then

$$\begin{aligned} \dot{y}_\epsilon(t) - \dot{x}_\epsilon(t) &= f(y_\epsilon(t)) - qE_\epsilon(t)y_\epsilon(t) - f(x_\epsilon(t)) + qE_\epsilon(t)x_\epsilon(t) \\ &= r(y_\epsilon(t) - x_\epsilon(t)) + \frac{r}{K}(x_\epsilon^2(t) - y_\epsilon^2(t)) - qE_\epsilon(t)(y_\epsilon(t) - x_\epsilon(t)) \end{aligned}$$

and therefore

$$\begin{aligned} |\dot{y}_\epsilon(t) - \dot{x}_\epsilon(t)| &\le (r + qE_\epsilon(t))|y_\epsilon(t) - x_\epsilon(t)| + \frac{r}{K}|x_\epsilon(t) - y_\epsilon(t)|2K \\ &= (3r + qE_M)|y_\epsilon(t) - x_\epsilon(t)|\,. \end{aligned}$$

Thus, by Gronwall lemma, one obtains

$$|y_\epsilon(t) - x_\epsilon(t)| \le e^{(3r+qE_M)t}\,|y - x|.$$

In replacing this inequality in the preceding bound of $|V(y) - V(x)|$, one obtains

$$|V(y) - V(x)| \le 2\epsilon + pqE_M\, e^{(3r+qE_M)T}(\frac{1}{\delta} - \frac{e^{-\delta T}}{\delta})|y - x|$$

and therefore, for $|y - x| \to 0$, one obtains

$$|V(y) - V(x)| \le 3\epsilon\,.$$

The continuity of $V(\cdot)$ on $[0, K]$ follows. □

3.3. **The control problem as a calculus of variations problem.** An important property of the solution path of the Cauchy problem associated to dynamic (3.1),

$$\begin{cases} \dot{x}(t) = f(x(t)) - qEx(t) & \text{for all } t \ge 0, \\ x(0) = x_0\,. \end{cases}$$

is that if $x_0 \in (0, K]$, then $x(t) \in (0, K]$ for all $t \ge 0$: only for the control $E(\cdot) = qE_M$, the value $x(t)$ goes asymptotically to 0 when t goes to $+\infty$.

Thus, for any $x_0 \in (0, K]$, we can express $E(t)$ as

$$E(t) = \frac{f(x(t)) - \dot{x}(t)}{qx(t)} \quad \text{for all} \quad t \ge 0$$

and thereby the problem (P) as a problem of the calculus of variations:

$$(vp) \qquad v(x_0) := \sup_{\mathrm{Adm}(x_0)} \int_{t=0}^{+\infty} e^{-\delta t}(p - \frac{c}{qx(t)})(f(x(t)) - \dot{x}(t))dt$$

where

$$\begin{aligned} \mathrm{Adm}(x_0) = \{\, & x : [0, \infty[\to [0, K], \text{ locally absolutly continuous, such that} \\ & x(0) = x_0 \text{ and } f(x) - qE_M x \le \dot{x} \le f(x) \text{ almost every } t\,\} \end{aligned}$$

In this case, the sup-value in (vp) coincides with the value $V(x_0)$ defined in (3.3).

When $x_0 = 0$, the admissible set Adm(0) is reduced to the origin, as for the control approach, and therefore we can set $v(0) = 0$.

In resume, the function $V(\cdot)$ of problem (P) coincides with the function $v(\cdot)$ of problem (vp) on the whole interval $[0, K]$. In particular (from Proposition 3.1), the function $v(\cdot)$ is continuous on this interval too.

Remark 3.2. We underline that the calculus of variations problem (vp) is of the type of the problems studied in the first part and correspond to a problem (VP) where $A(x) = (p - \frac{c}{qx})f(x)$ and $B(x) = -(p - \frac{c}{qx})$. Moreover we have that $f^+(0) = f^-(0) = 0$, a situation excluded in the first part of this paper. Thus in this case too, we have obtained the continuity of the value function on $[a, b]$.

3.4. **The value function as a continuous solution of some Hamilton-Jacobi equation.** Let $A(\cdot)$ and $B(\cdot)$ given by the expressions in the preceding remark. Then, for all $x_0 \in (0, K]$,

$$V(x_0) = \sup_{\mathrm{Adm}(x_0)} \int_{t=0}^{+\infty} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt.$$

Let $x_0 \in (0, K]$ and $\phi \in C^1([0, K])$, a test function, such that x_0 is a maximum of $V(\cdot) - \phi(\cdot)$ on $[0, K]$. Without loss of generality we can assume that $(V - \phi)(x) \leq (V - \phi)(x_0) = 0$.

Then from the dynamical programmation principle:

$$V(x) = \sup_{\mathrm{Adm}(x)} \int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt + e^{-\delta T} V(x(T))$$

we deduce

$$\sup_{\mathrm{Adm}(x_0)} \int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt + e^{-\delta T} V(x(T)) - \phi(x_0) = 0$$

and so

$$\sup_{\mathrm{Adm}(x_0)} \int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt + e^{-\delta T} \phi(x(T)) - \phi(x_0) \geq 0$$

Now from:

$$\sup_{\mathrm{Adm}(x_0)} \frac{1}{T} \int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt + \frac{e^{-\delta T} \phi(x(T)) - \phi(x_0)}{T} \geq 0$$

as T tends to 0 we obtain:

$$\sup_{\dot{x}(0) \in [f^-(x_0), f^+(x_0)]} [A(x_0) + B(x_0)\dot{x}(0) - \delta\phi(x_0) + \phi'(x_0)\dot{x}(0)] \geq 0$$

and therefore

$$\delta V(x_0) - A(x_0) - \sup_{\dot{x}(0)} [(B(x_0) + \phi'(x_0))\dot{x}(0) \geq 0.$$

Since $\dot{x}(0) \in [f^-(x_0), f^+(x_0)]$, we deduce

$$\delta V(x_0) - A(x_0) - \max[(B(x_0) + \phi'(x_0))f^-(x_0),\, (B(x_0) + \phi'(x_0))f^+(x_0)] \leq 0\,.$$

This says that the value function $V(\cdot)$ is a **viscosity subsolution** on $(0,K]$ of the following Hamilton-Jacobi equation:

$$(HJ) \qquad \delta V(x) + H(x, V'(x)) = 0$$

where,

$$H(x,p) = -A(x) - \max[(B(x)+p)f^-(x),\, (B(x)+p)f^+(x)]\,.$$

Let us now prove that $V(\cdot)$ is also a **viscosity supersolution** on $(0,K]$ of this same equation (HJ). Let $x_0 \in (0,K]$ and $\phi \in C^1((0,K])$ be such that x_0 is a minimum on $(0,K]$ of $V(\cdot)-\phi(\cdot)$. Similar to the previous situation, we can assume, without loss of generality, that $(V-\phi)(x) \geq (V-\phi)(x_0) = 0$ for all $x \in (0,K]$. From the dynamic programming principle we have :

$$\phi(x_0) = V(x_0) \geq \int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt + e^{-\delta T}V(x(T))$$

and therefore

$$-\frac{1}{T}\int_{t=0}^{T} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt \geq \frac{e^{-\delta T}\phi(x(T)) - \phi(x_0)}{T}\,.$$

As T tends to 0 we deduce:

$$\delta\phi(x_0) - A(x_0) - (B(x_0) + \phi'(x_0))\dot{x}(0) \geq 0\,.$$

Therefore

$$\delta V(x_0) - A(x_0) - \max[(B(x_0)+\phi'(x_0))f^-(x_0),\, (B(x_0)+\phi'(x_0))f^+(x_0)] \geq 0$$

and so $V(\cdot)$ is a viscosity supersolution of

$$\delta V(x) + H(x, V'(x)) = 0\,.$$

In resume, we have proved the following proposition:

Proposition 3.3. *The value function $V(\cdot)$ is a continuous solution of viscosity on $(0,K]$ of the following Hamilton-Jacobi equation:*

$$(HJ) \qquad \delta V(x) + H(x, V'(x)) = 0$$

where,

$$H(x,p) = -A(x) - \max[(B(x)+p)f^-(x),\, (B(x)+p)f^+(x)]\,.$$

3.5. **Turnpike optimality.** It is well known that the interior solutions of problem (VP), using the same notation of Remark 3.2. must satisfy the Euler-Lagrange first order condition:

$$(EL) \qquad C(x) := A'(x) + \delta B(x) = 0\,.$$

In this setting, the interior notion is based in the uniform topology on the space of continuous paths $C^0([0,\infty[, [0,K])$. Thus, for any $x_0 \in (0,K]$, a path $x(\cdot)$ belongs to the interior of $\mathrm{Adm}(x_0)$ if and only if hold the following two conditions:

- $x(0) = x_0$
- For all $t > 0$, one has $\dot{x}(t) \in (\, f^-(x(t)), f^+(x(t))\,)$, with $f^-(x) = f(x) - qE_M x$ and $f^+(x) = f(x)$.

As we saw in the previous section, the admissible set Adm(0) is reduced to the origin and therefore its interior is reduced to empty set.

Proposition 3.4. *Assume also that* $pqK > c$*. Then, the equation* (EL) *has exactly one zero on* $(0, K)$ *.*

Proof. Since $A(x) = (p - \frac{c}{qx})f(x)$ and $B(x) = -(p - \frac{c}{qx})$, then

$$C(x) = \frac{cr}{qx}(1 - \frac{x}{K}) + (p - \frac{c}{qx})(r - \frac{2rx}{K} - \delta).$$

Thus, $C(x) = 0$ if and only if

$$P(x) := \frac{cr}{q}(1 - \frac{x}{K}) + (px - \frac{c}{q})(r - \delta - \frac{2rx}{K}) = 0\,.$$

Since P is a concave function on R^1 (and therefore continuous) satisying $P(0) = (c\delta)/q > 0$ and $P(K) = (pK - c/q)(-\delta - r) < 0$, then there exists a unique $\bar{x}$ belonging to $(0, K)$ such that $P(\bar{x}) = 0$. The result follows. □

Remark 3.5.

- If $pqK - c \leq 0$, then for all $x \in [0, K]$, one has $pqx - c \leq 0$ and thereby $V(x) = 0$ for all $x \in [0, K]$. In this case the exploitation of the ressource present no interest.
- Since the objetive function of problem (VP) is not concave, the necessary optimality condition given by the Euler-Lagrange condition is not sufficient.

3.6. **The most rapid approach path: MRAP.** The following proposition guarantees the existence and uniqueness of the MRAPs. Denotes by $\bar{x}$ the unique solution of equation (EL) on $(0, K)$.

Proposition 3.6. *For each initial point* $x_0 \in (0, K]$*, there exists a unique admissible path* $MRAP(x_0, \bar{x})$ *belonging to* $Adm(x_0)$*.*

Proof. Given $x_0 \in (0, K]$, let us consider the following Cauchy problem

$$(CP)\left\{\begin{array}{l} \dot{x}(t) = f(x(t)) - qE_M x(t)\chi_{(0,\bar{x}]}(x_0) \;\text{ for all } t \geq 0, \\ x(0) = x_0 \end{array}\right.$$

where $\chi_{(0,\bar{x}]}(\cdot)$ denotes the characteristic function of interval $(0, \bar{x}]$:

$$\chi_{(0,\bar{x}]}(x_0) = \left\{\begin{array}{l} 1 \;\text{ if } x_0 \in (0, \bar{x}], \\ 0 \;\text{ otherwise }. \end{array}\right.$$

Let us denote by $x(\cdot)$, the unique solution of system (CP). Then, define the nonnegative real number (indeed it is strictly positive if $x_0 \neq \bar{x}$),

$$\tau(x_0) = \inf[\, t \geq 0 : x(t) = \bar{x}\,]$$

and thereby the path $x^*(\cdot)$ such that for all $t \geq 0$,

$$x^*(t) = \left\{\begin{array}{l} x(t) \;\text{ if } t \leq \tau(x_0), \\ \bar{x} \;\text{ if } t \geq \tau(x_0). \end{array}\right.$$

From the definition, the path $x^*(\cdot)$ satisfies the properties of More Rapid Approach Paths and also belongs to $\mathrm{Adm}(x_0)$. The result follows □

The particular situation when $x_0 = 0$ necessites to precise the definition of the function $T(\cdot)$ introduced before. Let us define the function $T : [0, K] \to R^1$ by

$$T(x_0) = \begin{cases} J[MRAP(x_0, \bar{x})] & \text{if} \quad x_0 \in (0, K], \\ 0 & \text{if} \quad x_0 = 0\,. \end{cases}$$

In particular, for $x_0 = \bar{x}$, we obtain

$$T(\bar{x}) = \frac{\bar{E}}{\delta}(pq\bar{x} - c)$$

with $\bar{E}$ satisfying $f(\bar{x}) - q\bar{E}\bar{x} = 0$.

3.7. **The explicit expression of function T.** From definition, for all $x_0 \in (0, K]$, one has

$$T(x_0) = J[MRAP(x_0, \bar{x})] = \int_0^{\tau(x_0)} e^{-\delta t}(pqx(t) - c)E(t)dt + \alpha(x_0)\,,$$

where,

$$\alpha(x_0) = \frac{e^{-\delta\tau(x_0)}}{\delta}\bar{E}(pq\bar{x} - c)\,.$$

From definition, the valued $\tau(x_0)$ is obtained from $x(\tau(x_0)) = \bar{x}$.

In order to obtain the explicit expression of function T, we must calculate $\tau(x_0)$ in terms of donnes. For that, we must solve the following Cauchy problem:

$$\text{(3.7.1)} \qquad \begin{cases} \dot{x} = rx - qEx - \frac{r}{K}x^2 \\ x(0) = x_0\,, \end{cases}$$

with $E = 0$ or E_M, depending if x_0 belongs to $(0, \bar{x}]$ or $(\bar{x}, K]$, respectively.

By the change of variable $x = 1/y$, equation (3.7.1) is transformed to

$$\begin{cases} \dot{y} = (qE - r)y + \frac{r}{K} \\ y(0) = y_0 = 1/x_0 \end{cases}$$

whose solution path is given by

$$y(t) = \begin{cases} (\frac{1}{x_0} - \frac{1}{K})e^{-rt} + \frac{1}{K} & \text{if} \quad E = 0\,, \\ \frac{r}{K}t + \frac{1}{x_0} & \text{if} \quad E = E_M\,. \end{cases}$$

As $y(t) > 0$ for all $t \leq 0$, the solution path of system (3.7.1) is given by $x(t) = 1/y(t)$.

I.- **Assume that** $x_0 \in (0, \bar{x}]$: Then, $E = 0$. Then, the solution path of system (3.7.1) is given by

$$x(t) = 1/y(t) = \frac{1}{(\frac{1}{x_0} - \frac{1}{K})e^{-rt} + \frac{1}{K}}\,.$$

It follows from $x(\tau(x_0)) = \bar{x}$ that,

$$\tau(x_0) = \frac{1}{r}\ln\frac{\bar{x}(K - x_0)}{x_0(K - \bar{x})}$$

and therefore

$$T(x_0) = \frac{(pq\bar{x} - c)\bar{E}}{\delta}\left(\frac{x_0(K-\bar{x})}{\bar{x}(K-x_0)}\right)^{\delta/r} = \frac{A(\bar{x})}{\delta}e^{-\delta\tau(x_0)} .$$

II.- **Assume that** $x_0 \in (\bar{x}, K]$: Then, $E = E_M$ and, since $r = qE_M$, the solution path of system (3.7.1) is given by

$$x(t) = 1/y(t) = \frac{1}{\frac{r}{K}t + \frac{1}{x_0}}.$$

Similar to the previous argument we deduce that

$$\tau(x_0) = \frac{K}{r}(\frac{x_0 - \bar{x}}{x_0\bar{x}})$$

and therefore

$$T(x_0) = (pq\bar{x} - c)\frac{\bar{E}}{\delta}e^{-\delta\tau(x_0)} - \frac{rc}{q\delta}(1 - e^{-\delta\tau(x_0)}) + pqrx_0\int_0^{\tau(x_0)} \frac{e^{-\delta t}}{K + rx_0 t}dt .$$

From these expression we deduce in particular the continuity of the function $T(\cdot)$ on $[0, K]$. From the first expression, when x tends to 0, then $T(x)$ tends to 0 witch is the value we let for $T(0)$. Moreover the two expressions coincide at $\bar{x}$.

3.8. T **is a solution of equation** (HJ)**:** In this part we will prove that the funcion $T(\cdot)$ is also a solution on $(0, K]$ of equation (HJ) defined in Section 3.4. That is, we will prove that for all $x \in (0, K]$

$$\delta T(x) - A(x) - \max[(B(x) + T'(x))f^-(x),\ (B(x) + T'(x))f^+(x)] = 0 .$$

1) On $(0, \bar{x}]$ we have

$$T(x) = \frac{A(\bar{x})}{\delta}e^{-\delta\tau(x)}$$

that is differentiable with

$$T'(x) = \frac{A(\bar{x}}{f(x)}e^{-\delta\tau(x)}$$

which follows from $\tau'(x) = -\frac{1}{f(x)}$.

We then easily derive that

$$\delta T(x) - T'(x))f(x) = 0.$$

On the other hand we know that for all $x \in (0, K]$

$$A(x) + B(x)f(x) = 0.$$

Thus, for all $x \in (0, \bar{x}]$

$$\delta T(x) - A(x) - (B(x) + T'(x))f(x) = 0.$$

Now, from this last relation we deduce that for all $x \in (0, \bar{x}]$

$$\delta T(x) - A(x) = (B(x) + T'(x))f(x).$$

We have also

$$\delta T(x) - A(x) = T_2(x) = \int_0^{\tau(x)} e^{-\delta t} C(\tilde{x}(t))\dot{\tilde{x}}(t)dt > 0$$

where $\tilde{x}(\cdot)$ stands for the MRAP$(x, \bar{x})$. The last inequality comes from the fact that on $(0, \bar{x})$, the function $C(\cdot)$ and the value $\dot{\tilde{x}}(t) = f(\tilde{x}(t))$ are strictly positive.

Thus we deduce that

$$(B(x) + T'(x))f(x) > 0 \quad x \in (0, \bar{x})$$

As at point $\bar{x}$, the left hand side of this last relation is equal to zero, we deduce therefore that the function $T(\cdot)$ is solution on $(0, \bar{x}]$ of the following equation

$$\delta T(x) - A(x) - \max[(B(x) + T'(x))f^-(x),\, (B(x) + T'(x))f^+(x)] = 0\,,$$

which coincides with the equation (HJ).

2) With a same computation it can be proved that the corresponding expression for $T(\cdot)$ on $[\bar{x}, K]$ satisfies

$$\delta T(x) - A(x) - (B(x) + T'(x))f^-(x) = 0\,.$$

As $\delta T(x) - A(x) > 0$ on $(\bar{x}, K]$, then $B(x) + T'(x) < 0$ and therefore $T(\cdot)$ is solution on $[\bar{x}, K]$ of

$$\delta T(x) - A(x) - \max[(B(x) + T'(x))f^-(x),\, (B(x) + T'(x))f^+(x)] = 0\,.$$

From these two arguments, we deduce that the function $T(\cdot)$ is a solution on $(0, K]$ of the same Hamilton-Jacobi equation (HJ) that is satisfied by the value function $V(\cdot)$.

3.9. **MRAPs are optimal solutions.** We now can apply the same theorem on the uniqueness of the solutions of a PDE recalled in the first part.

- $T(\cdot)$ and $V(\cdot)$ are continuous on $[0, K]$.
- $T(0) = V(0)$ and $T(K) = V(K)$ (the last equality can easily be obtained using the Hartl and Feichtinger's result given in the first part).
- $T(\cdot)$ and $V(\cdot)$ are solutions of the same Hamilton-Jacobi equation.
- The Hamiltonian of this equation satisfies the condition given in the first part.

These items was proved in the preceding sections, thus $T(\cdot) = V(\cdot)$ on $[0, K]$ and we conclude that the MRAPs are optimal.

4. Generalised Shaefer's models

Many variations of the Shaefer's model can be construct.

(1) In [11], the authors proposed to study a management problem with new expressions for the growth function and the price of the harvest:

$$\begin{array}{rl} f(x) = & rx^\gamma\left(1 - \frac{x}{K}\right), \quad (\gamma > 1) \\ p(x) = & \frac{\bar{p}}{1+\alpha x^\beta}, \quad (\alpha > 0,\ \beta > 1)\,. \end{array}$$

In this situation the Euler-Lagrange equation $C(x) = 0$, possesses 3 positive solutions $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$. Assume that $qE_M = r$. Then our preceding results (in particular on the boundary conditions given in Section 2 apply and the proof given in [11] is now complete in this case. We recall that depending of the value of the actualisation factor δ we can obtain a competition between two turnpikes, i.e. from a specific initial condition x_0 the two curves $\mathrm{MRAP}(x_0, \bar{x}_1)$ and $\mathrm{MRAP}(x_0, \bar{x}_3)$ are optimal.

The case $qE_M < r$ proposed in [11] necessites a specific study that we don't give here.

(2) Management of a fishery with two zones. In [5], the author study a model with two marine zones, where growth and harvest exist. A limit case is given by the following situation. Assume that the evolution of the stocks are given by

$$\begin{aligned} \dot{x}_1 = &\quad \gamma(x_1 + x_2)\left(1 - \tfrac{1}{K}(x_1 + x_2)\right) - bx_1 - q_1 E_1 x_1 \\ \dot{x}_2 = &\quad bx_1 - q_2 E_2 x_2 \end{aligned}$$

where x_i and E_i denote stocks and efforts in each zone .

Assume moreover that a manager has to maximise the total revenue of the artisanal fishery (zone 1) and industrial (zone 2) with the two instruments E_1 and E_2 :

$$\max_{E_2(\cdot), E_1(\cdot)} \int_0^{+\infty} e^{-\delta t} \left[(p_2 q_2 x_2(t) - c_2)\, E_2(t) + (p_1 q_1 x_1(t) - c_1)\, E_1(t) \right] dt \,.$$

It is easy to derive that in this case the Euler-Lagrange equations are given by two algebraic equations.

To find the optimal solutions for this problem seems to be yet an open problem.

5. Conclusion

In this paper we visit again a singular problem of calculus of variations with infinite horizon by the help of the viscosity solutions. This approach was used in a preceding paper [11], where some precisions had been missed. We proved that the MRAP from any initial conditions (in a bounded interval) to some solutions of the (algebraic) Euler-Lagrange equation are optimal. To obtain this result we showed that the value function of the problem, $V(\cdot)$, and the value of the objective along a MRAP, $T(\cdot)$, are solutions of the same Hamilton-Jacobi equation. We derive then the optimality by using a uniqueness result of the solutions for this equation. For this uniqueness result we underline the important role played by the boundary conditions and we give assumptions that suffice to establish them. These boundary conditions play the role of the transversality conditions, an important question that arise in the Halkin-Pontryagin approach [1], [7]. We give new and straightforward proofs for the continuity of the functions $V(\cdot)$ and $T(\cdot)$. In a second part we study a fisheries management problem whose modelisation looks like the preceding problem. We underline that the presence of an equilibrium in the dynamic necessites a specific study and we solve this question. Our proof are straightforword and we don't use the results of the first part that are obtain with a new problem associated to the original calculus of variations problem.

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Eladio Ocaña Anaya
IMCA, Instituto de Matemática y Ciencias Afines, Universidad Nacional de Ingeniería, Calle Los Biólogos 245 Urb. San Cesar, La Molina, Lima 12, Perú
E-mail address: eocana@imca.edu.pe

Pierre Cartigny
INRA-SupAgro, UMR 729 Analyse des systèmes et Biométrie, F-34060 Montpellier & GREQAM, Université de la Méditerranée, 2 rue de la vieille Charité, 13002 Marseille, France
E-mail address: pierre.cartigny@supagro.inra.fr

Patrice Loisel
INRA-SupAgro, UMR 729 Analyse des systèmes et Biométrie, F-34060 Montpellier, France
E-mail address: patrice.loisel@supagro.inra.fr