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ON DERIVATIVES OF SET-VALUED MAPS AND OPTIMALITY CONDITIONS FOR SET OPTIMIZATION

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Abstract. We consider a set-valued optimization problem, which is an optimization problem with a set-valued objective map. We know that there are two types of criteria for solutions of the problem, one is the criterion of *vector optimization* and the other is the criterion of *set optimization*. In this paper, we consider a set optimization problem and investigate first order optimality conditions by using directional derivatives based on an embedding idea. Also we observe optimality conditions when the object set-valued map satisfies certain convexity assumptions.

1. Introduction

Throughout the paper, let E be a locally convex topological vector space over $\mathbb{R}$, let K be a closed convex cone in E, and assume K is solid and pointed, that is, $\mathrm{int}K \neq \emptyset$ and $K \cap (-K) = \{\theta\}$ where θ is the null vector of E. We define an order relation $\leq$ on E by

$$x, y \in E, \quad x \leq y \ \text{ if } \ y - x \in K.$$

For a given set-valued map F from a set X to E, we consider the following set-valued optimization problem:

$$\text{(P)} \quad \text{Minimize } F(x) \quad \text{ subject to } x \in X.$$

There are two types different criteria of solutions for the problem (P), one is the criterion of *vector optimization* (VP), and the other is the criterion of *set optimization* (SP). The criterion of (VP) is based on comparisons of all elements of all values of F, with respect to $\leq$, and an element $x_0 \in X$ is said to be an efficient solution of (VP) if there exists $y_0 \in F(x_0)$ such that $(y_0 - K) \cap \bigcup_{x \in X} F(x) = \{y_0\}$, or equivalently,

$$x \in X,\ y \in F(x),\ y \leq y_0 \implies y_0 \leq y.$$

The problem (VP) has been researched and developed by many authors, for example, see [2].

On the other hand, the criterion of (SP) is based on comparisons of all values of F with respect to a binary relation $\preceq$ on 2^E, which is called a *set relation*. For example, the following are set relations: for given nonempty subsets A, B of E, $A \preceq B$ means,

(1) $\forall x \in A$, $\forall y \in B$, $x \leq y$;
(2) $\exists x \in A$ such that $\forall y \in B$, $x \leq y$;
(3) $\forall y \in B$, $\exists x \in A$ such that $x \leq y$;

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(4) $\exists y \in B$ such that $\forall x \in A$, $x \leq y$;
(5) $\forall x \in A$, $\exists y \in B$ such that $x \leq y$;
(6) $\exists x \in A$, $\exists y \in B$ such that $x \leq y$.

An element $x_0 \in X$ is said to be a solution of set optimization (SP) with respect to set relation $\preceq$ if

$$x \in X,\ F(x) \preceq F(x_0) \implies F(x_0) \preceq F(x).$$

The problem (SP) is introduced by the author, and has been developed now, see [3, 4]. Though there are various types of set relations on 2^E, we define the following set relation $\leq_K^l$: for A, $B \in 2^E$,

$$A \leq_K^l B \ \text{ if } \ \mathrm{cl}(A+K) \supset B, \qquad \textit{cf. (3)}$$

and we observe the following notions of solutions in this paper.

Definition 1.1. An element $x_0 \in X$ is said to be a minimal solution of (SP) with respect to $\leq_K^l$ if

$$x \in X,\ F(x) \leq_K^l F(x_0) \implies F(x_0) \leq_K^l F(x).$$

The aim of the paper is to define directional derivatives of set-valued maps, and to consider first order optimality conditions for the set-valued optimization problem (SP). At first, we introduce an embedding space into which our minimization problem (SP) is embedded in Section 2; results of the section are studied in [5], and based on the embedding idea, we define directional derivatives of set-valued maps in Section 3. We give results about necessary and sufficient optimality conditions by using the directional derivatives in Section 4, and finally we observe optimality conditions under convexity or pseudoconvexity assumptions of the set-valued objective map in Section 5.

2. Introduction and preliminaries: an embedding idea

In this section, we introduce an ordered vector space $\mathcal{V}$ in which a certain subfamily $\mathcal{G}$ of 2^E is embedded; all results of this section are studied in [5]. By using these results, we define directional derivatives of set-valued maps in the next section.

A subset A of E is said to be K-convex if $A+K$ is convex, and A is said to be K^+-bounded if $\langle y^*, A\rangle$ is bounded from below for any $y^* \in K^+$, where K^+ be the positive polar cone of K, that is

$$K^+ = \{y^* \in E^* \mid \langle y^*, k\rangle \geq 0, \forall k \in K\}.$$

Let $\mathcal{G}$ be the family of all nonempty K-convex and K^+-bounded subsets of E. Then the following lemmas hold:

Lemma 2.1. *For any A, $B \in \mathcal{G}$,*

$$\mathrm{cl}(A+K) \supset B \textit{ if and only if } \inf\langle y^*, A\rangle \leq \inf\langle y^*, B\rangle \textit{ for all } y^* \in K^+.$$

Lemma 2.2 (cancellation law)**.** *For any A, B, $C \in \mathcal{G}$,*

$$\mathrm{cl}(A+C+K) = \mathrm{cl}(B+C+K) \textit{ if and only if } \mathrm{cl}(A+K) = \mathrm{cl}(B+K).$$

Define a binary relation $\equiv$ on $\mathcal{G}^2$: for each (A,B), $(C,D) \in \mathcal{G}^2$,

$$(A,B) \equiv (C,D) \text{ if } \operatorname{cl}(A+D+K) = \operatorname{cl}(B+C+K).$$

We can check $\equiv$ is an equivalence relation on $\mathcal{G}^2$ by using the cancellation low. Denote the quotient space $\mathcal{G}^2/\!\equiv$ by $\mathcal{V}$, that is

$$\mathcal{V} = \{[A,B] \mid (A,B) \in \mathcal{G}^2\},$$

where $[A,B] = \{(C,D) \in \mathcal{G}^2 \mid (A,B) \equiv (C,D)\}$, and define addition and scalar multiplication on $\mathcal{V}$ as follows:

$$[A,B] + [C,D] = [A+C, B+D],$$

$$\lambda \cdot [A,B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0, \end{cases}$$

then $(\mathcal{V}, +, \,\cdot\,)$ is a vector space over $\mathbb{R}$. Define

$$\mu(K) = \left\{ [A,B] \in \mathcal{V} \;\middle|\; B \leq_K^l A \right\},$$

then it is a pointed convex cone in $\mathcal{V}$, and order relation $\preceq_{\mu(K)}$ on $\mathcal{V}$ is defined as follows:

$$[A,B] \preceq_{\mu(K)} [C,D] \text{ if } [C,D] - [A,B] \in \mu(K).$$

Note that set optimization problem (SP) can be regarded as a vector optimization problem by an embedding idea. Assume that F is a map from X to $\mathcal{G}$. Then $x_0 \in E$ is a minimal solution of (SP) with respect to $\leq_K^l$ if and only if

$$\varphi \circ F(X) \cap (\varphi \circ F(x_0) - \mu(K)) = \{\varphi \circ F(x_0)\},$$

where φ is a function from $\mathcal{G}$ to $\mathcal{V}$ defined by $\varphi(A) = [A, \{\theta\}]$ for all $A \in \mathcal{G}$.

Finally, we introduce a norm $|\cdot|$ in a subspace of $\mathcal{V}$. Let W be a base of K^+, that is $\mathbb{R}_+ W = K$, and $\theta^* \notin W$. From Lemma 2.1,

$$|[A,B]| = \sup_{y^* \in W} |\inf \langle y^*, A\rangle - \inf \langle y^*, B\rangle|$$

is well-defined for any $[A,B] \in \mathcal{V}$, and it is a norm in

$$\mathcal{V}(W) = \{[A,B] \in \mathcal{V} \mid |[A,B]| < \infty\}.$$

Also $\mu(K)$ is closed in the normed space $(\mathcal{V}(W), |\cdot|)$.

3. Directional derivatives of set-valued maps for set optimization

Based on results in previous section, we define directional derivatives of set-valued maps for set optimization. In the rest of the paper, additionally assume that X is a convex set of a normed space $(Z, \|\cdot\|)$ over $\mathbb{R}$, F is a map from X to $\mathcal{G}$, and W is a w*-closed convex base of K^+ satisfying $\mathcal{V} = \mathcal{V}(W)$, that is, $|[A,B]| < \infty$ whenever $[A,B] \in \mathcal{V}$.

Now we define directional derivatives $CF(x,d)$ and $DF(x,d)$, and give examples of these derivatives:

Definition 3.1. Let $x \in X$ and $d \in Z$. Then

$$CF(x,d) = \left\{ [A,B] \in \mathcal{V} \;\middle|\; \exists \{\lambda_k\} \downarrow 0 \text{ s.t. } \frac{1}{\lambda_k}[F(x+\lambda_k d), F(x)] \to [A,B] \right\}$$

is said to be $\mathcal{V}$-directional derivative clusters of F at x in the direction d. If $CF(x,d)$ is a singleton, then the element is written by $DF(x,d)$ and called $\mathcal{V}$-directional derivative of F at x in the direction d, and F is said to be $\mathcal{V}$-directional differentiable at x in the direction d.

Example 3.2. Let $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be a set-valued map defined by

$$F(x) = \text{co}\,\{(|x|, -|x|+1), (-|x|+1, |x|)\}, \quad \forall x \in \mathbb{R},$$

and let $K = \mathbb{R}^2_+ = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$. Then F is $\mathcal{V}$-directional differentiable at all points in all directions. When $x_0 = 0$,

$$DF(x_0, d) = [\{(0,0)\}, |d|\,\text{co}\{(1,-1),(-1,1)\}], \quad \forall d \in \mathbb{R},$$

when $0 < x_0 < \frac{1}{2}$,

$$DF(x_0,d) = \begin{cases} [\{(0,0)\}, |d|\,\text{co}\{(1,-1),(-1,1)\}], & \text{if } d \geq 0, \\ [|d|\,\text{co}\{(1,-1),(-1,1)\}, \{(0,0)\}], & \text{if } d < 0, \end{cases}$$

and when $x_0 = \frac{1}{2}$,

$$DF(x_0,d) = [|d|\,\text{co}\{(1,-1),(-1,1)\}, \{(0,0)\}], \quad \forall d \in \mathbb{R}.$$

Note that the left part of $DF(x_0,d)$ shows the rate of increase of value of F at x_0 in the direction d, and the right part shows the rate of decrease.

Example 3.3. A set-valued map $F : X \to 2^E$ defined by

$$F(x) = g(x) + \sum_{i \in I} r_i(x) A_i, \quad x \in X,$$

where g is a function from X to E which is directional differentiable at $x_0 \in X$ in the direction $d \in Z$, I is a nonempty finite set, and for each $i \in I$, r_i is a function from X to $(0,\infty)$ which is directional differentiable at x_0 in the direction d, and $A_i \in \mathcal{G}$. Then F is $\mathcal{V}$-directional differentiable at x_0 in the direction d, and

$$\begin{aligned} DF(x_0,d) &= [g'(x_0,d), \{\theta\}] + \sum_{i \in I} r_i'(x_0,d)[A_i, \{\theta\}] \\ &= \left[g'(x_0,d) + \sum_{i \in I_+(d)} r_i'(x_0,d) A_i,\; -\sum_{i \in I_-(d)} r_i'(x_0,d) A_i \right] \end{aligned}$$

where $I_+(d) = \{i \in I \mid r_i'(x_0,d) > 0\}$ and $I_-(d) = \{i \in I \mid r_i'(x_0,d) < 0\}$.

4. Optimality conditions of a set optimization problem

The purpose of this section is to observe optimality conditions for minimal and weak minimal solutions of (SP) by using the directional derivatives defined in the previous section. At first, we define a binary relation on $\mathcal{G}$: for A, $B \in \mathcal{G}$,

$$A <^l_K B \;\text{ if }\; \exists V \subset E: \text{ a neighborhood of } \theta \;\text{ s.t. }\; A + K \supset B + V.$$

Then we have the following lemma:

Lemma 4.1. *For any A, $B \in \mathcal{G}$, $[A,B] \in \mathrm{Int}\mu(K)$ implies $B <_K^l A$, where* $\mathrm{Int}\mu(K)$ *is the set of all interior points with respect to $(\mathcal{V}, |\cdot|)$. The converse implication holds when topological vector space E is normable.*

Proof. If $[A,B] \in \mathrm{Int}\mu(K)$, then there exists $\varepsilon > 0$ such that $[A,B] + \varepsilon\mathcal{U} \subset \mu(K)$, where

$$\mathcal{U} = \{[C,D] \in \mathcal{V} \mid |[C,D]| \leq 1\}.$$

Since $\mathrm{int}K \neq \emptyset$, we can find $p \in K$ and a convex neighborhood V of θ in E such that $p+V \subset K$. Let $V_0 = \frac{\varepsilon}{|[V,\{\theta\}]|}V$, then we obtain $B+K \supset A+V_0$ since $[V_0, \{\theta\}] \in \varepsilon\mathcal{U}$. Hence we have $B <_K^l A$.

Conversely, if $B <_K^l A$, then there exists a neighborhood of V of θ such that $B+K \supset A+V$. Since W is a w*-closed convex base of K^+, then $M := \inf_{y^* \in W} \|y^*\|$ is positive by using separation theorem. Choose a positive number ε satisfying $\frac{\varepsilon}{M}U \subset V$, where U is the unit ball of E, then $B+K \supset A+\frac{\varepsilon}{M}U$. Now we can verify that $[A,B]+\varepsilon\mathcal{U} \subset \mu(K)$. Indeed, for all $[C,D] \in \varepsilon\mathcal{U}$, by using Lemma 2.1,

$$\begin{aligned}\inf\langle y^*, B+D\rangle &= \inf\langle y^*, B\rangle + \inf\langle y^*, D\rangle \\ &\leq \inf\langle y^*, A\rangle + \inf\left\langle y^*, \tfrac{\varepsilon}{M}U\right\rangle + \inf\langle y^*, C\rangle + \varepsilon \\ &= \inf\langle y^*, A+C\rangle,\end{aligned}$$

for any $y^* \in W$. Since W is a base of K^+, by using Lemma 2.1 again, we have

$$\mathrm{cl}(B+D+K) \supset A+C,$$

and consequently, $[A,B]+\varepsilon\mathcal{U} \subset \mu(K)$. This completes the proof. $\square$

Definition 4.2. An element $x_0 \in X$ is said to be a weak minimal solution of (SP) if

$$\nexists x \in X \text{ s.t. } F(x) <_K^l F(x_0).$$

Moreover, we define local solutions of (SP).

Definition 4.3. An element $x_0 \in X$ is said to be, a local minimal solution of (SP) if there exists N a neighborhood of x_0 such that

$$x \in N \cap X,\ F(x) \leq_K^l F(x_0) \implies F(x_0) \leq_K^l F(x),$$

and a local weak minimal solution of (SP) if there exists N a neighborhood of x_0 such that

$$\nexists x \in N \cap X \text{ s.t. } F(x) <_K^l F(x_0).$$

Now we have a result of a necessary condition of local weak optimality of (SP).

Theorem 4.4. *If x_0 be a local weak minimal solution of (SP), then we have*

$$CF(x_0, x-x_0) \cap (-\mathrm{Int}\,\mu(K)) = \emptyset, \qquad \forall x \in X.$$

Moreover, if F satisfying the following condition: for any $x \in Z$ and any open set V of E satisfying $F(x)+K \supset V$, there exists a neighborhood Y of x such that

$$F(y)+K \supset V, \quad \forall y \in Y,$$

then we have

$$CF(x_0, d) \cap (-\mathrm{Int}\,\mu(K)) = \emptyset, \qquad \forall d \in T_X(x_0) = \mathrm{cl}\bigcup_{\lambda>0}\frac{X-x_0}{\lambda}.$$

Proof. Assume that x_0 is a local weak minimal solution of (SP) and there exists $x_1 \in X \setminus \{x_0\}$ such that

$$CF(x_0, x_1 - x_0) \cap (-\text{Int}\mu(K)) \neq \emptyset.$$

Let $[A, B]$ be an element of $CF(x_0, x_1 - x_0) \cap (-\text{Int}\mu(K))$. By definition of $\mathcal{V}$-directional derivative clusters, there exists $\{\lambda_k\} \subset (0, \infty)$, which converges to 0, such that

$$\frac{1}{\lambda_k}[F(x_0 + \lambda_k(x_1 - x_0)), F(x_0)] \to [A, B], \quad k \to \infty,$$

then we have $F(x_0 + \lambda_k(x_1 - x_0)) <_K^l F(x_0)$ and $x_0 + \lambda_k(x_1 - x_0) \in X$ for sufficient large k. This is a contradiction.

Next we show the latter part of the theorem. Assume that there exists a nonzero vector $d_0 \in T_X(x_0)$ such that

$$CF(x_0, d_0) \cap (-\text{Int}\mu(K)) \neq \emptyset,$$

then we can choose $\lambda_0 > 0$ such that $F(x_0 + \lambda_0 d_0) <_K^l F(x_0)$ in the similar way, that is

$$F(x_0 + \lambda_0 d_0) + K \supset F(x_0) + V$$

for some open set V of E. By using the assumption, there exists a neighborhood Y of $x_0 + \lambda_0 d_0$ satisfying

$$F(y) + K \supset F(x_0) + V, \quad \forall y \in Y.$$

Now we can find $d \in \bigcup_{\lambda > 0} \frac{X - x_0}{\lambda}$ such that $x_0 + \lambda_0 d \in Y$, hence

$$F(x_0 + \lambda_0 d) + K \supset F(x_0) + V,$$

that is, $F(x_0 + \lambda_0 d) <_K^l F(x_0)$. This is a contradiction. □

Also we have a result of a sufficient condition of local optimality of (SP).

Theorem 4.5. *Assume that Z is a finite dimensional space, and F is $\mathcal{V}$-directional derivative at $x_0 \in X$ in each direction. Moreover, we assume that*

$$DF(x_0, d) = \lim_{t \downarrow 0} \frac{1}{\lambda}[F(x_0 + \lambda d), F(x_0)]$$

converges uniformly and continuous with respect to d on the unit ball. If

$$DF(x_0, d) \notin -\mu(K), \qquad \forall d \in T_X(x_0) \setminus \{\theta\},$$

then x_0 is a local minimal solution of (SP).

Proof. If x_0 is not local minimal solution of (SP), then there exists a sequence $\{x_n\} \subset X$ converges to x_0 such that $F(x_n) \leq_K^l F(x_0)$ and $F(x_0) \not\leq_K^l F(x_n)$. For all $n \in \mathbb{N}$, since $x_n \neq x_0$, let $d_n = (x_n - x_0)/\|x_n - x_0\|$. Then we can choose a subsequence $\{d_{n'}\}$ of $\{d_n\}$ and $d_0 \in \mathbb{R}^n$ such that $\|d_0\| = 1$ and $d_{n'} \to d_0$. Now we show

$$\frac{1}{\|x_{n'} - x_0\|}[F(x_{n'}), F(x_0)] \to DF(x_0, d_0).$$

Indeed, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ and for any $d \in \mathbb{R}^n$ with $\|d\| = 1$,

$$\left| \frac{1}{\lambda}[F(x_0 + \lambda d), F(x_0)] - DF(x_0, d) \right| < \frac{\varepsilon}{2}.$$

Also there exists $k \in \mathbb{N}$ such that, for any $n' \geq k$,

$$|DF(x, d_{n'}) - DF(x, d_0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |x_{n'} - x_0| < \lambda_0,$$

therefore, for any $n' \geq k$,

$$\left| \frac{1}{\|x_{n'} - x_0\|}[F(x_{n'}), F(x_0)] - DF(x_0, d_0) \right| < \epsilon.$$

Since

$$\frac{1}{\|x_{n'} - x_0\|}[F(x_{n'}), F(x_0)] \in -\mu(K),$$

and $\mu(K)$ is closed, then we have $DF(x_0, d_0) \in -\mu(K)$. However this is a contradiction, because $d_0 \in T_X(x_0) \setminus \{\theta\}$. This completes the proof. □

Example 4.6. We discuss Example 3.2, in the standpoint of the above optimality conditions. At first, we can check easily that no (global) minimal, and no (global) weak minimal solution of (SP), 0 is the only local minimal solution of (SP), and for each $x \in \mathbb{R}$, x is a local weak minimal solution of (SP).

When $x_0 = 0$, $DF(x_0, d) \in -\mu(K)$ holds if and only if

$$\mathbb{R}^2_+ \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\},$$

but it does not hold when $d \neq 0$. From Theorem 4.5, we have x_0 is a local minimal solution of (SP). Also when $0 < x_0 < \frac{1}{2}$, $DF(x_0, d) \in -\mu(K)$ holds if and only if

$$\begin{cases} \mathbb{R}^2_+ \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\}, & \text{if } d \geq 0, \\ \mathbb{R}^2_+ + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0), & \text{if } d < 0. \end{cases}$$

and $\mathbb{R}^2_+ + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0)$ is always true when $d < 0$. This is consistent with Theorem 4.5, since x_0 is not local minimal solution. Moreover, $DF(x_0, d) \in -\mathrm{Int}\mu(K)$ holds if and only if

$$\begin{cases} (0, \infty)^2 \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\}, & \text{if } d \geq 0 \\ (0, \infty)^2 + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0), & \text{if } d < 0 \end{cases}$$

does not hold for each $d \in \mathbb{R}$. This is also consistent with Theorem 4.4, since x_0 is a local weak minimal solution of (SP).

5. Convexity and pseudoconvexity of set-valued maps and globality of solutions

In this section, we observe local and and global minimal solutions when the objective map is convex or pseudoconvex. At first, we introduce the convexity of set-valued maps for set optimization.

Definition 5.1. A set valued map F from X to $\mathcal{G}$ is $\mathcal{V}$-convex if, for any $x, y \in X$, $\lambda \in (0, 1)$,

$$\mathrm{cl}(F((1 - \lambda)x + \lambda y) + K) \supset (1 - \lambda)F(x) + \lambda F(y).$$

Clearly the inequality is equivalent to

$$\varphi\circ F((1-\lambda)x+\lambda y)\preceq_{\mu(K)}(1-\lambda)\cdot\varphi\circ F(x)+\lambda\cdot\varphi\circ F(y),$$

that is, $\mathcal{V}$-convexity is strongly connected with the cone convexity, see [2]. The convexity of set-valued maps assures local minimal solutions are also global ones.

Proposition 5.2. *Assume that F is $\mathcal{V}$-convex. If $x_0 \in X$ is a local minimal solution of (SP), then it is also minimal solution of (SP), and moreover if $x_0 \in X$ is a local weak minimal solution of (SP), then it is also weak minimal solution of (SP).*

Proof. If $x_0 \in X$ is a local minimal solution of (SP), and there exists $x_1 \in X$ such that

$$F(x_1)\leq^l_K F(x_0) \quad \text{and} \quad F(x_0)\not\leq^l_K F(x_1),$$

then, for each $\lambda \in (0,1)$, we have

$$F((1-\lambda)x_0+\lambda x_1)\leq^l_K(1-\lambda)F(x_0)+\lambda F(x_1)\leq^l_K F(x_0),$$

and also,

$$F(x_0)\not\leq^l_K F((1-\lambda)x_0+\lambda x_1)$$

for all $\lambda \in (0,1)$. When we choose λ sufficiently small, the above formula contradicts with $x_0 \in X$ is a local minimal solution of (SP). If $x_0 \in X$ is a local weak minimal solution of (SP), and there exists $x_1 \in X$ such that $F(x_1) <^l_K F(x_0)$, we can choose a neighborhood V of θ satisfying

$$F(x_1)+K\supset F(x_0)+V.$$

From the convexity, for each $\lambda \in (0,1)$, we have

$$\mathrm{cl}(F((1-\lambda)x_0+\lambda x_1)+K)\supset(1-\lambda)F(x_0)+\lambda F(x_1)\supset F(x_0)+\lambda V.$$

This contradicts with $x_0 \in X$ is a local weak minimal solution of (SP). □

Then we have a corollary concerned with a sufficient condition of optimality.

Corollary 5.3. *Under the same conditions in Theorem 4.5, if F is $\mathcal{V}$-convex, then x_0 is a global minimal solution of (SP).*

When F is $\mathcal{V}$-convex, if $x \in X$ and $x+td \in X$ for some $t>0$,

$$\lambda\mapsto\frac{1}{\lambda}[F(x+\lambda d),F(x)]$$

is increasing on $(0,t]$ with respect to $\preceq_{\mu(K)}$. Therefore if $CF(x,d)\neq\emptyset$, then F is $\mathcal{V}$-directional differentiable at x in the direction d and

$$[F(x+d),F(x)]\succeq_{\mu(K)}CF(x,d),$$

see [5]. By using the property, we have a result concerned with a sufficient condition of weak optimality.

Theorem 5.4. *Assume that topological vector space E is normable, and F is $\mathcal{V}$-convex and $\mathcal{V}$-directional differentiable at x_0 in the direction $x-x_0$ for any $x \in X$. If the condition*

$$DF(x_0,x-x_0)\notin-\mathrm{Int}\mu(K),\quad\forall x\in X$$

holds, then x_0 is a global weak minimal solution of (SP).

Proof. If x_0 is not, then there exists $x \in X$ satisfying $F(x) <^l_K F(x_0)$, and then

$$[F(x), F(x_0)] \in -\mathrm{Int}\mu(K)$$

from Lemma 4.1. Since F is $\mathcal{V}$-convex, we have

$$[F(x), F(x_0)] \succeq_{\mu(K)} DF(x_0, x - x_0),$$

and hence $DF(x_0, x - x_0) \in -\mathrm{Int}\mu(K)$. This is a contradiction to the assumption of this theorem. $\square$

Next, we define the pseudoconvexity of set-valued maps for set optimization.

Definition 5.5. F is said to be $\mathcal{V}$-pseudoconvex if, for any $x, y \in X$, $CF(x, y-x) \neq \emptyset$ and

$$CF(x, y - x) \cap (-\mathrm{Int}\mu(K)) = \emptyset \Longrightarrow [F(y), F(x)] \notin -\mathrm{Int}\mu(K).$$

It is clear that F is $\mathcal{V}$-pseudoconvex when F is $\mathcal{V}$-convex and $\mathcal{V}$-directional differentiable. If F is a real-valued differentiable function, since $CF(x, y - x) = \langle \nabla F(x), y - x \rangle$ and $-\mathrm{Int}\mu(K) = (-\infty, 0)$, the implication is equivalent to

$$\langle \nabla F(x), y - x \rangle \geq 0 \Longrightarrow F(y) \geq F(x),$$

this is the condition of the usual pseudoconvexity. Next we have a perfect characterization between $\mathcal{V}$-pseudoconvexity and local weak minimality of (SP).

Theorem 5.6. *Assume that topological vector space E is normable and F is $\mathcal{V}$-pseudoconvex. Then x_0 is a local weak minimal solution of (SP) if and only if the following condition holds:*

$$CF(x_0, x - x_0) \cap (-\mathrm{Int}\mu(K)) = \emptyset, \quad \forall x \in X.$$

Proof. It is clear that if x_0 is a local weak minimal solution of (SP) then the condition $CF(x_0, x-x_0) \cap (-\mathrm{Int}\mu(K)) = \emptyset$ holds for all $x \in X$ from Theorem 4.4. Conversely, assume that the condition $CF(x_0, x - x_0) \cap (-\mathrm{Int}\mu(K)) = \emptyset$ holds for all $x \in X$, then from the definition of $\mathcal{V}$-pseudoconvexity, we have $[F(y), F(x)] \notin -\mathrm{Int}\mu(K)$ for any $x \in X$. If x_0 is not local weak minimal solution of (SP), there exists $x_1 \in X$ such that $F(x_1) <^l_K F(x_0)$, then $[F(x_1), F(x_0)] \in -\mathrm{Int}\mu(K)$ from Lemma 4.1. This is a contradiction. $\square$

Example 5.7. Let $g : (0, 1) \to \mathbb{R}^2$, $r : (0, 1) \to \mathbb{R}$ defined by

$$g(x) = (x, 1 - x), \quad r(x) = \min\{x, 1 - x\}, \quad \forall x \in (0, 1),$$

let $F : (0, 1) \to 2^{\mathbb{R}^2}$ defined by

$$F(x) = g(x) + r(x)U, \quad \forall x \in (0, 1),$$

where U is the unit ball in $\mathbb{R}^2$, let $K = \mathbb{R}^2_+ = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$, and choose W be a closed convex base of K^+. Then we can check that the map F is $\mathcal{V}$-convex, and $\frac{1}{2}$ is the unique minimal solution and every points of $(0, 1)$ are weak minimal solutions of the set optimization problem.

Also we can calculate directional derivatives:

$$DF(x, d) = \begin{cases} [\{(d, -d)\}, |d|U] & \text{if } (x, d) \in (0, \frac{1}{2}] \times (-\infty, 0) \cup [\frac{1}{2}, 1) \times [0, \infty), \\ [\{(d, -d)\} + |d|U, \{(0, 0)\}] & \\ \text{if } (x, d) \in (0, \frac{1}{2}) \times [0, \infty) \cup (\frac{1}{2}, 1) \times (-\infty, 0), & \end{cases}$$

and we have

(i) $[\{(d,-d)\}, |d|U] \in -\mu(K)$ if and only if $(d,-d) + K \supset |d|U$,
(ii) $[\{(d,-d)\}, |d|U] \in -\mathrm{Int}\mu(K)$ if and only if $(d,-d) + \mathrm{int}K \supset |d|U$,
(iii) $[\{(d,-d)\}+|d|U, \{(0,0)\}] \in -\mu(K)$ if and only if $(d,-d)+|d|U+K \ni (0,0)$,
(iv) $[\{(d,-d)\}+|d|U, \{(0,0)\}] \in -\mathrm{Int}\mu(K)$ if and only if $(d,-d)+|d|U+\mathrm{int}K \ni (0,0)$,

from Lemma 4.1. It is easy to show that conditions (i), (ii), and (iv) do not hold when $d \neq 0$, but condition (iii) always holds. These results show us, $\frac{1}{2}$ is the unique minimal solution from Theorem 4.5 and Corollary 5.3, and every points of $(0,1)$ are weak minimal solutions from Theorem 5.4 or Theorem 5.6.

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