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REGULARIZATION METHOD FOR NONMONOTONE EQUILIBRIUM PROBLEMS

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Abstract. We consider a general equilibrium problem in a finite-dimensional space setting and propose a new sufficient condition for existence of solutions, which also provides well-definiteness and convergence of the regularization method without additional monotonicity assumptions.

1. Introduction and preliminaries

Let K be a nonempty convex and closed set in a Euclidean space E and let $f : K \times K \to R$ be an equilibrium bifunction, i.e. $f(x, x) = 0$ for every $x \in K$. Then one can define the general equilibrium problem (EP for short) that is to find a point $x^* \in K$ such that

$$f(x^*, y) \geq 0 \quad \forall y \in K. \tag{1.1}$$

EPs are a suitable and common format for investigation of various applied problems arising in Economics, Mathematical Physics, Transportation, Communication Systems, Engineering and other fields; see Refs. [3], [7]. At the same time, EPs are closely related with other general problems in Nonlinear Analysis, such as fixed point, game equilibrium, variational inequality, and optimization problems.

For instance, if we set

$$f(x, y) = \langle G(x), y - x\rangle,$$

where $G : K \to E$ is a given mapping, then EP (1.1) reduces to the variational inequality problem (VI for short): Find a point $x^* \in K$ such that

$$\langle G(x^*), y - x^*\rangle \geq 0 \quad \forall y \in K. \tag{1.2}$$

In turn, if K is a convex cone, then (1.2) reduces to the complementarity problem (CP for short): Find a point $x^* \in E$ such that

$$x^* \in K, G(x^*) \in K^*, \langle x^*, G(x^*)\rangle = 0, \tag{1.3}$$

where

$$K^* = \{q \in E \mid \langle q, x\rangle \geq 0 \quad \forall x \in K\}$$

is the dual cone for K. Besides, EPs involve more general classes of problems. In fact, if we set

$$f(x, y) = \langle G(x), y - x\rangle + \varphi(y) - \varphi(x),$$

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where $G : K \to E$ is a given mapping and $\varphi : K \to R$ is a given function, EP(1.1) reduces to the so-called mixed variational inequality problem (MVI for short): Find a point $x^* \in K$ such that

$$\langle G(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in K. \tag{1.4}$$

In turn, problem (1.4) coincides with the problem of minimizing the function φ over K if we set $G \equiv 0$. Additionally, if we set

$$f(x, y) = \sup_{t \in T(x)} \langle t, y - x \rangle,$$

where $T : K \to \Pi(E)$ is a given multi-valued mapping with nonempty, convex and compact values, and $\Pi(E)$ denotes the family of all subsets of E, then EP(1.1) becomes equivalent to the multi-valued (or generalized) variational inequality problem (GVI for short): Find a point $x^* \in K$ such that

$$\exists t^* \in T(x^*), \langle t^*, y - x^* \rangle \geq 0 \quad \forall y \in K. \tag{1.5}$$

A great number of works are devoted to solution methods for all the above problems; see e.g. Refs. [8], [27], [14], [10] and references therein. The regularization approach is one of the most popular and fruitful among such methods; see Refs. [29], [4]. However, most convergence results for iterative solution methods, including those for regularization ones, require certain (strong) monotonicity assumptions. In particular, regularization based methods for general monotone EPs were substantiated in Refs. [22],[16], [19]. Indeed, the monotonicity provides two basic properties of the regularization method:

(i) *Existence and uniqueness of a solution of any perturbed problem;*

(ii) *Strong convergence of solutions of perturbed problems to the minimal norm solution of the initial problem.*

However, the monotonicity condition seems too restrictive for many applied problems. For this reason, substantiation of the regularization method without monotonicity will give new opportunities in solving very complicated nonlinear problems formulated in one of the formats (1.1)–(1.5). Of course, one cannot expect to maintain both properties (i) and (ii) in full, but derivation of certain partial well-posedness and convergence properties simultaneously is also not a trivial task.

For instance, if the set K is a cone or a cone segment, we can replace the usual norm monotonicity with order monotonicity, which is weaker essentially, and utilize this approach for CP (1.3) and VIs (1.2), (1.4), and (1.5). Under the condition that the cost mapping is P_0, property (i) was established for linear CPs (see Ref. [8]), for nonlinear CPs in Ref. [9] (see also Ref. [10], Section 12.2), for MVIs in Refs. [13], [23], and for GVIs in Ref. [20]. Moreover, being based on suitable extensions of the concept of a P_0-mapping, the same property was obtained for so-called partitionable VIs; see Refs. [1], [2]. However, it was shown in Ref. [9], Example 4.6, that the convergence property (ii) does not hold in general even for single-valued CPs, that is, one has to suggest an additional condition to guarantee the boundedness of the sequence of solutions of perturbed problems. In Ref. [9], the convergence was proved under the boundedness and nonemptiness of the solution set of the initial problem, which seems however rather restrictive. Moreover, regularization technique is applicable for systems of variational inequalities or equilibrium problems involving nonmonotone (or nonconvex) terms; see e.g. Ref. [17], Section 5.

We can extract two main directions in obtaining weakened sufficient conditions for convergence of the regularization method. The first approach consists in replacing the initial problem by a similar problem with reduced bounded feasible set; see Refs. [21], [20]. Unlike the optimization problems, VIs require then certain conditions which prevent the appearance of new superfluous solutions of the reduced problem. The second approach consists in utilizing so-called coercivity conditions, which are destined for deriving existence results on unbounded feasible sets, for providing convergence of the regularization method. Of course, such a condition should not yield the boundedness of the solution set.

In case VI (1.2) with box-constrained set, the following condition was proposed in Ref. [28]:

(A1) *There exists a point* $x^0 \in K$ *such that the set*

$$\left\{x \in K \mid \langle G(x), x - x^0 \rangle < 0\right\}$$

is bounded.

The more general condition was proposed in Refs. [15], [18] (see also Ref. [20]) for general VIs:

(A2) *There exists a nonempty compact set* D *such that for any point* $x \in K \setminus D$ *there is a point* $y \in K \bigcap D$ *such that*

$$\langle G(x), x - y \rangle \geq 0.$$

Clearly, **(A1)** implies **(A2)**, but **(A2)** yields the existence of solutions of VI (1.2) under continuity of G and its solution set need not be bounded; see Ref. [25]. It was shown in Ref. [18] that **(A2)** then also yields the existence of solution of all perturbed problems and this sequence has limit points so that all these points are solutions of the initial problem. Observe that **(A2)** is rather mild. It was shown in Ref. [5] that it becomes necessary for existence of solutions of pseudomonotone VIs.

However, we are interested in further weakening the sufficient conditions for the regularization method.

In this paper, we present a new coercivity condition, which extends and modifies the conditions suggested in Refs. [12], [6], and apply it to substantiate the regularization method for equilibrium problems. Since all the problems (1.2)–(1.5) are particular cases of EP (1.1), we thus describe a general approach to solve all these problems without monotonicity assumptions.

2. Coercivity conditions

We first recall that a function $\varphi : K \to R$ is said to be

(a) *quasiconvex* if for each pair of points $x, y \in K$ and for all $\alpha \in [0, 1]$ it holds that

$$\varphi(\alpha x + (1 - \alpha)y) \leq \max\left\{\varphi(x), \varphi(y)\right\};$$

(b) *explicitly quasiconvex* if it is quasiconvex and for each pair of points $x, y \in K, \varphi(x) \neq \varphi(y)$ and for all $\alpha \in (0,1)$ it holds that

$$\varphi(\alpha x + (1-\alpha)y) < \max\{\varphi(x), \varphi(y)\};$$

(c) *convex*, if for each pair of points $x, y \in K$ and for all $\alpha \in [0,1]$, it holds that

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y);$$

(d) *strongly convex* with constant $\kappa > 0$, if for each pair of points $x, y \in K$ and for all $\alpha \in [0,1]$, it holds that

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y) - 0.5\kappa\alpha(1-\alpha)\|x-y\|^2.$$

We shall consider problem (1.1) under the following basic assumptions.

(B) *K is a nonempty, convex and closed subset of a finite-dimensional space $E, f : K \times K \to R$ is an equilibrium bifunction, $f(\cdot, y)$ is upper semicontinuous for each $y \in K, f(x, \cdot)$ is convex for each $x \in K$.*

We now recall the well-known Ky Fan inequality from Ref. [11].

Proposition 2.1. If X is a nonempty, convex and compact subset of a real topological vector space, $\Phi : X \times X \to R$ is an equilibrium bifunction, $\Phi(\cdot, y)$ is upper semicontinuous for each $y \in X$, and $\Phi(x, \cdot)$ is quasiconvex for each $x \in X$, then there exists a point $x^* \in X$ such that

$$\Phi(x^*, y) \geq 0 \quad \forall y \in X.$$

By a simple adjustment of this property we give an existence result for problem (1.1) on bounded sets.

Proposition 2.2. If **(B)** holds and K is bounded, then EP (1.1) has a solution.

Now we turn to the unbounded case. First we extend **(A2)** to EPs.

(C1) *There exists a nonempty compact set D such that for any point $x \in K \setminus D$ there is a point $y \in K \bigcap D$ such that $f(x, y) \leq 0$.*

In Ref. [6], the following more general condition was suggested. Set

$$K_r = \{x \in K \mid \|x\| \leq r\}.$$

(C2) *There exists a number $r > 0$ such that for any point $x \in K \setminus K_r$ there is a point $y \in K$, $\|y\| < \|x\|$ such that $f(x, y) \leq 0$.*

Clearly, **(C1)** implies **(C2)**. However, we shall utilize another condition, which generalizes **(C2)**. Set

$$W_r = \{x \in K \mid \mu(x) \leq r\}$$

for a given function $\mu : E \to R$.

(H) *There exist a strongly convex and lower semicontinuous function $\mu : E \to R$ and a number $r > 0$ such that for any point $x \in K \setminus W_r$ there is a point $y \in K, \mu(y) < \mu(x)$, such that $f(x, y) \leq 0$.*

Clearly, **(H)** reduces to **(C2)** if we set $\mu(x) = 0.5\|x\|^2$. However, these conditions are not equivalent as the following simple example illustrates.

Example 2.3. *Let us consider the particular case of EP (1.1), namely, VI (1.2) when $E = R^2$,*

$$G(x) = (x_1^2 + x_2^2)\begin{pmatrix}1\\0\end{pmatrix}.$$

Clearly, G is continuous and quasimonotone, but not pseudomonotone; see Ref. [14]*, Definition 1.1.1. If we set $K = R^2$, then VI (1.2) has the unique solution $x^* = (0,0)^T$. However,* **(C2)** *does not hold along the ray*

$$\{x \mid x_1 \leq 0, x_2 = 0\}.$$

Next, let μ be a strongly convex quadratic function. Then **(H)** *may also not hold on R^2 but along some other ray. Therefore, there exists $K \subset R^2$ so that* **(H)** *holds, but this is not the case for* **(C2)** *and conversely. For instance, if we take the function $\mu(x) = x_1^2 + 2x_1x_2 + 2x_2^2$,* **(H)** *does not hold along the ray*

$$\{x \mid x_1 + 2x_2 = 0, x_1 \leq 0\}.$$

Therefore, we can utilize **(H)** *for instance for VI (1.2) with the feasible set*

$$K = \left\{x \in R^2 \mid x_1 + 4x_2 \leq 0\right\},$$

but **(C2)** *can not be utilized here.*

Set

$$U_r = \{x \in K \mid \mu(x) < r\}.$$

We now obtain a basic property of solutions of EPs on reduced sets.

Proposition 2.4. Suppose **(B)** holds and, for some $\rho > 0$, there exist

$$x^\rho \in W_\rho \quad \text{such that} \quad f(x^\rho, y) \geq 0 \quad \forall y \in W_\rho \tag{2.1}$$

and $z \in U_\rho$ such that $f(x^\rho, z) \leq 0$. Then x^ρ is a solution of EP (1.1).

Proof. Set

$$\varphi(x) = f(x^\rho, x),$$

then

$$0 = \varphi(x^\rho) \leq \varphi(y) \quad \forall y \in W_\rho,$$

moreover, $z \in W_\rho$, hence

$$0 = \varphi(x^\rho) \leq \varphi(z) \leq 0.$$

Therefore, x^ρ and z are minimizers for the function φ over W_ρ. Suppose that there exists a point $x' \in K \setminus W_\rho$ such that

$$\varphi(x') < \varphi(z),$$

then set $x(\alpha) = \alpha x' + (1-\alpha)z$. Clearly, $x(\alpha) \in K$ for each $\alpha \in (0,1)$. By strong convexity of μ, we have

$$\begin{array}{rcl} \mu[x(\alpha)] & \leq & \alpha\mu(x') + (1-\alpha)\mu(z) - 0.5\kappa\alpha(1-\alpha)\|x' - z\|^2 \\ & < & \mu(z) + \alpha[\mu(x') - \mu(z)] \leq \rho \end{array}$$

for $\alpha > 0$ small enough. Then $x(\alpha) \in W_\rho$ for $\alpha > 0$ small enough, but

$$\varphi[x(\alpha)] \leq \alpha\varphi(x') + (1-\alpha)\varphi(z) < \alpha\varphi(z) + (1-\alpha)\varphi(z) = \varphi(z), \tag{2.2}$$

which is a contradiction. Therefore

$$\varphi(x^\rho) = \varphi(z) \leq \varphi(y) \quad \forall y \in K,$$

or equivalently,

$$f(x^\rho, y) \geq f(x^\rho, x^\rho) = 0 \quad \forall y \in K,$$

i.e. x^ρ solves EP (1.1). □

We are now ready to establish the existence result for EP (1.1).

Theorem 2.5. If **(B)** and **(H)** are fulfilled, then EP (1.1) has a solution.

Proof. Since **(H)** holds, choose any $\rho > r$, then the set W_ρ is nonempty, convex and compact due to the strong convexity of μ. From Proposition 2.2 we obtain that there exists a solution x^ρ of problem (2.1). The result now follows from Proposition 2.4. □

Remark 2.6. The result of Theorem 2.5 remains true if E is a reflexive Banach space since this is the case for the Ky Fan inequality as Proposition 2.1 states. Of course, we should utilize then the weak topology. Moreover, we can somewhat weaken the assumption on $f(x,\cdot)$. If we replace in **(B)** the convexity of $f(x,\cdot)$ by the explicit quasiconvexity of $f(x,\cdot)$ for each $x \in K$, then the assertion of Proposition 2.2 remains true. Next, in the proof of Proposition 2.4 we now replace the chain in (2.2) by the following:

$$\varphi[x(\alpha)] < \max\left\{\varphi(x'), \varphi(z)\right\} = \varphi(z)$$

and so on. Therefore, the assertions of Proposition 2.4 and Theorem 2.5 remain true as well.

It has been mentioned that **(A2)** becomes necessary for existence of a solution of VI with pseudomonotone cost mapping G; see Ref. [5]. Moreover, **(C2)** is also necessary for existence of a solution of EP (1.1) if f is pseudomonotone; see Ref. [6]. We now present a similar result for condition **(H)** under some other assumption. Namely, extending the asymptotic pseudomonotonicity concept from Ref. [24], we say that a bifunction $f : K \times K \to R$ is *asymptotically μ-pseudomonotone* if there exists a number ρ such that for each pair of points $x, y \in K$ with $\max\{\rho, \mu(x)\} < \mu(y)$ it holds that

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0.$$

Theorem 2.7. Let **(B)** be fulfilled and let f be asymptotically μ-pseudomonotone, where $\mu : E \to R$ is a strongly convex and lower semicontinuous function. If EP (1.1) has a solution, then **(H)** holds true.

Proof. Let x^* solve EP (1.1). Choose $r > \max\{\rho, \mu(x^*)\}$ where ρ is associated with the asymptotic μ-pseudomonotonicity of f. Take an arbitrary point $x \in K \setminus W_r$, then $\mu(x) > \max\{\rho, \mu(x^*)\}$ and $f(x^*, x) \geq 0$, therefore $f(x, x^*) \leq 0$, and **(H)** holds, as desired. □

We conclude that condition **(H)** becomes necessary for existence of a solution of EP (1.1) under the essentially weaker assumption than the usual pseudomonotonicity.

3. Regularization method

We now intend to substantiate the regularization method under the condition **(H)**. Given a number $\varepsilon > 0$, we consider the problem of finding a point $z^\varepsilon \in K$ such that

$$f(z^\varepsilon, y) + \varepsilon\left[\mu(y) - \mu(z^\varepsilon)\right] \geq 0 \quad \forall y \in K; \tag{3.1}$$

which is nothing but the perturbed problem for EP (1.1). For brevity, set

$$f_\varepsilon(x,y) = f(x,y) + \varepsilon\left[\mu(y) - \mu(x)\right].$$

Theorem 3.1. If **(B)** and **(H)** are fulfilled, then EP (3.1) has a solution for each $\varepsilon > 0$.

Proof. We will show that **(H)** is true for f_ε as well. Suppose that there exists $x' \in K \setminus W_r$ such that

$$f_\varepsilon(x', y) > 0 \quad \forall y \in K, \mu(y) < \mu(x').$$

However, there is $y' \in K, \mu(y') < \mu(x')$ such that $f(x', y') \leq 0$ and we have

$$\begin{aligned} 0 < f_\varepsilon(x', y') &= f(x', y') + \varepsilon\left[\mu(y') - \mu(x')\right] \\ &\leq \varepsilon\left[\mu(y') - \mu(x')\right] < 0. \end{aligned}$$

which is a contradiction. Therefore, **(H)** holds true for f_ε. Since $f_\varepsilon(\cdot, y)$ is upper semicontinuous for each $y \in K$, and $f_\varepsilon(x, \cdot)$ is convex for each $x \in K$, the result follows now from Theorem 2.5 with $f = f_\varepsilon$. □

We are now in a position to establish the main result of this paper.

Theorem 3.2. Suppose conditions **(B)** and **(H)** are fulfilled. Then:
(i) EP (1.1) has a solution;
(ii) EP (3.1) has a solution for each $\varepsilon > 0$;
(iii) Each sequence $\{z^{\varepsilon_k}\}$ of solutions of EP (3.1) has limit points and if $\{\varepsilon_k\} \searrow 0$ all these limit points are solutions of EP (1.1).

Proof. Since (i) and (ii) follow from Theorems 2.5 and 3.1, respectively, we have to prove only (iii). Let z^ε be a solution of EP (3.1) for some $\varepsilon > 0$. If $z^\varepsilon \in K \setminus W_r$, then there exists $y^\varepsilon \in K$, $\mu(y^\varepsilon) < \mu(z^\varepsilon)$ such that $f(z^\varepsilon, y^\varepsilon) \leq 0$. We have

$$\begin{aligned} 0 &\leq f(z^\varepsilon, y^\varepsilon) + \varepsilon\left[\mu(y^\varepsilon) - \mu(z^\varepsilon)\right] \\ &\leq \varepsilon\left[\mu(y^\varepsilon) - \mu(z^\varepsilon)\right] < 0, \end{aligned}$$

which is a contradiction. Hence $z^\varepsilon \in W_r$ for each $\varepsilon > 0$ and the sequence $\{z^{\varepsilon_k}\}$ is bounded. Therefore, it has limit points. Let z' be a limit point for $\{z^{\varepsilon_k}\}$ with $\{\varepsilon_k\} \searrow 0$, i.e.

$$z' = \lim_{k_s \to +\infty} z^{\varepsilon_{k_s}}.$$

Then, by definition,

$$f(z^{\varepsilon_{k_s}}, y) + \varepsilon_{k_s}\left[\mu(y) - \mu(z^{\varepsilon_{k_s}})\right] \geq 0 \quad \forall y \in K,$$

but the sequence $\{\mu(z^{\varepsilon_{k_s}})\}$ must be bounded below. Therefore, for each $y \in K$, we obtain

$$f(z', y) \geq \limsup_{k_s \to +\infty} f(z^{\varepsilon_{k_s}}, y) \geq 0,$$

i.e. z' solves EP (1.1) and (iii) is true. □

In the case when μ is a continuously differentiable function, the perturbed EP (3.1) can be replaced by the following problem: Find $z^\varepsilon \in K$ such that

$$f(z^\varepsilon, y) + \varepsilon\langle \mu'(z^\varepsilon), y - z^\varepsilon\rangle \geq 0 \quad \forall y \in K; \tag{3.2}$$

moreover, both these problems have the same solution set. In fact, EP (3.2) implies EP (3.1) due to the convexity of μ. Conversely, let z^ε be a solution of EP (3.1), then it solves the problem of minimizing the function $\varphi + \psi$ over K, where

$$\varphi(y) = f(z^\varepsilon, y), \psi(y) = \varepsilon\mu(y).$$

But this problem is equivalent to the MVI: Find $z^\varepsilon \in K$ such that

$$\langle \psi'(z^\varepsilon), y - z^\varepsilon \rangle + \varphi(y) - \varphi(z^\varepsilon) \geq 0 \quad \forall y \in K;$$

see e.g. Ref. [26], Proposition 2.2.2; i.e. z^ε solves EP (3.2). At the same time, the cost bifunctions in (3.1) and (3.2) possess somewhat different properties. Indeed, $f_\varepsilon(x, \cdot)$ is strongly convex, but this is not the case for the bifunction

$$\Phi_\varepsilon(x, y) = f(x, y) + \varepsilon\langle \mu'(x), y - x \rangle$$

in general. In turn, suppose that f is monotone, i.e.

$$f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in K.$$

Then so is f_ε, however, Φ_ε is then strongly monotone, i.e.

$$\begin{aligned} \Phi_\varepsilon(x, y) + \Phi_\varepsilon(x, y) &= f(x, y) + f(y, x) + \varepsilon\langle \mu'(x) - \mu'(y), y - x \rangle \\ &\leq -\varepsilon\kappa\|x - y\|^2. \end{aligned}$$

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