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WEAK*-TOPOLOGY AND ALAOGLU'S THEOREM ON HYPERSPACE

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Abstract. Let X be a Banach space and X^* its dual space. The classical Alaoglu theorem states that closed balls B_r^* of X^* are weak*-compact. Suppose now $W^*CC(X^*)$ is the collection of all non-empty weak*-compact, convex subsets of X^*. We shall define a certain weak*-topology $\mathcal{T}_w^*$ on the hyperspace $W^*CC(X^*)$. If X is separable, we shall prove that closed balls $\mathcal{B}_r^*$ of $W^*CC(X^*)$ are weak*-compact ($\mathcal{T}_w^*$-compact).

1. Introduction

Let X be a Banach space and X^* its topological dual. Let $BCC(X)$; $WCC(X)$; $CC(X)$ denote the collection of all non-empty bounded closed convex subsets; weakly compact, convex subsets; and compact convex subsets of X respectively. Sequential weak convergence on $BCC(X)$ has been introduced and studied by De-Blasi and Myjak [3]. Other notions of weak convergence has also been studied ([1], [8], [9], [10]). On the other hand, the concept of weak topology on $CC(X)$ and $WCC(X)$ has been introduced and studied by Hu and company ([4], [5]). Suppose now $W^*CC(X^*)$ is the collection of all non-empty weak*-compact, convex subsets of X^*. We shall define a certain weak*-topology $\mathcal{T}_w^*$ on $W^*CC(X^*)$, and investigate which properties that the underlying space X^* possesses can be extended to the hyperspace $W^*CC(X^*)$. If X is separable, we shall prove that closed balls $\mathcal{B}_r^* = \{A \in W^*CC(X^*) | h(A, \{0\}) \leq r\}$ of the hyperspace $W^*CC(X^*)$ are weak*-compact ($\mathcal{T}_w^*$-compact) where h is the Hausdorff metric on $W^*CC(X^*)$.

2. Notations and preliminaries

Let X be a Banach space, X^* its topological dual and $BCC(X)$ be the collection of all non-empty bounded, closed, convex subsets of X. For $A, B \in BCC(X)$, define $N(A, \varepsilon) = \{x \in X : d(x, a) = \|x - a\| < \varepsilon \text{ for some } a \in A\}$ and $h(A, B) = \inf\{\varepsilon > 0 : A \subset N(B, \varepsilon) \text{ and } B \subset N(A, \varepsilon)\}$, equivalently $h(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\}$. Then h is known as the Hausdorff metric of the hyperspace $(BCC(X), h)$. Now let $CC(X)$ be the collection of all non-empty weakly compact, convex subsets of X and $WCC(X)$ be the collection of all non-empty weakly compact, convex subsets of X. For general X, we have $CC(X) \subsetneq WCC(X) \subsetneq BCC(X)$. If $\dim(X) < \infty$, we have $CC(X) = WCC(X) = BCC(X)$. Weak topologies on $CC(X)$ and $WCC(X)$ have been introduced and investigated related to some fixed point theorems ([4], [5]). To continue our discussion, we let $\mathbb{C}$ denote the complex plane, and $CC(\mathbb{C})$ the collection of all non-empty compact,

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convex subsets of $\mathbb{C}$. First, observe that for each $x^* \in X^*$, it follows from the weak continuity and linearity of x^* that for each non-empty weakly compact, convex subset A of $\mathbb{C}$ (i.e., $A \in WCC(X)$), we have $x^*(A) \in CC(\mathbb{C})$ (i.e., $x^*(A)$ is a compact, convex subset of $\mathbb{C}$). Thus each x^* maps the space $WCC(X)$ into $CC(\mathbb{C})$.

Lemma 2.1.

(a) *Suppose* $A, B \in WCC(X)$. *Then* $h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$ *for each* $x^* \in X^*$.

(b) *Suppose* $A^*, B^* \in W^*CC(X^*)$. *Then* $h(x(A^*), x(B^*)) \leq \|x\| h(A^*, B^*)$ *for each* $x \in X$.

Proof. Let $r > h(A, B)$. Then $A \subset N(B; r)$ and $B \subset N(A; r)$. Hence for each $a \in A$, there exists $b \in B$ such that $\|a - b\| < r$ and consequently $\|x^*(a) - x^*(b)\| \leq \|x^*(a - b)\| \leq \|x^*\|\|a - b\| < \|x^*\| \cdot r$, which in turn implies that $x^*(A) \subset N(x^*(B); \|x^*\| r)$. Similarly, $x^*(B) \subset N(x^*(A); \|x^*\| r)$. Hence $h(x^*(A), x^*(B)) \leq \|x^*\| r$, which implies that $h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$ and the proof is complete. □

Now let $W^*CC(X^*)$ be the collection of all non-empty, weak*-compact, convex subset of X^*. And since (X, τ_w) and (X^*, τ_w^*) are locally convex topological vector spaces, it follows immediately from Hahn-Banach Theorem that we have the following

Lemma 2.2.

(a) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in X^*$, *where* $A, B \in WCC(X)$.

(b) $A^* = B^*$ *if and only if* $x(A^*) = x(B^*)$ *for each* $x \in X$, *where* $A^*, B^* \in W^*CC(X^*)$.

Recall that the weak topology τ_w (or the X^*-topology) on X is defined to be the weakest topology which makes each $x^* : (X, \tau_w) \to (\mathbb{C}, |\cdot|)$ continuous. It follows from Lemma 2.1 that each $x^* : (WCC(X), h) \to (CC(\mathbb{C}), h)$ is continuous. Thus we may define the weak topology $\mathcal{T}_w$ (or the X^*-topology) on $WCC(X)$ to be the weakest topology such that each $x^* : (WCC(X), \mathcal{T}_w) \to (CC(\mathbb{C}), h)$ is continuous. In general, for any $F \subset X^*$, we shall define the weak topology $\mathcal{T}_F$ (or the F-topology) on $WCC(X)$ to be the weakest topology such that each $f : (WCC(X), \mathcal{T}_F) \to (CC(\mathbb{C}), h)$ is continuous for each $f \in F$. Similarly, if $F \subset X$ we define the F-topology $\mathcal{T}_F$ (X-topology $\mathcal{T}_w^*$) on $W^*CC(X^*)$ to be the weakest topology such that each $f : (W^*CC(X^*), \mathcal{T}_F) \to (CC(\mathbb{C}), h)$ (each x: $W^*CC(X^*), \mathcal{T}_w^*) \to (CC(\mathbb{C}), h)$) is continuous for each $f \in F$. A typical weak neighborhood of $A \in WCC(X)$ is denoted by $\mathcal{W}(A; x_1^*, \ldots, x_n^*; \varepsilon) = \{B \in WCC(X) : h(x_i^*(B), x_i^*(A)) < \varepsilon$ for $i = 1, 2, \ldots, n\}$, and a weak*-neighborhood of $A^* \in W^*CC(X^*)$ is denoted by $\mathcal{W}^*(A^*; x_1, x_2, \ldots, x_n; \varepsilon) = \{B^* \in W^*CC(X^*) : h(x_i(B^*), x_i(A^*)) < \varepsilon$ for $i = 1, 2, \ldots, n\}$. Let $\overline{X} = \{\overline{x} = \{x\} : x \in X\}$, (i.e., $\overline{X}$ is the hyperspace consisting of singletons). Then $(\overline{X}, h)$ may be identified with $(X, \|\ \|)$; and $(\overline{X}, \mathcal{T}_w)$ may be identified with (X, τ_w) naturally. Thus theorems on hyperspaces are extensions of their counterparts on original underlying spaces. We shall use small letters to denote elements of X and X^*; capital letters to denote elements of $WCC(X)$, $W^*CC(X^*)$ as well as other subsets of X and X^*; script letters to denote subsets of $WCC(X)$

and $W^*CC(X^*)$ respectively. Thus $B[0,r] = \{x \in X : \|x\| \leq r\}$ and $B^*[0,r]$ are closed balls of X and X^*; $\mathcal{B}[0,r] = \{A \in WCC(X) : h(A,\{0\}) \leq r\}$ and $\mathcal{B}^*[0,r]$ are closed balls of $WCC(X)$ and $W^*CC(X^*)$ respectively. Observe that from the way the weak topology $\mathcal{T}_w$ on $WCC(X)$ and the weak*-topology $\mathcal{T}_w^*$ on $W^*CC(X^*)$ are defined, the range space is $(CC(\mathbb{C}), h)$ rather than $(\mathbb{C}, |\cdot|)$, and consequently, properties of the space $(CC(\mathbb{C}), h)$ will be essential in later discussions. These properties are evidently special cases of the following more general Lemma 2.3.

Lemma 2.3.

(a) *The hyperspace* $(BCC(X), h)$ *is a complete metric space.*

(b) *If* $A_n, A \in BCC(X)$ *and* A_n *converges to* A, *then* A *is the collection of all subsequential limit points of* $\{A_n\}$ *(i.e.,* $A = \{a \in X : a = \lim_{i\to\infty} a_{n_i}$, *where* $a_{n_i} \in A_{n_i}\}$*).*

(c) *If* $\dim(X) < \infty$, *then every bounded sequence* $\{A_n\} \subseteq BCC(X)$ *has a subsequence* $\{A_{n_i}\}$ *such that* $\lim_{i\to\infty} A_{n_i} = A$.

Suppose X is a complex Banach space. Then X is also a real Banach space. For each $A \in WCC(X)$ and each real linear functional u, let $S_A(u) = \sup\{u(a) : a \in A\}$ be the support functional of A. Note that each u is now a function that maps $WCC(X)$ into $CC(\mathbb{R})$, where $CC(\mathbb{R})$ is the collection of all non-empty compact convex subsets of $\mathbb{R}$ (i.e., closed bounded intervals of $\mathbb{R}$). And for $[a_1, a_2], [b_1, b_2] \in CC(\mathbb{R})$, $h([a_1, a_2], [b_1, b_2]) = \max\{|b_2 - a_2|, |b_1 - a_1|\}$. DeBlasi and Myjak [3] defined that A_n converges weakly to A if and only if $S_{A_n}(u) \to S_A(u)$ for each real linear functional u. We shall establish that $\mathcal{T}_w$-convergence is equivalent to the weak convergence in the sense of DeBlasi and Myjak [3].

Lemma 2.4. *Let* X *be a complex Banach space and* $A, A_n \in WCC(X)$. *Then* $x^*(A_n) \to x^*(A)$ *for each complex linear functional* x^* *on* X *if and only if* $S_{A_n}(u) \to S_A(u)$ *for each real linear functional* u *on* X.

Proof. Suppose $x^*(A_n) \to x^*(A)$ with $x^* = u + iv$. Then u, v are real linear functionals on X and $u, v : WCC(X) \to CC(\mathbb{R})$. Thus $u(A_n) = [a_n, b_n]$ converges to $u(A) = [a, b]$ in the range space $(CC(\mathbb{R}), h)$. Consequently $h([a_n, b_n], [a, b]) = \max(|b_n - b|, |a_n - a|) \to 0$ as $n \to \infty$, which in turn implies $S_{A_n}(u) = b_n$ converges to $S_A(u) = b$.

On the other hand, suppose there exists some complex linear functional x^* such that $x^*(A_n) \not\to x^*(A)$ with $x^* = u + iv$. Then there exists $\varepsilon > 0$ and a subsequence $\{A_{n_k}\}$ such that $h(x^*(A_{n_k}), x^*(A)) \geq \varepsilon$ for $i = 1, 2, \ldots$, which in turn implies that either (a) $x^*(A) \not\subset N(x^*(A_{n_k}), \varepsilon)$ or (b) $x^*(A_{n_k}) \not\subset N(x^*(A), \varepsilon)$. It is elementary (but tedious) to show that either $S_{A_n}(u) \not\to S_A(u)$ or $S_{A_n}(v) \not\to S_A(v)$. Either way, we get a contradiction. Hence the lemma is proved. ☐

3. Main results

Suppose X is a Banach space, X^* its topological dual, $(WCC(X), \mathcal{T}_w)$ and $(W^*CC(X^*), \mathcal{T}_w^*)$ their corresponding hyperspaces. A subset $\mathcal{K} \subset WCC(X)$ is *bounded* (or originally bounded) if and only if there exists $M < \infty$ such that $\sup\{h(A, \{0\}) : A \in \mathcal{K}\} \leq M$. $\mathcal{K}$ is *weakly bounded* (or $\mathcal{T}_w$-bounded) if and only if for each $x^* \in X^*$, there exists $M_{x^*} < \infty$ such that $\sup\{h(x^*(A), \{0\}) : A \in \mathcal{K}\} \leq M_{x^*}$.

$\mathcal{K}$ is *weakly sequentially complete* if and only if every $\mathcal{T}_w$-Cauchy sequence $\{A_n\}$ of $\mathcal{K}$ converges to some $A \in \mathcal{K}$. $\mathcal{K}$ is *weakly sequentially compact* if and only if every infinite sequence $\{A_n\} \subset \mathcal{K}$ has a subsequence $\{A_{n_i}\}$ such that $\{A_{n_i}\}$ $\mathcal{T}_w$-converges to some $A \in \mathcal{K}$. A subset $\mathcal{K}^* \subset W^*CC(X^*)$ is *weak*-bounded*, *weak*-sequentially complete*, and *weak*-compact* are defined analogously. We shall now state and prove the following analog of the Uniform Boundedness Principle on hyperspaces. Like the Uniform Boundedness Principle, it is a very useful tool.

Theorem 3.1.

(a) *A subset $\mathcal{K} \subset WCC(X)$ is weakly bounded if and only if $\mathcal{K}$ is bounded.*
(b) *A subset $\mathcal{K}^* \subset W^*CC(X^*)$ is weak*-bounded if and only if $\mathcal{K}^*$ is bounded.*

Proof. We shall prove the non-trivial part of (b). Suppose $\mathcal{K}^*$ is weak*-bounded. Then for each $x \in X$, $\sup\{h(x(A^*), x(\{0\})) : A^* \in \mathcal{K}^*\} \le M_x < \infty$. Note that $h(x(A^*), x(\{0\})) = \sup\{\|x(a^*)\| : a^* \in A^*\}$. Thus if we set $K^* = \bigcup_{A^* \in \mathcal{K}^*} A^* = \bigcup_{A^* \in \mathcal{K}^*} \{a^* : a^* \in A^*\} \subseteq X^*$, we have $\sup\{h(x(A^*), x(\{0\})) : A^* \in \mathcal{K}^*\} = \sup_{A^* \in \mathcal{K}^*} [\sup\{\|x(a^*)\| : a^* \in A^*\}] = \sup\{\|x(a^*)\| : a^* \in K^*\} \le M_x < \infty$. Thus $K^* \subset X^*$ is a collection of linear functionals that is pointwise bounded at each $x \in X$. It follows now from the uniform boundedness principle that K^* is a bounded subset of X^*, i.e., $\sup\{\|a^*\| : a^* \in K^*\} \le N < \infty$ for some N. Now, for each $A^* \in \mathcal{K}^*$, we have $h(A^*, \{0\}) = \sup\{\|a^*\| : a^* \in A^*\} \le \sup\{\|a^*\| : a^* \in K^*\} \le N$ proving that $\mathcal{K}^*$ is a bounded subset of $(W^*CC(X^*), h)$. □

Corollary 3.2.

(a) *Suppose $\mathcal{K}^* \subset W^*CC(X^*)$ is $\mathcal{T}_w^*$-compact. Then $\mathcal{K}^*$ is $\mathcal{T}_w^*$-closed and bounded.*
(b) *Suppose $\mathcal{K} \subset WCC(X)$ is $\mathcal{T}_w$-compact. Then $\mathcal{K}$ is $\mathcal{T}_w$-closed and bounded.*

Proof. To prove (a), we let $\mathcal{K}^*$ be a $\mathcal{T}_w^*$-compact subset. Since $x : (W^*CC(X^*), \mathcal{T}_w^*) \to (CC(\mathbb{C}), h)$ is continuous, we have $x(\mathcal{K}^*)$ is a compact subset of $CC(\mathbb{C})$ and hence bounded for each $x \in X$. Hence $\mathcal{K}^*$ is weak*-bounded which in turn implies that $\mathcal{K}^*$ is bounded. Similarly, we may prove (b). □

Corollary 3.3.

(a) *Suppose $\{A_n^*\}$ is a $\mathcal{T}_w^*$-Cauchy sequence of $W^*CC(X^*)$. Then $\{A_n^*\}$ is bounded. Moreover, if A_n^* is $\mathcal{T}_w^*$-convergent to A^*, we have $h(A^*, \{0\}) \le \liminf_{n\to\infty} h(A_n^*, \{0\})$.*
(b) *Suppose $\{A_n\}$ is a $\mathcal{T}_w$-Cauchy sequence of $WCC(X)$. Then $\{A_n\}$ is bounded. Moreover if A_n is $\mathcal{T}_w$-convergent to A, we have $h(A, \{0\}) \le \liminf_{n\to\infty} h(A_n, \{0\})$.*

Proof. To prove (b), we let $\{A_n\}$ be a $\mathcal{T}_w$-Cauchy sequence. It follows that for each $x^* \in X^*$, $\{x^*(A_n)\}$ is a Cauchy sequence of the metric space $(CC(\mathbb{C}), h)$, and hence bounded. Thus $\{A_n\}$ is weakly bounded and it follows from Theorem 3.1 that $\{A_n\}$ is bounded. Suppose now $\{A_n\}$ is $\mathcal{T}_w$-convergent to A, and if $\liminf_{n\to\infty} h(A_n, \{0\}) < \alpha < h(A, \{0\})$. Then there exists some subsequence $\{A_{n_k}\}$ such that $h(A_{n_k}, \{0\}) < \alpha$. On the other hand, $h(A, \{0\}) = \sup\{\|a\| : a \in A\} > \alpha$ implies the existence of some $a_0 \in A$ such that $\|a_0\| > \alpha$. By Hahn-Banach Theorem, there exists some $x^* \in X^*$ with $\|x^*\| = 1$, and $|x^*(a_0)| = \|a_0\|$ and consequently, $h(x^*(A), \{0\}) = \sup\{|x^*(a)| : a \in A\} \ge |x^*(a_0)| = \|a_0\| > \alpha$. But, it follows from Lemma 2.1 that

$h(x^*(A_{n_k}), \{0\}) \leq \|x^*\| h(A_{n_k}, \{0\}) \leq h(A_{n_k}, \{0\}) < \alpha$. Hence $x^*(A_{n_k})$ does not converge to $x^*(A)$. That is a contradiction and the theorem is proved. Similar arguments establishes part (a). □

Theorem 3.4. *Suppose X is a separable Banach space. Then the closed ball $\mathcal{B}_r^* = \{A^* \in W^*CC(X^*) : h(A^*, \{0\}) \leq r\}$ of the hyperspace $(W^*CC(X^*), h)$ is weak*-sequentially compact (i.e., $\mathcal{T}_w^*$-sequentially compact).*

Proof. First, we show that $(W^*CC(X^*), \mathcal{T}_w^*)$ is sequentially complete. For that purpose, let $\{A_n^*\} \subset W^*CC(X^*)$ be $\mathcal{T}_w^*$-Cauchy. Then for each $x \in X$, $\{x(A_n^*)\}$ is a Cauchy sequence in $(CC(\mathbb{C}), h)$ and it follows from Lemma 2.3 (known as Blaschke's Convergence Theorem) that there exists some $D_x \in CC(\mathbb{C})$ such that $x(A_n^*)$ converges to D_x. Also, it follows from Corollary 3.3 that $\{A_n^*\}$ is bounded. Thus there exists some $r > 0$ such that $h(A_n^*, \{0\}) \leq r$ for $n = 1, 2, \ldots$. Consequently $A_n^* \subset B^*[0, r] \subset X^*$. Let $A^* = [\bigcap_{x \in X} (x^{-1})(D_x)] \cap B^*[0, r]$. Claim that $A^* \neq \varnothing$. For that purpose, we let $a_n^* \in A_n^* \subset B^*[0, r]$. X is separable implies that $B^*[0, r]$ is weak*-sequentially compact and hence $\{a_n^*\}$ has a subsequence $\{a_{n_i}^*\}$ such that $\{a_{n_i}^*\}$ weak*-converges to some $a^* \in B^*[0, r]$. Thus for each $x \in X$, we have $x(a_{n_i}^*)$ converges to $x(a^*)$. It follows from Lemma 2.3 that $x(a^*) \in D_x$ or $a^* \in x^{-1}(D_x)$ for each $x \in X$ which in turn implies that $a^* \in A^*$ showing that $A^* \neq \varnothing$. Next D_x is closed, convex and $x : X^* \to \mathbb{C}$ is weak*-continuous imply that $(x^{-1})(D_x)$ is weak*-closed and convex. Consequently A^* is a bounded, weak*-closed, convex set and hence weak*-compact by Alaoglu's Theorem. Thus $A^* \in W^*CC(X^*)$. Finally we shall show that A_n^* weak*-converges to A^*, i.e., $x(A_n^*)$ converges to $x(A^*)$ for each $x \in X$. Since $x(A_n^*)$ converges to D_x, it suffices to show that $x(A^*) = D_x$ for each $x \in X$. Fix $x_0 \in X$, we have $x_0(A^*) = x_0\{[\bigcap_{x \in X} (x^{-1})(D_x)] \cap B^*[0, r]\} \subset \{\bigcap_{x \in X} x_0[x^{-1}(D_x)] \cap x(B^*[0, r])\} \subset x_0[x_0^{-1}(D_{x_0})] = D_{x_0}$. On the other hand, let $d \in D_{x_0} = \lim_{n \to \infty} x_0(A_n^*)$. It follows then from Lemma 2.3 that there exists $a_{n_i}^* \in A_{n_i}^* \subseteq B^*[0, r]$ such that $x_0(a_{n_i}^*)$ converges to d. Again $B^*[0, r]$ is weak*-sequentially compact implies that $\{a_{n_i}^*\}$ has a subsequence $\{a_{n_i}^*\}$ (relabelling to simplify the notation) such that $\{a_{n_i}^*\}$ weak*-converges to some $a^* \in B^*[0, r]$. That is $\lim_{n \to \infty} x(a_{n_i}^*) = x(a^*)$. By Lemma 2.3, $x(a^*) \in D_x$ for each $x \in X$ which in turn implies that $a^* \in (x^{-1})(D_x)$ for each $x \in X$ and hence $a^* \in A^*$. Now that we have $\lim_{i \to \infty} x_0(a_{n_i}^*) = d$ as well as $\lim_{i \to \infty} x_0(a_{n_i}^*) = x_0(a^*)$, it follows $d = x_0(a^*) \in x_0(A^*)$ and hence $D_{x_0} \subset x_0(A^*)$. Thus $x_0(A^*) = D_{x_0}$ and since $x_0 \in X$ is arbitrary, we have $x(A^*) = D_x$ for each $x \in X$ and consequently $(W^*CC(X^*), \mathcal{T}_w^*)$ is sequentially complete.

Next, we let $\{x_i\}$ be a countable everywhere dense subset of X and $A_n^* \in \mathcal{B}_r^*$. Since $h(A_n^*, \{0\}) \leq r$, we have $h(x_1(A_n^*), \{0\}) \leq \|x_1\| h(A_n^*, \{0\}) \leq \|x_1\| r$, it follows from Blaschke's theorem that $\{x_1(A_n^*)\}$ has a convergent subsequence $\{x_1(A_{1n}^*)\}$ such that $x_1(A_{1n}^*)$ converges to $D_1 \in CC(\mathbb{C})$. Inductively, we construct a subsequence $\{A_{(i+1)n}^*\}$ of $\{A_{in}^*\}$ such that $x_{i+1}(A_{(i+1)n}^*)$ converges to $D_{i+1} \in CC(\mathbb{C})$. Consider the diagonal sequence $\{A_{nn}^*\}$. Claim that $\{A_{nn}^*\}$ is $\mathcal{T}_w^*$-Cauchy (i.e., $x(A_{nn}^*)$ is Cauchy in $(CC(\mathbb{C}), h)$ for each $x \in X$). Since $\{x_i\}$ is dense, for any given $\varepsilon > 0$ and $x \in X$, there exists some x_i such that $\|x_i - x\| < \varepsilon/(3r)$. Also

$\{x_i(A^*_{nn})\}$ is Cauchy implies that there exists some N such that $m, n \geq N$ implies $h(x_i(A^*_{mm}), x_i(A^*_{nn})) < \varepsilon/3$. Hence $h(x(A^*_{mm}), x(A^*_{nn})) \leq h(x(A^*_{mm}), x_i(A^*_{mm})) + h(x_i(A^*_{mm}), x_i(A^*_{nn})) + h(x_i(A^*_{nn}), x(A^*_{nn})) \leq \|x - x_i\| h(A^*_{mm}, \{0\}) + h(x_i(A^*_{mm}), x_i(A^*_{nn})) + \|x_i - x\| h(A^*_{nn}, \{0\}) < (\varepsilon/3r) \cdot r + \varepsilon/3 + (\varepsilon/3r) \cdot r < \varepsilon$ and the claim is proved. It follows now from the previous part of this proof that there exists some $A^* \in W^*CC(X^*)$ such that $\{A^*_{nn}\}$ $\mathcal{T}^*_w$-converges to A^*. It follows from Corollary 3.3 that $h(A^*, \{0\}) \leq \liminf_{n\to\infty} h(A^*_{nn}, \{0\}) \leq r$. Hence $A^* \in \mathcal{B}^*_r$ and the theorem is proved. □

We need the following lemmas to obtain our final results.

Lemma 3.5. *Suppose X is a Banach space and $F \subset G \subset X^*$ such that F is a norm dense subset of G. Then the restrictions of the F-topology $\mathcal{T}_F$ and the G-topology $\mathcal{T}_G$ are equivalent when restricted to bounded subsets of the hyperspace $WCC(X)$.*

Proof. Since $F \subset G$, we have $\mathcal{T}_F \subset \mathcal{T}_G$. Hence, it suffices to show that if $\{A_\alpha\} \subset WCC(X)$ is a net such that $\sup_\alpha\{h(A_\alpha, \{0\}) \leq r\}$ and A_α converges to A in $\mathcal{T}_F$, then A_α converges to A in $\mathcal{T}_G$. For that purpose, let $g \in G$ and $\varepsilon > 0$ be given. Since F is norm dense in G, we may choose $f \in F$ sucht that $\|f - g\| < \frac{\varepsilon}{3r}$. Since A_α converges to A in $\mathcal{T}_F$, we may choose α_0 sucht that $\alpha \geq \alpha_0$ implies that $h(f(A_\alpha), f(A)) < \varepsilon/3$. We then have $h(g(A_\alpha), g(A)) \leq h(g(A_\alpha), f(A_\alpha)) + h(f(A_\alpha), f(A)) + h(f(A), g(A)) \leq \|g - f\| h(A_\alpha, \{0\}) + h(f(A_\alpha), f(A)) + \|f - g\| h(A, \{0\}) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ whenever $\alpha \geq \alpha_0$. Thus $g(A_\alpha)$ converges to $g(A)$ in $(CC(\mathbb{C}), h)$ for every $g \in G$. Consequently A_α converges to A in $\mathcal{T}_G$ and the proof is complete. □

Lemma 3.6. *Suppose $F = \{f_1, f_2, \ldots, f_n, \ldots\} \subset X^*$ is a countable family that separates points of $WCC(X)$ (i.e. for $A, B \in WCC(X)$ with $A \neq B$, there exists $f \in F$ such that $h(f(A), f(B)) > 0$). Then the F-topology $\mathcal{T}_F$ on $WCC(X)$ is metrizable.*

Proof. $d(A, B) = \sum_{n=1}^\infty \frac{h(f_n(A), f_n(B))}{2^n[1 + h(f_n(A), f_n(B))]}$. Suppose $A, B \in WCC(X)$ with $A \neq B$. Since F separates points, it follows that there exists some $f_n \in F$ such that $h(f_n(A), f_n(B)) > 0$ which in turn implies that $d(A, B) > 0$. Consequently, $d(A, B) = 0$ if and only if $A = B$. The remaing properties to establish that d is a metric can be routinely verified. Now, suppose $\mathcal{B}_d(A; \varepsilon) = \{B \in WCC(X) \mid d(B, A) < \varepsilon\}$ is given, choose k large enough such that $\sum_{n=k+1}^\infty \frac{1}{2^n} < \frac{\varepsilon}{2}$. Claim that $\mathcal{W}(A; f_1, f_2, \ldots, f_k; \frac{\varepsilon}{4}) \subseteq \mathcal{B}_d(A, \varepsilon)$. To preve the claim, we let $B \in \mathcal{W}(A)$ and we have $h(f_n(A), f_n(B)) < \frac{\varepsilon}{4}$ for $n = 1, 2, \ldots, k$. Hence $\sum_{n=1}^k \frac{1}{2^n} \cdot \frac{h(f_n(A), f_n(B))}{[1 + h(f_n(A), f_n(B))]} \leq \sum_{n=1}^k \frac{1}{2^n} \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{4} \sum_{n=1}^k \frac{1}{2^n} < \frac{\varepsilon}{4} \cdot 2 = \frac{\varepsilon}{2}$. Also $\sum_{n=k+1}^\infty \frac{1}{2^n} \cdot \frac{h(f_n(A), f_n(B))}{[1 + h(f_n(A), f_n(B))]} \leq \sum_{n=k+1}^\infty \frac{1}{2^n} < \frac{\varepsilon}{2}$. Consequently $d(A, B) = \sum_{n=1}^k \frac{1}{2^n} \cdot \frac{h(f_n(A), f_n(B))}{[1 + h(f_n(A), f_n(B))]} + \sum_{n=k+1}^\infty \frac{1}{2^n} \cdot \frac{h(f_n(A), f_n(B))}{[1 + h(f_n(A), f_n(B))]} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and the claim is proved. □

Conversely, suppose a $\mathcal{T}_F$-neighborhood $\mathcal{W}(A; f_{n_1}, \ldots, f_{n_j}; \varepsilon)$ is given. Let $k = \max(n_1, \ldots, n_j)$, then $\mathcal{W}(A; f_1, \ldots, f_k; \varepsilon) \subset \mathcal{W}(A; f_{n_1}, \ldots, f_{n_j}; \varepsilon)$. Claim that $\mathcal{B}_d(A; \varepsilon/2^k(1+\varepsilon)) \subset \mathcal{W}(A; f_{n_1}, \ldots, f_{n_j}; \varepsilon)$. Indeed, if $d(A, B) = \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{h(f_n(A), f_n(B))}{[1 + h(f_n(A), f_n(B))]} < \frac{\varepsilon}{2^k(1+\varepsilon)}$, then $\frac{h(f_n(A), f_n(B))}{1 + h(f_n(A), f_n(B))} < \frac{\varepsilon}{1+\varepsilon}$, which in turn implies that $h(f_n(A), f_n(B) < \varepsilon$

for $n = 1, 2, \ldots, k$. Thus $B \in \mathcal{W}(A; f_1, \ldots, f_k; \varepsilon) \subset \mathcal{W}(A; f_n, \ldots, f_{n_j}; \varepsilon)$. Hence the F-topology $\mathcal{T}_F$ and the metric d on $WCC(X)$ are equivalent.

Theorem 3.7. *Suppose X is a separable Banach space. Then the weak*-topology $\mathcal{T}_w^*$ of $W^*CC(X^*)$ restricted to $\mathcal{B}_1^* = \{A \in W^*CC(X^*) : h(A, \{0\}) \leq 1\}$ is metrizable.*

Proof. Suppose X is separable and $F \subset X$ is a countable norm dense subset of X. Since $F \subset X \subset X^{**}$, it follows from Lemma 3.6 that the F-topology $\mathcal{T}_F$ on $W^*CC(X^*)$ is metrizable. Also it follows from Lemma 3.5 that $\mathcal{T}_F$ and $\mathcal{T}_w^*$ (i.e., the X-topology) when restricted to the bounded let $\mathcal{B}_1^*$ are equivalent. Hence the theorem is proved. □

Finally, we have the following theorem which is an extension of the classical Alaoglu theorem under the additional condition that X is separable.

Theorem 3.8. *Suppose X is a separable Banach space. Then the closed ball $\mathcal{B}_1^* \subset W^*CC(X^*)$ is weak*-compact (i.e., $\mathcal{T}_w^*$-compact).*

Proof. By Theorem 3.7, $(\mathcal{B}_1^*, \mathcal{T}_w^*)$ is metrizable. Also by Theorem 3.4, $(\mathcal{B}_1^*, \mathcal{T}_w^*)$ is sequentially compact. Thus $(B_1^*, \mathcal{T}_w^*)$ is compact since compactness and sequentially compact are equivalent on metric space and the proof is complete. □

Corollary 3.9. *Suppose X is a reflexive separable Banach space. Then the closed ball $\mathcal{B}r$ of $WCC(X)$ is weakly compact as well as weakly sequentially compact.*

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