Journal of Nonlinear and Convex Analysis
Volume 10, Number 1, 2009, 51–72

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

OPTIMAL COMPATIBILITY PROBLEM AND ITS APPLICATIONS

ELENA ROVENSKAYA

Abstract. In this paper we consider the problem of finding the minimum value of a scalar parameter at which depending on this parameter an equation has a solution in a given set which also depends on that parameter. A space of arguments is infinite. We call this problem as optimal compatibility problem. Under considered assumptions the optimization problem is not convex. We suggest an iteration method to solving the optimal compatibility problem based on the idea of the method of extremal shifting by N.N.Krasovskiy known in the game theory. We provide an application of the method to solving two classes of optimal control problems.

1. Introduction

In this paper we consider a problem of finding the minimum value of a scalar parameter p at which depending on this parameter an equation $F(p,x) = b(p)$ has a solution in a given set $X(p)$; the solution itself is to be found too. A space of agruments x is supposed to be infinite. We call this problem as *optimal compatibility problem*. Such problems appear in various applied problems (e.g., problems of optimization of networks of insurance companies [8], problems of optimization of portfolios of innovation projects [9]), and in the study of parametric families of operator equations [10]. Besides a standard optimization problem, i.e., the problem of finding the minimum value of a function subject to constraints of equality and inequality types, can be written as an optimal compatibility problem.

The work is devoted to constructing an iteration method for solving described above optimal compatibility problem formulated as an optimization problem.

Under considered in this paper assumptions the latter is not convex, that is why standard optimization methods, e.g., those of gradient type [12] may be not applicable for it. Existing approaches for solving non-convex problems, e.g., methods of fine and barrair functions [4], homotopic methods [20], are rather general but are hardly constructively realized. A type of convex problem usually leads to specific difficulties on a way of verification of constructive algorithms for its solving. In a view of that specified approaches devoted to particular classes of non-convex optimization problems are being developed and this work represent one of them.

We suggest an iteration method for solving the optimal compatibility problem based on the idea of the method of extremal shifting by N. Krasovskiy [6] known in the game theory. In earlier works [9] – [11], [17] the optimal compatibility problem was considered for the case of linear function $F(p,\cdot)$. In this work we expand the

2000 *Mathematics Subject Classification.* Primary 90C26.

Key words and phrases. Non-convex optimization, numerical algorithms, application to optimal control.

Copyright © 2009 Yokohama Publishers http://www.ybook.co.jp

approach on non-linear case but conquered to a number of assumptions. The most important of them is an assumption on the convexity of sets $F(p, X(p))$. By means of appropriate randomization of an argument (see [7]) it allows to formulate an expanded problem in a sense equivalent to the optimal compatibility problem. For that expanded problem we suggest an iteration algorithm which generalizes the method built in [10] for the optimal compatibility problem in the linear case.

It turns out that some optimal control problems can be written as optimal compatibility problems. The basic tool for solving optimal control problems is Pontryagin maximum principle [13]. In many cases it allows to get a solution of an optimal control problem or reveal its analytic structure. However a significant number of optimal control problems lie outside the sphere of its effective application. First of all, these are optimal control problems with phase (as well as mixed) constraints: the maximum principle for them has a complicated form and can hardly be effectively applied in concrete cases [1, 2]. Another rather universal approach to solving optimal control problems, including those with phase constraints, is provided by the method of dynamic programming [3] whose numerical realization however usually leads to a great amount of calculations.

In this work we provide an effective application of the iteration algorithm for the optimal compatibility problem for solving two classes of optimal control problems: a time minimum problem with phase constraint and a problem of optimization of mixed constraints both for controllable systems affine with respect to a control.

Results presented in this paper are detaily represented in author's works [14] – [16]. The regularization algorithm for the optimal compatibility problem is given in [18].

2. Problem formulation

Let X be a normalized space with a norm $|\cdot|_X$, in which we will consider a non-empty set X_0. Let Y be a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_Y$ and a norm $|\cdot|_Y$, standardly based on that scalar product; p_0, p^0 be such real numbers that $p_0 \geq p^0$ (we admit $p^0 = \infty$; in that case by $[p, p^0]$ we mean an interval $[p, \infty)$); $F(\cdot,\cdot)$ be a function acting from $[p_0, p^0] \times X_0$ to Y, $b(\cdot)$ be a function acting from $[p_0, p^0]$ to Y and $X(p)$ ($p \in [p_0, p^0]$) be a one-parametric family of sub-sets of a set X_0. We consider the following optimization problem

$$\begin{array}{rcl} p & \to & \min, \\ F(p,x) & = & b(p), \\ x & \in & X(p), \\ p & \in & [p_0, p^0], \end{array} \tag{2.1}$$

which implies finding the minimum value of a parameter $p \in [p_0, p^0]$, such that the system $F(p, x) = b(p)$ is compatible in $X(p)$.

In what follows we believe that there exists an admissible element for problem (2.1). We will denote the optimal value of problem (2.1) by p_*, and a set of all its solutions (p_*, x_*) – by $\{p_*\} \times X_*$.

In accordance to [7], we will call a function $G : [p_0, p^0] \times X_0 \mapsto Y$ by a *compactificator* if for any sequence (p_k, x_k) from $[p_0, p^0] \times X_0$ such that $|G(p_k, x_k)|_Y \to 0$ and $p_k \to \bar{p}$, a sequence (x_k) is compact in X.

We will consider problem (2.1) under the following assumptions:

(A1) multi-valued function $p \mapsto X(p)$ is continuous, i.e., if $x_k \in X(p_k)$ $(k = 0, 1, \ldots)$ $p_k \to \bar{p}$ and $x_k \to \bar{x}$, then $\bar{x} \in X(\bar{p})$;

(A2) functions $p \mapsto b(p)$ and $(p, x) \mapsto F(p, x)$ are continuous;

(A3) a function $(p, x) \mapsto F(p, x) - b(p)$ is a compactificator.

Lemma 2.1. *Let assumptions (A1) – (A3) be satisfied. Then for any sequence* (p_k, x_k) *from* $[p_0, p^0] \times X_0$ *such that* $p_k \to \bar{p}$, $|F(p_k, x_k) - b(p_k)|_Y \to 0$ *and* $x_k \in X(p_k)$ $(k = 0, 1, \ldots)$, *a sequence* (x_k) *is compact in* X *and* $\operatorname{dist}(x_k, X(\bar{p})) \to 0$.*

Obviously from Lemma 2.1 it follows

Corollary 2.2. *Let assumptions (A1) – (A3) be satisfied. Then*

1) *for any* $\bar{p} \in [p_0, p^0]$ *a set* $E = \{(p, x) \in [p_0, \bar{p}] \times X(p) : F(p, x) = b(p)\}$ *is a compact in* $R^1 \times X_0$;

2) *problem (2.1) has a solution;*

3) *a set* X_* *is non-empty compact in* X_0.

In further analysis the following assumption plays an important role:

(A4) for each $p \in [p_0, p_*]$ a set $F(p, X(p)) = \{F(p, x) : x \in X(p)\}$ is convex.

3. Expanded problem

By means of randomization of an argument [7] Assumption (A4) allows to formulate a problem with a linearized equality constraint in a sense equivalent to problem (2.1). Namely, let Σ be a set of all sub-sets of a space X. By Dirak measure concentrated in a point $x \in X_0$, as usual (see, e.g., [19]) we mean such a function $\delta_x : \Sigma \mapsto [0, 1]$ that $\delta_x(S) = 1$, if $x \in S$, and $\delta_x(S) = 1$, if $x \notin S$ $(S \in \Sigma)$. We will consider probabilistic measures μ on Σ, which are finite convex combinations of Dirak measures:

$$\mu = \sum_{i=1}^{m} \alpha_i \delta_{x_i}, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1, \quad x_1, \ldots, x_m \in X_0. \tag{3.1}$$

By M we denote a set of all measures μ on Σ of a form (3.1). For all $p \in [p_0, p^0]$ and all $\mu \in M$ of a form (3.1) we set

$$F(p, \mu) = \int F(p, x)\mu(dx) = \sum_{i=1}^{m} \alpha_i \int F(p, x)\delta_{x_i}(dx) = \sum_{i=1}^{m} \alpha_i F(p, x_i).$$

Note that $F(p, \delta_x) = F(p, x)$ $(p \in [p_0, p^0]$, $x \in X_0)$ and thus, identifying δ_x with x $(x \in X_0)$, a function $F(\cdot, \cdot)$ receive arguments lying in $[p_0, p^0] \times M$ instead of $[p_0, p^0] \times X_0$ as it was the initial case.

Lemma 3.1. *Let Assumption (A4) be satisfied. Then for all* $p \in [p_0, p^0]$ *a function* $\mu \mapsto F(p, \mu)$ *is linear on* M *in a sense that for any* $\mu_1, \mu_2 \in M$ *and any* $\lambda \in [0, 1]$

$$F(p, \lambda\mu_1 + (1 - \lambda)\mu_2) = \lambda F(p, \mu_1) + (1 - \lambda)F(p, \mu_2).$$

---

*We don't provide proofs for all theorems and lemmas in this paper. A reader can find them in author's works [14] – [18].

For any $p \in [p_0, p^0]$ we set

$$M(p) = \{\mu \in M : \ \mu(X(p)) = 1\}$$

and consider the following *expanded* optimization *problem*:

$$\begin{array}{rcl} p & \to & \min, \\ F(p,\mu) & = & b(p), \\ \mu & \in & M(p), \\ p & \in & [p_0, p^0]. \end{array}$$

Each admissible element (p, x) of problem (2.1) gives rise to an admissible element (p, δ_x) of problem (3).

That is why a set of all admissible elements of problem (3) is not empty. We denote the optimal value of problem (3) by $\hat{p_*}$.

Theorem 3.2. *Let assumptions (A1) – (A4) be satisfied. Then* $p_* = \hat{p_*}$ *and if* (p_*, x_*) *is a solution of problem (2.1), then* (p_*, δ_{x_*}) *is a solution of problem (3).*

Now we aim at constructing an iteration algorithm for solving problem (2.1). We introduce the following additional assumptions:

(A5) a multi-valued function $p \mapsto X(p)$ increases monotonically on $[p_0, p_*]$, i.e., $X(p_1) \subset X(p_2)$ for any $p_1 \in [p_0, p_*]$, $p_2 \in [p_1, p_*]$;

(A6) for any $p \in [p_0, p_*]$ a set $F(p, X(p))$ is closed in Y;

(A7) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $p_1 \in [p_0, p_*]$, $p_2 \in [p_1, p_*]$: $p_2 - p_1 < \delta$ and any $x \in X(p_1)$ it holds that $|F(p_1, x) - F(p_2, x)|_Y < \varepsilon$;

(A8) a set $\bigcup_{p \in [p_0, p_*]} F(p, X(p))$ is bounded in Y;

(A9) for any $l \in Y$ a function $p \mapsto c(l|F(p, X(p)))$ is continuous on $[p_0, p_*]$.

Here and in what follows $c(\cdot|W)$ is a support function of a non-empty set $W \subset Y$:

$$c(l|W) = \sup_{y \in W} \langle l, y \rangle \quad (l \in Y).$$

For any $p \in [p_0, p^0]$ and any measure $\mu \in M(p)$ we set

$$R(p, \mu) = \{x \in X(p) : F(p, x) = F(p, \mu)\}.$$

Let us note that assumption (A4) implies $R(p, \mu) \neq \emptyset$ for all $p \in [p_0, p_*]$ and $\mu \in M(p)$.

4. Algorithm

An iteration algorithm solving problem (2.1) which we suggest below is recalculating a triple (p_k, μ_k, x_k), where (p_k, x_k) is a current approximation of a solution of problem (2.1) and $\mu_k \in M$ is a subsidiary element related to expanded problem (3).

Let us consider the following iteration algorithm. At zero step we chose

$$\mu_0 \in M(p_0), \quad x_0 \in R(p_0, \mu_0) \tag{4.1}$$

and a triple (p_0, μ_0, x_0) is accepted as a initial element of the algorithm sequence. At a step $k+1$ by means of an element $(p_k, \mu_k, x_k) \in [p_0, p^0] \times M \times X_0$ we set an element

$(p_{k+1}, \mu_{k+1}, x_{k+1}) \in [p_0, p^0] \times M \times X_0$. Namely, we find a solution (p_{k+1}, ν_{k+1}) of the following optimization problem:

$$\begin{array}{rcl} p & \to & \min, \\ p & \in & [p_k, p^0], \\ \langle F(p_k, \mu_k) & - & b(p_k), F(p,\nu) - b(p)\rangle_Y \leq 0, \\ \nu & \in & M(p); \end{array} \tag{4.2}$$

we define

$$\tau_{k+1} = \arg\min_{\tau\in[0,1]} |F(p_{k+1}, (1-\tau)\mu_k + \tau\nu_{k+1}) - b(p_{k+1})|_Y^2 \tag{4.3}$$

and set

$$\mu_{k+1} = (1-\tau_{k+1})\mu_k + \tau_{k+1}\nu_{k+1}, \tag{4.4}$$

$$\begin{array}{lcl} x_{k+1} & \in & R(p_{k+1}, \mu_{k+1}) \quad \text{if} \quad R(p_{k+1}, \mu_{k+1}) \neq \emptyset, \\ x_{k+1} & \in & X(p_{k+1}) \quad \text{if} \quad R(p_{k+1}, \mu_{k+1}) = \emptyset. \end{array} \tag{4.5}$$

Note that one can find τ_{k+1} (4.3) explicitly. Namely,

$$\tau_{k+1} = \begin{cases} 0, & \tau^*_{k+1} < 0, \\ \tau^*_{k+1}, & \tau^*_{k+1} \in [0,1], \\ 1, & \tau^*_{k+1} > 1, \end{cases} \qquad \tau^*_{k+1} = \frac{\langle q_{k+1}, b(p_{k+1}) - F(p_{k+1}, \mu_k)\rangle_Y}{|q_{k+1}|_Y^2} \tag{4.6}$$

$$(q_{k+1} = F(p_{k+1}, \nu_{k+1}) - F(p_{k+1}, \mu_k) \neq 0),$$

$$\tau_{k+1} \in [0,1] \quad (q_{k+1} = 0). \tag{4.7}$$

Lemma 4.1. *Let assumptions (A1) – (A6) and (A9) hold. Then a sequence* (p_k, μ_k, x_k) *is defined by (4.1) – (4.7) correctly, i.e., for all* $k = 0, 1, \dots$ *problem (4.2) has a solution* (p_{k+1}, ν_{k+1}) *and the element* τ_{k+1} *(4.3) exists. Moreover for all* $k = 0, 1, \dots$

$$p_0 \leq \ldots \leq p_k \leq \ldots \leq p_*, \tag{4.8}$$

$$\mu_k \in M(p_k), \quad x_k \in R(p_k, \mu_k). \tag{4.9}$$

By $\operatorname{dist}(z, X_*)$ we will denote a distance in X between an element z and a set $X_* : \operatorname{dist}(z, X_*) = \inf_{x\in X_*} |z - x|_X$.

The main result of this work is comprised in

Theorem 4.2. *Let assumptions (A1) – (A9) be satisfied and a sequence* (p_k, μ_k, x_k) *be defined by algorithm (4.1) – (4.7). Then* $p_k \to p_*$ *and* $\operatorname{dist}(x_k, X_*) \to 0$.

For proving Theorem 4.2 we need two Lemmas.

Lemma 4.3. *Let* $\alpha > 0$, $\beta_k \geq 0$, $\gamma_k \geq 0$, $\gamma_{k+1} \leq (1 - \alpha\gamma_k)\gamma_k + \beta_k$ $(k = 0, 1, \dots)$ *and* $\beta_k \to 0$. *Then* $\gamma_k \to 0$.[†]

The following Lemma plays a key role:

[†]One can find the proof of this Lemma in [7].

Lemma 4.4. *Let conditions of Theorem 4.2 be satisfied. Then*

$$p_k \to \bar{p} \in [p_0, p_*], \tag{4.10}$$

$$\lim_{k\to\infty} |F(p_k,\mu_k) - b(p_k)|_Y = 0. \tag{4.11}$$

Proof. From (4.8) it follows that (4.10). Let us prove (4.11). We set

$$H(p,\mu) = F(p,\mu) - b(p),\ \ \mu_k(\tau) = \mu_k + \tau(\nu_{k+1} - \mu_k). \tag{4.12}$$

For all natural k and any $\tau \in [0,1]$ we have

$$\begin{aligned} |H(p_{k+1},\mu_k(\tau))|_Y^2 &= |H(p_{k+1},\mu_k + \tau(\nu_{k+1}-\mu_k))|_Y^2 \\ &= |(1-\tau)H(p_{k+1},\mu_k + \tau H(p_{k+1},\nu_{k+1}))|_Y^2 \\ &\leq (1-2\tau)|H(p_{k+1},\mu_k)|_Y^2 \\ &+ 2\tau(1-\tau)\langle H(p_{k+1},\mu_k), H(p_{k+1},\nu_{k+1})\rangle_Y \\ &+ \tau^2\{|H(p_{k+1},\mu_k)|_Y^2 + |H(p_{k+1},\nu_{k+1})|_Y^2\}. \end{aligned} \tag{4.13}$$

Since $\mu_k \in M(p_k)$ and $\nu_{k+1} \in M(p_{k+1})$, from Assumption (A4) we get $F(p_k,\mu_k) \in F(p_k, X(p_k))$ and $F(p_{k+1},\nu_{k+1}) \in F(p_{k+1}, X(p_{k+1}))$. According to Lemma 4.1 $p_k, p_{k+1} \in [p_0, p_*]$, hence

$$F(p_k,\mu_k), F(p_{k+1},\nu_{k+1}) \in E = \bigcup_{p\in[p_0,p_*]} F(p, X(p)).$$

Thanks to Assumption (A8) a set E is bounded in Y. From this fact and Assumption (A2) we conclude that

$$|H(p_k,\mu_k)|_Y \leq L, \quad |H(p_{k+1},\nu_{k+1})|_Y \leq L, \tag{4.14}$$

where $L \geq 0$ is such that $L \geq |y|_Y + |b(p)|_Y$ for all $y \in E$ and $p \in [p_0, p_*]$. Using estimates in the right-hand side of (4.14), substituting (p_{k+1},μ_k) by (p_k,μ_k), and adding the corresponding difference, we continue (4.13) as follows:

$$|H(p_{k+1},\mu_k(\tau))|_Y^2 \leq (1-2\tau)|H(p_k,\mu_k)|_Y^2 + \alpha_k(\tau) + \beta_k(\tau) + 2L^2\tau^2;$$

where

$$\begin{aligned} \alpha_k(\tau) &= 2\tau(1-\tau)\langle H(p_k,\mu_k), H(p_{k+1},\nu_{k+1})\rangle_Y, \\ \beta_k(\tau) &= (1-2\tau)\left(|H(p_{k+1},\mu_k)|_Y^2 - |H(p_k,\mu_k)|_Y^2\right) \\ &+ 2\tau(1-\tau)\langle H(p_{k+1},\mu_k) - H(p_k,\mu_k), H(p_{k+1},\nu_{k+1})\rangle_Y \leq \beta_k, \end{aligned}$$

$$\begin{aligned} \beta_k &= \left(|H(p_{k+1},\mu_k)|_Y^2 - |H(p_k,\mu_k)|_Y^2\right) \\ &+ 2\langle H(p_{k+1},\mu_k) - H(p_{k+1},\mu_k), H(p_{k+1},\nu_{k+1})\rangle_Y. \end{aligned} \tag{4.15}$$

Since (p_{k+1},ν_{k+1}) is an admissible element in problem (4.2) $\alpha_k(\tau) \leq 0$. Therefore

$$|H(p_{k+1},\mu_k(\tau))|_Y^2 \leq (1-2\tau)|H(p_k,\mu_k)|_Y^2 + \beta_k + 2L^2\tau^2.$$

According to (4.3) $\tau_{k+1} = \arg\min_{\tau\in[0,1]} |H(p_{k+1},\mu_k(\tau)|_Y^2$, hence

$$|H(p_{k+1},\mu_{k+1})|_Y^2 = |H(p_{k+1},\mu_k(\tau_{k+1}))|_Y^2$$

$$\leq \min_{\tau\in[0,1]}\{(1-2\tau)|H(p_k,\mu_k)|_Y^2+\beta_k+2L^2\tau^2\}.$$

The minimum value of a quadratic function of an argument τ, appearing in a right-hand side, is approached at a point $\tau_{k+1}=|H(p_k,\nu_k)|_Y^2/2L^2$. And

$$\min_{\tau\in[0,1]}\left\{(1-2\tau)|H(p_k,\mu_k)|_Y^2+2L^2\tau^2\right\}=\left(1-\frac{|H(p_k,\mu_k)|_Y^2}{2L^2}\right)|H(p_k,\mu_k)|_Y^2+\beta_k.$$

Finally,

$$|H(p_{k+1},\mu_{k+1})|_Y^2\leq\left(1-\frac{|H(p_k,\mu_k)|_Y^2}{2L^2}\right)|H(\mu_k)|_Y^2+\beta_k. \tag{4.16}$$

Let us show that $\beta_k\to 0$. To do so it is sufficient to check that

$$|F(p_{k+1},\mu_k)-F(p_k,\mu_k)|_Y\to 0, \tag{4.17}$$

$$|b(p_{k+1})-b(p_k)|_Y\to 0 \tag{4.18}$$

(which follows from formula for β_k (4.15), notations (4.12) and estimates (4.14)). The convergence (4.18) follows from continuity of the function $b(\cdot)$ and convergence $p_k\to\bar p$ (see (4.10)). Let us prove (4.17). In accordance to Lemma 4.1 $\mu_k\in M(p_k)$, hence $\mu_k=\sum_{i=1}^{m_k}\alpha_i^{(k)}\delta_{x_i^{(k)}}$, where $\alpha_1^{(k)},\ldots,\alpha_{m_k}^{(k)}\in[0,1]$, $\sum_{i=1}^{m_k}\alpha_i^{(k)}=1$, $x_1^{(k)},\ldots,x_{m_k}^{(k)}\in X(p_k)$. Then

$$\begin{aligned}|F(p_{k+1},\mu_k)-F(p_k,\mu_k)|_Y &= \left|\sum_{i=1}^{m_k}\alpha_i^{(k)}F(p_{k+1},x_i^{(k)})-\sum_{i=1}^{m_k}\alpha_i^{(k)}F(p_k,x_i^{(k)})\right|_Y\\ &\leq \sum_{i=1}^{m_k}\alpha_i^{(k)}\left|F(p_{k+1},x_i^{(k)})-F(p_k,x_i^{(k)})\right|_Y.\end{aligned}$$

Because of convergence $p_k\to\bar p$ and Assumption (A7)

$$\max_{i=1,\ldots,m_k}|F(p_{k+1},x_i^{(k)})-F(p_k,x_i^{(k)})|_Y\to 0.$$

From the last fact we get convergence (4.17). Thus we proved that $\beta_k\to 0$.

Now let us set $|H(p_k,\mu_k)|_Y^2=\gamma_k$ and rewrite (4.16) as follows

$$\gamma_{k+1}\leq\left(1-\frac{\gamma_k}{2L^2}\right)\gamma_k+\beta_k.$$

According to Lemma 4.3 $\gamma_k\to 0$ as $k\to\infty$ which finalize the proof. □

Proof of Theorem 4.2. According to Lemma 4.4 we have (4.10) and (4.11); and besides owing to (4.8) $p_k\leq\bar p\leq p_*$ $(k=0,1,\ldots)$. By Lemma 4.1 $\mu_k\in M(p_k)$ and $x_k\in R(p_k,\mu_k)$, i.e., (see (3)) $x_k\in X(p_k)$ and $F(p_k,\mu_k)=F(p_k,x_k)$. From (4.10) it follows that

$$|F(p_k,x_k)-b(p_k)|_Y^2\to 0. \tag{4.19}$$

Thus a sequence (p_k,x_k) satisfies to conditions of Lemma 2.1. By this Lemma a sequence (x_k) is compact in X and

$$\operatorname{dist}(x_k,X(\bar p))\to 0. \tag{4.20}$$

Let $\bar{x}$ is an arbitrary limit of the sequence (x_k), i.e., a limit in X of some subsequence (x_{k_j}) From (4.20) and closeness of $X(\bar{p})$ (a consequence of Assumption (A1)) it follows that $\bar{x} \in X(\bar{p})$. From (4.19), Assumption (A1) and continuity of functions $F(\cdot,\cdot)$ and $b(\cdot)$ (Assumption (A2)) it follows that $F(\bar{p},\bar{x}) = b(\bar{p})$. Thus $\bar{x}$ is an admissible element of problem (2.1). Hence $\bar{p} \geq p_*$. As it was stated above $\bar{p} \leq p_*$, which leads to $\bar{p} = p_*$. Therefore $(p_*, \bar{x})$ is a solution of problem (2.1). Since $\bar{x}$ is an arbitrary limit of the sequence (x_k), then $\mathrm{dist}(x_k, X_*) \to 0$. which finalize proving the Theorem. □

5. Algorithm concretization

In this section we are aimed at rewriting the algorithm (4.1) – (4.7) in constructive terms. Namely, we rewrite problem (4.2) in a simplified form, i.e., as a scalar optimization problem with subsequent calculation of an extreme element in the space X. Consider the following iteration algorithm, which recalculates elements $(p_k, \mu_k, x_k) \in [p_0, p^0] \times M \times X_0$. At zero step we chose

$$x_0 \in X(p_0), \quad \mu_0 = \delta_{x_0}. \tag{5.1}$$

A triple (p_0, μ_0, x_0) is chosen as an initial element of algorithm's sequence. At step $k+1$ by means of an element $(p_k, \mu_k, x_k) \in [p_0, p^0] \times M \times X_0$ an element $(p_{k+1}, \mu_{k+1}, x_{k+1}) \in [p_0, p^0] \times M \times X_0$ is calculated. Namely, we set

$$l_k = b(p_k) - F(p_k, \mu_k), \tag{5.2}$$

$$\varphi_k(p) = c(l_k | F(p, X(p))) - \langle l_k, b(p) \rangle_Y \quad (p \in [p_k, p^0]), \tag{5.3}$$

and find a number

$$p_{k+1} = \min\{p \in [p_k, p^0] : \ \varphi_k(p) \geq 0\} \tag{5.4}$$

and an element

$$v_{k+1} \in X(p_{k+1}) \tag{5.5}$$

such that

$$\langle l_k, F(p_{k+1}, v_{k+1}) \rangle_Y = c(l_k | F(p_{k+1}, X(p_{k+1}))). \tag{5.6}$$

We set

$$\nu_{k+1} = \delta_{v_{k+1}}, \tag{5.7}$$

The value τ_{k+1} is defined by (4.6) – (4.7) and calculated by

$$\mu_{k+1} = (1 - \tau_{k+1})\mu_k + \tau_{k+1}\nu_{k+1}, \tag{5.8}$$

$$\begin{array}{c} x_{k+1} \in R(p_{k+1}, \mu_{k+1}) \quad \text{if} \quad R(p_{k+1}, \mu_{k+1}) \neq \emptyset, \\ x_{k+1} \in X(p_{k+1}) \quad \text{if} \quad R(p_{k+1}, \mu_{k+1}) = \emptyset. \end{array} \tag{5.9}$$

Let us note that

$$\mu_k = \sum_{i=0}^{k} \alpha_i^{(k)} \delta_{v_i},$$

where

$$v_0 = x_0, \quad \alpha_0^{(0)} = 1, \tag{5.10}$$

$$\alpha_i^{(k+1)} = (1-\tau_{k+1})\alpha_i^{(k)} \quad (i = 0, \dots, k), \quad \alpha_{k+1}^{(k+1)} = \tau_{k+1}. \tag{5.11}$$

That is why one can rewrite the algorithm (5.1) – (5.11) in the following equivalent form. At zero step we set

$$\alpha_0^{(0)} = 1, \quad x_0 \in X(p_0), \quad v_0 = x_0, \tag{5.12}$$

and we chose a set $(p_0, \alpha_0^{(0)}, v_0, x_0)$ as an initial element of algorithm's sequence. At step $k+1$ by means of a set $(p_k, \alpha_0^{(k)}, \dots, \alpha_k^{(k)}, v_0, \dots, v_k, x_k)$, where

$$p_k \in [p_0, p^0], \quad \alpha_0^{(k)}, \dots, \alpha_k^{(k)} \in [0,1], \quad v_0, \dots, v_k, \quad x_k \in X_0,$$

we calculate a set $(p_{k+1}, \alpha_0^{(k+1)}, \dots, \alpha_{k+1}^{(k+1)}, v_0, \dots, v_{k+1}, x_{k+1})$, where

$$p_{k+1} \in [p_0, p^0], \quad \alpha_0^{(k+1)}, \dots, \alpha_{k+1}^{(k+1)} \in [0,1], \quad v_0, \dots, v_{k+1}, x_{k+1} \in X_0.$$

Namely, we set

$$l_k = b(p_k) - \sum_{i=1}^{k} \alpha_i^{(k)} F(p_k, v_i), \tag{5.13}$$

$$\varphi_k(p) = c(l_k | F(p, X(p))) - \langle l_k, b(p)\rangle_Y \quad (p \in [p_k, p^0]) \tag{5.14}$$

and find a number

$$p_{k+1} = \min\{p \in [p_k, p^0] : \varphi_k(p) \geq 0\} \tag{5.15}$$

and an element

$$v_{k+1} \in X(p_{k+1}) \tag{5.16}$$

such that

$$\langle l_k, F(p_{k+1}, v_{k+1})\rangle_Y = c(l_k | F(p_{k+1}, X(p_{k+1}))). \tag{5.17}$$

The value τ_{k+1} is defined by

$$\tau_{k+1} = \begin{cases} 0, & \tau_{k+1}^* < 0, \\ \tau_{k+1}^*, & \tau_{k+1}^* \in [0,1], \\ 1, & \tau_{k+1}^* > 1, \end{cases} \qquad \tau_{k+1}^* = \frac{\langle q_{k+1}, b(p_{k+1}) - F(p_{k+1}, \mu_k)\rangle_Y}{|q_{k+1}|_Y^2} \tag{5.18}$$

$$\left(q_{k+1} = F(p_{k+1}, v_{k+1}) - \sum_{i=0}^{k} \alpha_i^{(k)} F(p_{k+1}, v_i) \neq 0\right),$$

$$\tau_{k+1} \in [0,1] \quad (q_{k+1} = 0). \tag{5.19}$$

We set

$$\alpha_i^{(k+1)} = (1-\tau_{k+1})\alpha_i^{(k)} \quad (i = 0, \dots, k), \quad \alpha_{k+1}^{(k+1)} = \tau_{k+1}. \tag{5.20}$$

An element x_{k+1} is found from

$$x_{k+1} \in F^{-1}(p_{k+1}, w_{k+1}) \quad \text{if} \quad F^{-1}(p_{k+1}, w_{k+1}) \neq \emptyset, \\ x_{k+1} \in X(p_{k+1}) \quad \text{if} \quad F^{-1}(p_{k+1}, w_{k+1}) = \emptyset, \tag{5.21}$$

where

$$w_{k+1} = \sum_{i=0}^{k+1} \alpha_i^{(k)} F(p_{k+1}, v_i), \tag{5.22}$$

$$F^{-1}(p, w) = \{x \in X(p) : F(p, x) = w\}. \tag{5.23}$$

Remark 5.1. The main operation at step $k+1$ of this algorithm comprises solving one-dimensional minimization problem (5.15) and subsequent finding an extreme element (5.16) – (5.17). The operation (5.22), (5.23) on finding an approximation x_{k+1} of a solution's component (lying in X_*) may by difficult. One can ignore it if only an approximation of optimal value p_* of problem (2.1) is to be found.

The following Lemma states the equivalence of the algorithm (5.12) – (5.23) to the basic algorithm (4.1) – (4.7).

Lemma 5.2. *Let assumptions (A1) – (A9) be satisfied. Then a sequence $(p_k, \alpha_0^{(k)}, \ldots, \alpha_k^{(k)}, v_0, \ldots, v_k, x_k)$ is defined by (5.12) – (5.23) correctly and a sequence (p_k, μ_k, x_k), defined by (5.12) – (5.23), is satisfied to (4.1) – (4.7) of the basic algorithm and for all $k = 0, 1, \ldots$ it holds (see (5.22)) $F(p_{k+1}, x_{k+1}) = w_{k+1}$, where w_{k+1} is defined by (5.23).*

Lemma 5.2 and Theorem 4.2 on convergence of the basic algorithm imply the convergence of the algorithm (5.12) – (5.23).

Theorem 5.3. *Let assumptions (A1) – (A9) be satisfied and a sequence $(p_k, \alpha_0^{(k)}, \ldots, \alpha_k^{(k)}, v_0, \ldots, v_k, x_k)$ is defined by the algorithm (5.12) – (5.23). Then $p_k \to p_*$ and* $\operatorname{dist}(x_k, X_*) \to 0$.

6. Application to optimal control problems

Let us consider a n-dimensional controllable system

$$\dot{z}(t) = f(z(t), t) + g(z(t), t)u(t), \tag{6.1}$$

functioning on the time interval $[0, T]$ ($T > 0$). Here $(z, t) \mapsto f(z, t)$ is a vector-function defined on $R^n \times [0, T]$ whose values lie in R^n, $(z, t) \mapsto g(z, t)$ is a matrix-function of dimension $m \times n$ defined on $R^n \times [0, T]$; $Q(\cdot)$ is a multi-valued function acting from the interval $[0, T]$ into a class of all non-empty sub-sets of $R^n \times R^m$. By a control we mean, as usual, any measurable bounded function $u(\cdot) : [0, T] \mapsto R^m$. We suppose the initial condition to hold:

$$z(0) = z^0, \tag{6.2}$$

where $z^0 \in R^n$.

In this section we consider two classes of optimal control problems settled for system (6.1), (6.2): time minimum problem and problem of optimization of mixed constraint. We show that these problems can be reformulated as optimal compatibility problems and provide a scheme of applying the algorithm (5.12) – (5.23) to find their solutions.

6.1. **Time minimum problem.** Let us consider a problem of finding a minimum time which it takes to system (6.1) to pass from the given initial state (6.2) to the given final state $z^1 \in R^n$ whereas a mixed constraint $(z(t), u(t)) \in Q(t)$ $(t \in [0,T])$ on a state variable $z(t)$ and a control variable $u(t)$ holds:

$$\begin{array}{rcl} p & \longrightarrow & \min, \\ \dot{z}(t) & = & f(z(t),t) + g(z(t),t)u(t), \\ z(0) & = & z^0, \quad z(p) \;=\; z^1, \\ (z(t),u(t)) & \in & Q(t), \\ (t & \in & [0,T]). \end{array} \tag{6.3}$$

Let us believe that for problem (6.3) there exists an admissible controllable process.

We denote the optimal value of problem (6.3) – minimum time – via p_*, and a set of its solutions – via $\{p_*\} \times \Pi_*$, where $\Pi_* \subset \Pi$.

In further analysis we will consider problem (6.3) under the following assumptions

(B1) $(z,t) \mapsto f(z,t)$ and $(z,t) \mapsto g(z,t)$ are continuous functions on $R^n \times [0,T]$;

(B2) $Q(t)$ is a convex, closed and bounded subset of $R^n \times R^m$ for all $t \in [0,T]$;

(B3) a multi-valued function $t \mapsto Q(t)$ is measurable (see [19]);

(B4) a set $\bigcup_{t\in[0,T]} Q(t)$ is bounded.

Theorem 6.1. *Let Assumptions (B1) – (B4) be satisfied. Then there exists a solution of problem (6.3).*

The fact that at the time moment $p \in [0,T]$ a trajectory $z(\cdot)$ of system (6.1) under a control $u(\cdot)$ comes to the given final state z^1, can be written in a form of integral equations:

$$\begin{array}{rcll} z(t) - \displaystyle\int_0^t (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau & = & z^0 & (t \in [0,T]), \\ z(t) + \displaystyle\int_t^p (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau & = & z^1 & (t \in [0,p]), \\ z(t) - \displaystyle\int_p^t (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau & = & z^1 & (t \in [p,T]). \end{array}$$

On the other hand, if a measurable function $z(\cdot) : [0,T] \mapsto R^n$ and a control $u(\cdot)$ satisfy to these equations for a.a. $t \in [0,T]$, then $z(\cdot)$ a.a. coincides with a trajectory under a control $u(\cdot)$, coming to the state z^1 at the time moment p. This observations allows to reformulate problem (6.3) as a optimal compatibility problem (2.1) in a space of measurable expansions of controllable processes.

Namely, let us introduce a Hilbert space

$$Y = L^2([0,T], R^n \times R^n) \tag{6.4}$$

and a normalized space

$$X = L^2([0,T],R^n) \times L^2_w([0,T],R^m), \tag{6.5}$$

where $L^2_w([0,T],R^m)$ is a space $L^2([0,T],R^m)$, accompanied with a weak norm (see [19]). We extract a bounded set X_0 in $L^2([0,T],R^n) \times L^2([0,T],R^m)$ such that all admissible controllable processes $x = (z(\cdot),u(\cdot))$ lie in X_0 (the existence of such a set is guaranteed by Assumption (B4)).

For all $p \in [0,T]$ we set

$$F_1(p,x)(t) = z(t) - \int_0^t (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \quad (t \in [0,T]), \tag{6.6}$$

$$F_2(p,x)(t) = \begin{cases} z(t) + \int_t^p (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau & (t \in [0,p]) \\ z(t) - \int_p^t (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau & (t \in [p,T]) \end{cases} \tag{6.7}$$

$$(x = (z(\cdot),u(\cdot)))$$

and introduce a function $(p,x) \mapsto F(p,x) : [0,T] \times X_0 \mapsto Y$, setting

$$F(p,x) = F(p,x)(\cdot) = (F_1(p,x)(\cdot), F_2(p,x)(\cdot)). \tag{6.8}$$

We put $b = (b_1(\cdot), b_2(\cdot)) \in Y$ as

$$b_1(t) = z^0, \quad b_2(t) = z^1 \quad (t \in [0,T]). \tag{6.9}$$

And, finally, we introduce a set

$$G = \{x = (z(\cdot),u(\cdot)) \in X_0 : (z(t),u(t)) \in Q(t)\ (t \in [0,T])\}. \tag{6.10}$$

Let us notice that $x = (z(\cdot),u(\cdot)) \in X_0$ almost always coincides with an admissible controllable process if and only if $F(p,x) = b$ and $x \in G$. Therefore, the following holds:

Theorem 6.2. *Problem (6.3) and problem (2.1), where $X(p) = G$, $b(p) = b$, are equivalent in the following sense:*

(i) *optimal values of problems (6.3) and (2.1) coincide;*

(ii) *a couple $(p_*, x) \in [0,T] \times X_0$, where $x = (z(\cdot),u(\cdot))$, solves problem (2.1) if and only if there exists an optimal controllable process $(z_*(\cdot),u_*(\cdot))$ such that $(z(t),u(t)) = (z_*(t),u_*(t))$ for a.a. $t \in [0,T]$.*

In further analysis we will consider problem (6.3) under the following subsidiary assumption

(B5) for each $p \in [0,p_*]$ a set $F(p,G) = \{F(p,x) : x \in G\}$ is convex.

Remark 6.3. Let us notice that there exists a class of problems for which Assumptions (B1) – (B5) are satisfied – time minimum problems for bilinear controllable systems with static state constraint:

$$\begin{aligned} p &\to \min, \\ \dot z^{(i)}(t) &= \sum_{j=1}^m [a_{ij}(t)z^{(j)}(t) + b_{ij}(t)u^{(j)}(t)z^{(j)}(t)] \quad (i = 1,\dots,n), \\ z(0) &= z^0, \quad z(p) = z^1, \\ u(t) &= (u^{(1)}(t),...,u^{(m)}(t)) \in U, \end{aligned} \tag{6.11}$$

$$z(t) \quad = \quad (z^{(1)}(t), ..., z^{(n)}(t)) \in Z,$$

where $z(t) = (z^{(1)}(t), \ldots, z^{(n)}(t)) \in R^n$ is a state vector, $u(t) = (u^{(1)}(t), \ldots, u^{(m)}(t)) \in U \subset R^m$ is a control vector, $z^0, z^1 \in R^n$ are the given initial and final states of the system, geometric constraint on a control is a m-dimensional parallelogram:

$$U = \prod_{j=1}^{m} [u_-^{(j)}, u_+^{(j)}],$$

where $u_-^{(j)} \leq u_+^{(j)}$ $(j = 1, \ldots, m)$. We believe that $t \mapsto a_{ij}(t)$ and $t \mapsto b_{ij}(t)$ $(i = 1, \ldots, n,\ \ j = 1, \ldots, m)$ are continuous scalar functions on $[0, T]$ and the set Z defining a state constraint is convex, closed and bounded in R^n and, besides, $Z \subset R^n_+$, where

$$R^n_+ = \{z = (z^1, \ldots, z^n) \in R^n : z^i > 0,\ i = 1, \ldots, n\}.$$

It is clear that problem (6.11) has a form of considered in this section problem (6.3). Corresponding to the approach described above we introduce functional spaces X and Y by (6.5), (6.4). We introduce a function $F(\cdot, \cdot)$ like (6.6) – (6.8), we put a vector $b \in Y$ like (6.9), and according to (6.10) we set

$$G = \{x = (z(\cdot), u(\cdot)) \in X_0 : z(t) \in Z,\ u(t) \in U\ (t \in [0, T])\}.$$

One can check that for the problem (6.3) of a form (6.11) Assumptions (B1) – (B5) holds.

Let us come back to time minimum problem (6.3) and optimization problem (2.1) equivalent to it (see Theorem 6.1) where functional spaces X and Y are defined by (6.5), (6.4), $X(p) = G$ is determined by (6.10), $b(p) = b$ is set by (6.9), a function $F(\cdot, \cdot)$ has a form (6.6) – (6.8).

We aim at applying the algorithm (5.12) – (5.23) of solving a problem (2.1) for solving time minimum problem (6.3). For that it is sufficient to prove that Assumptions (B1) – (B5) formulated for problem (6.3) lead to Assumptions (A1) – (A9) satisfied for equivalent problem (2.1).

Lemma 6.4. *Let assumptions (B1) – (B5) (6.3) be satisfied. Then for equivalent problem (2.1) assumptions (A1) – (A9) hold.*

Proof. 1. In order to prove (A1) one should prove the closedness of a set G. Let $x_k = x_k(\cdot) = (z_k(\cdot), u(\cdot)) \in G$, $x_k \to \bar{x} = \bar{x}(\cdot) = (\bar{z}(\cdot), \bar{u}(\cdot))$, i.e., $z_k(\cdot) \to \bar{z}(\cdot)$ in $L^2([0, T], R^n)$, $u_k(\cdot) \to \bar{u}(\cdot)$ in $L^2_w([0, T], R^m)$, $(z_k(t), u_k(t)) \in Q(t)$ $(t \in [0, T])$, from what it follows that

$$(\bar{z}(t), \bar{u}(t)) \in Q(t) \quad (t \in [0, T]).$$

2. Let us prove continuity of a function $(p, x) \mapsto F(p, x)$ (assumption (A2)). Let $p_k \to \bar{p}$ and $x_k \to \bar{x}$ in X (see (6.5)), i.e., $z_k(\cdot) \to \bar{z}(\cdot)$ in $L^2([0, T], R^n)$ $u_k(\cdot) \to \bar{u}(\cdot)$ in $L^2_w([0, T], R^m)$. Let us prove that $F(p_k, x_k) \to F(\bar{p}, \bar{x})$. Using a formula for $F_1(\cdot, \cdot)$ (6.6) and continuity of functions $f(\cdot, \cdot)$, $g(\cdot, \cdot)$ (assumption (B1)) we see that

$$|F_1(p_k, x_k) - F_1(\bar{p}, \bar{x})|_{L^2([0,T], R^n)} \to 0. \tag{6.12}$$

Let is show that $F_2(p_k, x_k) \to F_2(\bar{p}, \bar{x})$. We have

$$|F_2(p_k, x_k) - F_2(\bar{p}, \bar{x})| \leq |F_2(p_k, x_k) - F_2(\bar{p}, x_k)| + |F_2(\bar{p}, x_k) - F_2(\bar{p}, \bar{x})|. \tag{6.13}$$

Owing to the convergence $x_k \to \bar{x}$ and continuity of functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ we get $F_2(\bar{p},x_k) \to F_2(\bar{p},\bar{x})$ (see (6.7)). Let us consider the first term in (6.13). Suppose that $p_k \le \bar{p}$. Then

$$\begin{aligned}
|F_2(p_k,x_k) - F_2(\bar{p},x_k)|^2_{L^2([0,T],R^n)} &\le \int_0^T |F_2(p_k,x_k)(t) - F_2(\bar{p},x_k)(t)|^2_{R^n} dt \\
&\le \int_0^{p_k} \left| \int_{p_k}^{\bar{p}} (f(z_k(\tau),\tau) + g(z_k(\tau),\tau)u_k(\tau))d\tau \right|^2_{R^n} dt + \\
&\quad \int_{p_k}^{\bar{p}} \left| \int_{p_k}^{\bar{p}} (f(z_k(\tau),\tau) + g(z_k(\tau),\tau)u_k(\tau))d\tau \right|^2_{R^n} dt + \\
&\quad \int_{\bar{p}}^{T} \left| \int_{p_k}^{\bar{p}} (f(z_k(\tau),\tau) + g(z_k(\tau),\tau)u_k(\tau))d\tau \right|^2_{R^n} dt \\
&= \int_0^{T} \left| \int_{p_k}^{\bar{p}} (f(z_k(\tau),\tau) + g(z_k(\tau),\tau)u_k(\tau))d\tau \right|^2_{R^n} dt \\
&\le C|p_k - \bar{p}|^2,
\end{aligned}$$

where C is a constant. The latter inequality holds thanks to the boundedness of a sequence $(z_k(\cdot),u_k(\cdot))$. Supposing that $p_k > \bar{p}$, analogously we get the same estimate. Thus $|F_2(p_k,x_k) - F_2(\bar{p},x_k)| \to 0$. Since (6.13) we have (6.12). Continuity of a function $F(\cdot,\cdot)$ (see (6.8)) is proved.

3. Let us prove that a function $(p,x) \mapsto F(p,x) - b$ is a compactificator (Assumption (A3)). Let $|F(p_k,x_k) - b| \to 0$ and $p_k \to \bar{p}$. Let us show that a sequence $(x_k) = ((z_k(\cdot),u_k(\cdot)))$ is compact in X. Taking into account a formula (6.6) – (6.8) for a function $F(\cdot,\cdot)$, we have $|F_1(p_k,x_k) - b_1|_{L^2([0,T],R^n)} \to 0$ or

$$\int_0^T \left| z_k(t) - \int_0^t (f(z_k(\tau),\tau) + g(z_k(\tau),\tau)u_k(\tau)d\tau - z^0 \right|^2_{R^n} dt \to 0. \tag{6.14}$$

We rewrite (6.14) in a following form

$$|z_k(\cdot) - d_k(\cdot)|_{L^2([0,T],R^n)} \to 0, \tag{6.15}$$

where

$$d_k(t) = \int_0^t (f(z_k(\tau),\tau) + (z_k(\tau),\tau)u_k(\tau))d\tau + z^0 \quad (t \in [0,T]).$$

A family of functions $(d_k(\cdot))$ $(k = 1,2,\dots)$ if uniformly bounded and equipotentionally continuous. According to Arzela Theorem [5] a sequence $(d_k(\cdot))$ is compact in $C([0,T],R^n)$, and, hence, it is compact in $L^2([0,T],R^n)$. Therefore there exists a subsequence $(d_{k_j}(\cdot))$, converging in $L^2([0,T],R^n)$ to an element $\bar{d}(\cdot)$. According to (6.15) $z_{k_j}(\cdot) \to \bar{d}(\cdot)$ in $L^2([0,T],R^n)$, and hence $(z_k(\cdot))$ is compact in $L^2([0,T],R^n)$. A sequence $(u_k(\cdot))$ is bounded thanks to Assumption (B4), i.e., it is compact in $L^2_w([0,T],R^m)$. Thus a sequence $(x_k) = (z_k(\cdot),u_k(\cdot))$ is compact in X (see (6.5)).

4. The convexity of the set $F(p,G)$ for each $p \in [0,T]$ (Assumption (A4)) holds owing to Assumption (B5).

5. Boundedness of the set $F(p, G)$ for all $p \in [0, T]$ (Assumption (A5)) is a consequence of closedness, convexity and boundedness of values of multi-valued function $t \mapsto Q(t)$ (Assumptions (B2), (B4)).

6. Let us prove that Assumption (A6) holds, i.e., for any $\varepsilon > 0$ there exists such $\delta > 0$ that for all $p_1, p_2 \in [0, T]$ such that $|p_2 - p_1| < \delta$, and any $x = (z(\cdot), u(\cdot)) \in G$

$$|F(p_1, x) - F(p_2, x)|_Y < \varepsilon \tag{6.16}$$

holds. Let us notice that $F_1(p_1, x) = F_1(p_2, x)$ (see (6.6)). Let us suppose for distinctness that $p_1 < p_2$. Then

$$\begin{aligned}
|F_2(p_1,x) - F_2(p_2,x)|^2_{L^2([0,T],R^n)} &\leq \int_0^T |F_2(p_1,x)(t) - F_2(\bar{p},x)(t)|^2_{R^n} dt \\
&\leq \int_0^{p_1} \left| \int_{p_1}^{p_2} (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \right|^2_{R^n} dt \\
&+ \int_{p_1}^{p_2} \left| \int_{p_1}^{p_2} (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \right|^2_{R^n} dt \\
&+ \int_{p_2}^{T} \left| \int_{p_1}^{p_2} (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \right|^2_{R^n} dt \\
&= \int_0^{T} \left| \int_{p_1}^{p_2} (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \right|^2_{R^n} dt \\
&\leq C|p_1 - p_2|^2,
\end{aligned}$$

where C is a constant. Supposing that $p_1 > p_2$ analogously we get that same estimate. Then putting $\delta = \varepsilon^{1/2}/C^{1/2}$ we get that $|p_1 - p_2| < \delta$ holds (6.16).

7. The structure (6.6) – (6.8) of the function $F(\cdot, \cdot)$ and the boundedness of the multi-valued function $t \mapsto Q(t)$ (Assumption (B4)) lead to the boundedness of the set $\bigcup_{p\in[0,T]} F(p, G)$ (Assumption (A7)).

8. Continuity of the support function $p \mapsto c(l|F(p, G))$ (Assumption (A9)) for all $y \in Y$ is followed from the continuity of the function $p \mapsto F(p, \cdot)$ (6.6) – (6.8). □

Lemma 6.4 and Theorem 6.2 allow to apply the algorithm (5.12) – (5.23) for solving problem (6.3). Concretizing this algorithm for the considered here particular problem (2.1) equivalent to time-minimum problem (6.3) we get the following. At zero step we put

$$p_0 = 0, \quad \alpha_0^{(0)} = 1, \quad (z_0(\cdot), u_0(\cdot)) \in G, \quad v_0^z(\cdot) = z_0(\cdot), \quad v_0^u(\cdot) = u_0(\cdot) \tag{6.17}$$

and we chose a set $(p_0, \alpha_0^{(0)}, v_0^z(\cdot), v_0^u(\cdot), z_0(\cdot), u_0(\cdot))$ as an initial element of algorithm's sequence. At step $k+1$ by means of a set

$$(p_k, \alpha_0^{(k)}, \dots, \alpha_k^{(k)}, v_0^z(\cdot), v_0^u(\cdot), \dots, v_k^z(\cdot), v_k^u(\cdot), z_k(\cdot), u_k(\cdot)),$$

where

$$p_k \in [0, T], \quad \alpha_0^{(k)}, \dots, \alpha_k^{(k)} \in [0, 1],$$
$$(v_0^z(\cdot), v_0^u(\cdot)), \dots, (v_k^z(\cdot), v_k^u(\cdot)), (z_k(\cdot), u_k(\cdot)) \in X_0,$$

we find a set

$$(p_{k+1}, \alpha_0^{(k+1)}, \dots, \alpha_{k+1}^{(k+1)}, v_0^z(\cdot), v_0^u(\cdot), \dots, v_{k+1}^z(\cdot), v_{k+1}^u(\cdot), z_{k+1}(\cdot), u_{k+1}(\cdot)),$$

where

$$p_{k+1} \in [0, T], \quad \alpha_0^{(k+1)}, \dots, \alpha_{k+1}^{(k+1)} \in [0, 1],$$
$$(v_0^z(\cdot), v_0^u(\cdot)), \dots, (v_{k+1}^z(\cdot), v_{k+1}^u(\cdot)), (z_{k+1}(\cdot), u_{k+1}(\cdot)) \in X_0.$$

Namely, we put

$$l_k(t) = \bar{z} - \sum_{i=1}^{k} \alpha_i^{(k)} F(p_k, v_i)(t), \quad (\bar{z} = (z^0, z^1),\ t \in [0, T]). \tag{6.18}$$

In series as $p \in [p_k, T]$, $t \in [0, T]$ we calculate values

$$m(p, t) = (l^1(t) + l^2(t), r^1(p, t) + r^2(p, t)), \tag{6.19}$$

$$r^1(p, t) = -\int_t^T l^1(\tau) d\tau, \quad r^2(p, t) = \begin{cases} \int_0^t l^2(\tau) d\tau,\ t \in [0, p], \\ -\int_t^T l^2(\tau) d\tau,\ t \in [p, T], \end{cases} \tag{6.20}$$

and a set

$$W(t) = \{w(t, z, u) \in R^{2n} : w(t, z, u) = (z, f(z, t) + g(z, t)u), (z, u) \in Q(t)\}, \tag{6.21}$$

by means of which we define a function $\varphi_k(\cdot)$ of a form

$$\varphi_k(p) = \int_0^T [c(m_k(p, t)|W(t)) - \langle l_k(t), \bar{z}\rangle_{R^{2n}}] dt \quad (p_k \le p \le T). \tag{6.22}$$

We find a number

$$p_{k+1} = \min\{p \in [p_k, T] : \varphi_k(p) \ge 0\} \tag{6.23}$$

and an extreme functional couple $v_{k+1} = (v_{k+1}^z(\cdot), v_{k+1}^u(\cdot)) \in G$ such that

$$\langle m_k(p_{k+1}, t), w(t, v_{k+1}^z(t), v_{k+1}^u(t))\rangle_{R^{2n}} = c(m_k(p_{k+1}, t)|W(t)) \quad (t \in [0, T]), \tag{6.24}$$

where $w(t, z, u) = (z, f(z, t) + g(z, t)u)$ $(z \in R^n, u \in R^m, t \in [0, T])$. The value τ_{k+1} is defined by

$$\tau_{k+1} = \begin{cases} 0, & \tau_{k+1}^* < 0, \\ \tau_{k+1}^*, & \tau_{k+1}^* \in [0, 1], \\ 1, & \tau_{k+1}^* > 1, \end{cases} \tag{6.25}$$

$$\tau_{k+1}^* = \frac{\int_0^T \langle q_{k+1}(t), \bar{z} - q_{k+1}(t) + F(p_{k+1}, v_{k+1})(t)\rangle_{R^{2n}} dt}{\int_0^T |q_{k+1}(t)|_{R^{2n}}^2 dt} \tag{6.26}$$

$$\left(q_{k+1}(t) = F(p_{k+1}, v_{k+1})(t) - \sum_{i=0}^{k} \alpha_i^{(k)} F(p_{k+1}, v_i)(t) \not\equiv 0 \right),$$

$$\tau_{k+1} \in [0, 1] \quad (q_{k+1}(t) \equiv 0). \tag{6.27}$$

We set

$$\alpha_i^{(k+1)} = (1 - \tau_{k+1}) \alpha_i^{(k)} \quad (i = 0, \dots, k), \quad \alpha_{k+1}^{(k+1)} = \tau_{k+1}. \tag{6.28}$$

Functions $z_{k+1}(\cdot)$, $u_{k+1}(\cdot)$ are found as a solution of the system of integral equations

$$z_{k+1}(t) - \int_0^t (f(z_{k+1}(\tau),\tau) + g(z_{k+1}(\tau),\tau)u_{k+1}(\tau))d\tau \;=\; \sum_{i=0}^{k+1} \alpha_i^{(k+1)} F_1(p_{k+1}, v_i)(t) \quad (t \in [0,T]), \tag{6.29}$$

$$\begin{cases} z_{k+1}(t) + \int_t^{p_{k+1}} (f(z_{k+1}(\tau),\tau) + g(z_{k+1}(\tau),\tau)u_{k+1}(\tau))d\tau \\ \qquad = \sum_{i=0}^{k+1} \alpha_i^{(k+1)} F_2(p_{k+1}, v_i)(t) \quad (t \in [0, p_{k+1}]), \\ z_{k+1}(t) - \int_{p_{k+1}}^{t} (f(z_{k+1}(\tau),\tau) + g(z_{k+1}(\tau),\tau)u_{k+1}(\tau))d\tau \\ \qquad = \sum_{i=0}^{k+1} \alpha_i^{(k+1)} F_2(p_{k+1}, v_i)(t) \quad (t \in [p_{k+1}, T]) \end{cases} \tag{6.30}$$

under the constraint $(z_{k+1}(t), u_{k+1}(t)) \in Q(t)$ $(t \in [0,T])$.

Theorem 6.2 on equivalence of problems (6.3) and (2.1), and Theorem 5.3 on convergence based on Lemma 6.4 imply convergence of algorithm (6.17) – (6.30) to the solution of time minimum problem (6.3).

Theorem 6.5. *Let* (i) *Assumption (B1) – (B5) are satisfied;*

(ii) *spaces X and Y are determined by (6.5), (6.4), a set G is given by (6.10), an element $b(p) = b$ is set by (6.9), and function $F(\cdot,\cdot)$ has a form (6.6) – (6.8);*

(iii) *a sequence (p_k, x_k) $x_k = (z_k(\cdot), u_k(\cdot))$ is defined by algorithm (6.17) – (6.30). Then $p_k \to p_*$ and $\mathrm{dist}_X(x_k, D_*) \to 0$.*

6.2. **Problem of mixed constraint optimization.** Let us consider again the controllable system (6.1) functioning on the rime interval $[0,T]$ from the initial state (6.2). Now we believe that a state vector $z(t)$ and a control vector $u(t)$ are satisfied to a mixed constraint $(x(t), u(t)) \in Q(p,t)$ $(t \in [0,T])$, depending on a scalar parameter $p \in [p_0, p^0]$. Here $Q(\cdot,\cdot)$ is a multi-valued function acting from $[0,T] \times [p_0, p^0]$ into a class of all non-empty sub-sets of $R^n \times R^m$, p_0, p^0 are given numbers. We consider a problem of finding the minimum value of a parameter $p \in [p_0, p^0]$ at which there exists a trajectory $z(\cdot)$ of system (6.1) under a control $u(\cdot)$, such that for all $t \in [0,T]$ $(x(t), u(t)) \in Q(p,t)$ holds; the trajectory is to be detected too:

$$\begin{aligned} p &\to \min, \\ \dot z(t) &= f(z(t),t) + g(z(t),t)u(t), \\ z(0) &= z^0, \\ (z(t), u(t)) &\in Q(p,t), \\ (t &\in [0,T]). \end{aligned} \tag{6.31}$$

We denote a set of all admissible controllable processes corresponding to a value p via $\Pi(p)$. We believe that for at least one value $p \in [p_0, p^0]$ a set $\Pi(p)$ is not empty.

We denote the optimal value of the problem (6.31) via p_*, a set of all its solutions – via $\{p_*\} \times \Pi_*$, where $\Pi_* = \Pi(p_*)$.

In further analysis we will consider problem (6.31) under the following assumptions:

(C1) functions $(z,t) \mapsto f(z,t)$ and $(z,t) \mapsto g(z,t)$ are continuous on $R^n \times [0,T]$;

(C2) $Q(p,t)$ is a convex, bounded and closed subset of $R^n \times R^m$ for all $p \in [p_0, p^0]$ and $t \in [0,T]$;

(C3) a multi-valued function $t \mapsto Q(p,t)$ is measurable for all $p \in [p_0, p^0]$; a multi-valued function $p \mapsto Q(p,t)$ is continuous for all $t \in [0,T]$, i.e., if $(z_k, u_k) \in Q(p_k, t)$ $(k = 1, 2, \dots)$ and $p_k \to \bar{p}$, $z_k \to \bar{z}$, $u_k \to \bar{u}$, then $(\bar{z}, \bar{u}) \in Q(\bar{p}, t)$;

(C4) a set $\bigcup_{(p,t) \in [p_0, \infty) \times [0,T]} Q(p,t)$ is bounded;

(C5) for any $t \in [0,T]$ a multi-valued function $p \mapsto Q(p,t)$ increases monotonically, i.e., $Q(p_1,t) \subset Q(p_2,t)$ for all $p_1 \geq p_0$, $p_2 \geq p_1$.

Theorem 6.6. *Let Assumptions (C1) – (C3), (C5) be satisfied. Then there exists a solution of problem (6.31).*

Similar to the approach described for time minimum problem (6.3) we aim at reformulating problem of mixed constraint optimization (6.31) as an optimization problem of a form (2.1) in a space of measurable expansions of controllable processes. We fix the normalized space X (6.5) and a bounded set $X_0 \subset X$, containing all admissible controllable processes (X_0 exists thanks to Assumption (C4)). We introduce a Hilbert space

$$Y = L^2([0,T], R^n); \tag{6.32}$$

and define a function $x \mapsto F(p,x) = F(p,x)(\cdot) : X \mapsto Y$, setting

(6.33)
$$F(p,x)(t) = z(t) - \int_0^t (f(z(\tau),\tau) + g(z(\tau),\tau)u(\tau))d\tau \quad (t \in [0,T],\ x = (z(\cdot), u(\cdot)));$$

note that in (6.33) we write $F(p,x)$ subjunctively to keep initial notations since in fact F does not depend on p in this case). We put $b = b(\cdot) \in Y$ as

$$b(t) = z^0 \quad (t \in [0,T]); \tag{6.34}$$

and finally for all $p \geq p_0$ we introduce a set

$$X(p) = \{x = (z(\cdot), u(\cdot)) \in X_0 : (z(t), u(t)) \in Q(p,t)\ (t \in [0,T])\}. \tag{6.35}$$

Let us notice that $x = (z(\cdot), u(\cdot)) \in X_0$ almost always coincides with a admissible controllable process corresponding to a parameter $p \in [p_0, p^0]$ if and only if $F(x) = b$ and $x \in X(p)$. Thus we have

Theorem 6.7. *Problems (6.31) and (2.1) are equivalent in the following sense*

(i) *optimal values of problems (6.31) and (2.1) coincide;*

(ii) *a functional couple $(p_*, x) \in [p_0, p^0] \times X_0$, $x = (z(\cdot), u(\cdot))$, solves problem (2.1) if and only if there exists an optimal controllable process $(z_*(\cdot), u_*(\cdot))$ such that $(z(t), u(t)) = (z_*(t), u_*(t))$ for a.a. $t \in [0,T]$.*

In further analysis we will consider problem (6.31) under the following subsidiary assumption:

(C6) for any $p \in [p_0, p_*]$ a set $F(X(p)) = \{F(x) : x \in X(p)\}$ is convex.

Remark 6.8. Let us notice that analogously to the case of the time minimum problem (see Remark 6.8) Assumptions (B1) – (B5) are satisfied for the problem of mixed constraint optimization for a class of bilinear controllable systems:

$$p \quad \to \quad \min,$$

$$\begin{aligned} \dot{z}^{(i)}(t) &= \sum_{j=1}^{m}[a_{ij}(t)z^{(j)}(t) + b_{ij}(t)u^{(j)}(t)z^{(j)}(t)] \quad (i = 1,\dots,n), \\ z(0) &= z^0, \\ u(t) &= (u^{(1)}(t),...,u^{(m)}(t)) \in U, \\ z(t) &= (z^{(1)}(t),...,z^{(n)}(t)) \in Z(p,t), \end{aligned} \tag{6.36}$$

where $z(t) = (z^{(1)}(t),\dots,z^{(n)}(t)) \in R^n$ is a state vector, $u(t) = (u^{(1)}(t),\dots,u^{(m)}(t)) \in U \subset R^m$ is a control vector, $z^0 \in R^n$ are the given initial state of the system, geometric constraint on a control is a m-dimensional parallelogram (6.3); functions $t \mapsto a_{ij}(t)$ and $t \mapsto b_{ij}(t)$ $(i = 1,\dots,n,\ j = 1,\dots,m)$ are continuous scalar functions on $[0,T]$. The set $Z(p,t)$ characterizing a state constraint in the problem (6.36) satisfies to

(D1) for all $p \in [p_0,p^0]$, $t \in [0,T]$ sets $Z(p,t)$ are convex, closed and bounded and lying in a positive ortant R^n_+ (6.3);

(D2) for any $t \in [0,T]$ a multi-valued function $p \mapsto Z(p,t)$ is continuous and increases monotonically;

(D3) a set $\bigcup_{(p,t)\in[p_0,p^0]\times[0,T]} Z(p,t)$ is bounded. Rewriting problem (6.36) as (6.31) like in Remark 6.8 one can prove that Assumptions (D1) – (D3) guarantee (C1) – (C6).

Let us come back to problem (6.31). Analogously to Lemma 7 we have

Lemma 6.9. *Let for problem (6.31) Assumptions (C1) – (C6) be satisfied. Then for the equivalent problem (2.1) Assumptions (A1) – (A9) hold.*

Lemma 6.9 and Theorem 6.6 allow to apply algorithm (5.12) – (5.23) for solving problem (6.31). Concretizing this algorithm for the considered here particular problem (2.1) equivalent to problem of mixed constraint optimization (6.31) we get the following. At zero step we put

$$x_0 = (z_0(\cdot), u_0(\cdot)) \in X(p_0), \quad w_0(\cdot) = F(x_0)(\cdot) \tag{6.37}$$

and we chose a set $(p_0, w_0(\cdot), x_0(\cdot))$ as an initial element of algorithm's sequence. At step $k+1$ by means of $(p_k, w_k(\cdot), x_k(\cdot)) \in [p_0,p^0] \times L^2([0,T],R^n) \times L^2([0,T],R^n \times R^m)$, $x_k(\cdot) = (z_k(\cdot),u_k(\cdot))$, we find a set $(p_{k+1}, w_{k+1}(\cdot), x_{k+1}(\cdot)) \in [p_0,p^0] \times L^2([0,T],R^n) \times L^2([0,T],R^n \times R^m)$, $x_{k+1}(\cdot) = (z_{k+1}(\cdot),u_{k+1}(\cdot))$. Namely we put

$$l_k(t) = z^0 - w_k(t) \quad (t \in [0,T]). \tag{6.38}$$

In series for all $p \in [p_k,T]$, $t \in [0,T]$ we calculate

$$m(t) = (l(t), r(t)) \quad r(t) = -\int_t^T l(\tau)d\tau, \tag{6.39}$$

and a set

$$\begin{aligned} W(p,t) &= \{w(t,z,u) \in R^{2n} : w(t,z,u) = (z, f(z,t) + g(z,t)u), (z,u) \in Q(p,t)\} \\ & \qquad (p \in [p_0,p^0], t \in [0,T]) \end{aligned} \tag{6.40}$$

by means of which we define a function $\varphi_k(\cdot)$ of a form

$$\varphi_k(p) = \int_0^T [c(m_k(t)|W(p,t)) - \langle l_k(t), \bar{z}\rangle_{R^{2n}}]dt \quad (p \in [p_k, p^0]). \tag{6.41}$$

We find a number

$$p_{k+1} = \min\{p \in [p_k, p^0] : \varphi_k(p) \geq 0\} \tag{6.42}$$

and a couple of measurable functions $v_{k+1} = (v^z_{k+1}(\cdot), v^u_{k+1}(\cdot))$ on $[0,T]$ such that $(v^z_{k+1}(t), v^u_{k+1}(t)) \in Q(p_{k+1}, t)$ and

$$\langle m_k(t), w(t, v^z_{k+1}(t), v^u_{k+1}(t))\rangle_{R^n} = c(m_k(t)|W(p_k,t)) \quad (t \in [0,T]). \tag{6.43}$$

The value τ_{k+1} is defined by

$$\tau_{k+1} = \begin{cases} 0, & \tau^*_{k+1} < 0, \\ \tau^*_{k+1}, & \tau^*_{k+1} \in [0,1], \\ 1, & \tau^*_{k+1} > 1, \end{cases} \tag{6.44}$$

$$\tau^*_{k+1} = \frac{\int_0^T \langle q_{k+1}(t), z^0 - w_k(t)\rangle dt}{\int_0^T |q_{k+1}(t)|^2 dt} \quad \left(\int_0^T |q_{k+1}(t)|^2 dt \not\equiv 0\right),$$

$$\left(q_{k+1}(t) = v^z_{k+1}(t) - \int_0^t (f(v^z_{k+1}(\tau),\tau) + g(v^z_{k+1}(\tau),\tau)v^u_{k+1}(\tau))d\tau - w_k(t)\right)$$

$$\tau_{k+1} \in [0,1] \quad \left(\int_0^T |q_{k+1}(t)|^2 dt \equiv 0\right). \tag{6.45}$$

We set

$$\begin{aligned} w_{k+1}(t) &= (1-\tau_{k+1})w_k(t) + \\ &\quad \tau_{k+1}\left(v^z_{k+1}(t) - \int_0^t (f(v^z_{k+1}(\tau),\tau) + g(v^z_{k+1}(\tau),\tau)v^u_{k+1}(\tau))d\tau\right), \end{aligned} \tag{6.46}$$

$$z_{k+1}(t) = (1-\tau_{k+1})z_k(t) + \tau_{k+1}v^z_{k+1}(t) \quad (t \in [0,T]); \tag{6.47}$$

a control $u_{k+1}(\cdot)$ is found as a solution of the integral equation

$$z_{k+1}(t) - \int_0^t (f(z_{k+1}(\tau),\tau) + g(z_{k+1}(\tau),\tau)u_{k+1}(\tau))d\tau = w_{k+1}(t) \quad (t \in [0,T]) \tag{6.48}$$

under constraint $(z_{k+1}(t), u_{k+1}(t)) \in Q(p_{k+1}, t)$ $(t \in [0,T])$.

Theorem 6.10. *Let* (i) *Assumptions (C1) – (C6) be satisfied;*

(ii) *functional spaces X and Y are defined by (6.5), (6.32), a set $X(p)$ is determined by (6.35), an element $b(p) = b$ is set by (6.34), a function $F(\cdot)$ has a form (6.33);*

(iii) *a sequence $(p_k, w_k(\cdot), x_k(\cdot))$ from $[p_0, p^0] \times Y \times X_0$, $x_k(\cdot) = (z_k(\cdot), u_k(\cdot))$ $(k = 0, 1, \ldots)$, is defined by algorithm (6.37) – (6.48).*

Then $p_k \to p_*$ and $\text{dist}_X(x_k, D_*) \to 0$.

References

[1] S. M. Aseev and A. V. Kryazhimskiy, *The Pontryagin Maximum Principle and Optimal Economic Growth Problems*, Moscow: Nauka Publ. - MAIK Nauka/Interperiodika, 2007.

[2] A. A. Arutyunov and S. M. Aseev, *Investigation of degeneracy of the maximum principle for optimal control problem with state constraints*, SIAM Journal Control Optimization **35** (1997), 930–952.

[3] R. E. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, 1957.

[4] A. Fiakko, G. Mc-Kormic, *Nonlinear programming. Methods of sequent non-conditional optimization* [PROVIDE REFERENCE!!!!]

[5] K. Iosida, *Functional Analysis*, Springer-Verlag, New York, 1965.

[6] N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, New York, 1988.

[7] A. V. Kryazhimsky, *Optimization problems with convex epigraphs. Application to optimal control*, Internatioanl Journal of Applied Mathematics and Computer Science **11** (2001), 101–129.

[8] A. V. Kryazhimsky, B. V. Digas and Yu. M. Ermoliev, *Guaranteed optimization in insurance of catastrophic risks*, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1998, IR-98-082.

[9] A. V. Kryazhimsky and S. V. Paschenko, *Distribution of resources and problem of optimial compatibility* , Proceedings of International Conference on Distrubuted Systems "Optimization and Economic-Environmental Applications". Ekaterinburg, 30 May - 2 June, 2000. Ekaterinburg. Urals Branch, RAS. 2000, pp. 210–212.

[10] A. V. Kryazhimskii and S. V. Paschenko, *On the problem of optimal compatibility*, Journal Inverse Ill-Posed Problems **9** (2001), 283–300.

[11] A. V. Kryazhimskiy and S. V. Pashenko, *On solving a linear time minimum problem with mixed constraints*, VINITI. Summary of Science and Technology: Modern Mathematics and its applications, **90** (2002), 232–260 (in Russian).

[12] E. Polak, *Optimization: Algorithms and Consistent Approximations*, Springer-Verlag, New York, 1997.

[13] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, New York: Interscience, 1962.

[14] E. A. Rovenskaya, *On solving the problem of optimal compatibility parameter value for one kind of equations in Banach Space*, Journal of Computational Mathematics and Mathematical Physics, **44** (2004), 2150–2166.

[15] E. A. Rovenskaya, *On solving the minimum-time problem with state constraint for a simple model of one-leg jumping Robot*, Nonlinear Dynamics and Control **5** (2005), 186–212 (in Russian).

[16] E. A. Rovenskaya, *On a numerical method for solving the minimum-time problem with state constraint for a simple model of one-leg jumping robot*, Problems of Dynamical Control, Faculty of Computational Mathematics and Cybernetics, Moscow State University **1** (2005), 253–267 (in Russian).

[17] E. A. Rovenskaya, *On Solving the Problem of State Constraints Optimization for a Linear Control System*, Mathematical Models in Economics and Ecology, MAKS Press, Moscow, 2004, pp. 75–78 (in Russian).

[18] E. A. Rovenskaya, *The regularization of the problem of finding optimal compatibility parameter value for a class of equations in Banach space*, Abstracts of International Seminar "The Control Theory and the Theory of Generalized Solutions of Hamilton-Jacoby Equations", Ekaterinburg, Ural University, 2006, 2, pp. 165–172 (in Russian).

[19] J. Warga, *Optimal Control of Differential and Functional Equations*, Academi Press, New York, 1972.

[20] W. I. Zangwill and C. B. Garcia, *Pathways to Solutions, Fixed Points and Equilibria*, Prentice Hall, Englewood Cliffs, 1981.

Manuscript received October 9, 2007

revised August 2, 2008

Elena Rovenskaya
Lomonosov Moscow State Univeristy, Faculty of Computational Mathematics and Cybernetics, Leninskie Gory, 2-nd Educational Building, GSP-1, 119991, Moscow, Russia
International Institute for Applied System Analysis, Dynamic Systems Program, Schlossplatz 1, A-2361, Laxenburg, Austria
E-mail address: eroven@mail.ru