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FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS OF NONEXPANSIVE TYPE IN BANACH SPACES

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Abstract. In this paper, we introduce two nonlinear mappings of nonexpansive type which are connected with resolvents of maximal monotone operators in a Banach space. We first study some properties of these mappings. Next, we prove fixed point theorems and convergence theorems for these mappings.

1. Introduction

Let E be a smooth Banach space and let E^* be the dual of E. The function $V : E \times E \to \mathbb{R}$ is defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$, where J is the normalized duality mapping from E into E^*. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [21] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping $T : C \to C$ is called relatively nonexpansive [3, 19] if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$V(p, Tx) \leq V(p, x)$$

for each $x \in C$ and $p \in F(T)$. A mapping $T : C \to C$ is called generalized nonexpansive [8, 9] if $F(T) \neq \emptyset$ and

$$V(Tx, p) \leq V(x, p)$$

for each $x \in C$ and $p \in F(T)$. Many researcher have studied the asymptotic behavior of these mappings; see [3–5, 8–13, 15, 16, 18, 19, 21] and the references mentioned there.

Recently, Kohsaka and Takahashi [17] introduced the following mapping: A mapping $T : C \to C$ is of firmly nonexpansive type if

$$V(Tx, x) + V(Ty, y) + V(Tx, Ty) + V(Ty, Tx) \leq V(Ty, x) + V(Tx, y)$$

for each $x, y \in C$. The class of firmly nonexpansive type mappings contains the class of firmly nonexpansive mappings in Hilbert spaces and the class of relative resolvents of maximal monotone operators in Banach spaces. Further, they studied the existence of a fixed point for a firmly nonexpansive type mapping in a smooth Banach space.

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In this paper, motivated by Kohsaka and Takahashi [17], we introduce two nonlinear mappings of nonexpansive type which are connected with resolvents of maximal monotone operators in a smooth Banach space. We first study some properties of these mappings and show an important example of these mappings. Next, we prove fixed point theorems and convergence theorems for these mappings.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

A Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\} (=: S(E))$. In this case, the norm of E is said to be Gâteaux differentiable. The space E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S(E)$, the limit (2.1) is attained uniformly for $x \in S(E)$. The norm of E is said to be Fréchet differentiable if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit (2.1) is attained uniformly for $x, y \in S(E)$.

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^* \rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [6, 24] for more details).

The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. We also know the following properties (see [6, 23, 24] for details):

(1) J is monotone and $Jx \neq \emptyset$ for each $x \in E$.
(2) If E is reflexive, then J is surjective.
(3) If E is strictly convex, then J is one to one and strictly monotone.
(4) If E is smooth, then J is single valued and norm to weak* continuous.
(5) If E is smooth, strictly convex and reflexive, then the duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$.
(6) If E has a Fréchet differentiable norm, then J is norm to norm continuous.

(7) If E has a uniformly Gâteaux differentiable norm, then J is norm to weak* uniformly continuous on each bounded subset of E.
(8) If E is uniformly smooth, then the duality mapping J is norm to norm uniformly continuous on each bounded set of E.
(9) If E is uniformly convex, then E is reflexive and strictly convex.
(10) E is uniformly convex if and only if E^* is uniformly smooth.

Let E be a smooth Banach space and consider the following function studied in Alber [1] and Kamimura and Takahashi [14]:

$$V(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$. It is obvious from the definition of V that

$$(\|x\| - \|y\|)^2 \le V(x,y) \le (\|x\| + \|y\|)^2 \tag{2.2}$$

for each $x, y \in E$. We also know that

$$V(x,y) = V(x,z) + V(z,y) + 2\langle x - z, Jz - Jy\rangle \tag{2.3}$$

for each $x, y, z \in E$ (see [14]). It is also easy to see that if E is additionally assumed to be strictly convex, then

$$V(x,y) = 0 \;\Leftrightarrow\; x = y.$$

See [19] for more details. The following lemmas are well-known.

Lemma 2.1 ([14]). Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2 ([14]). Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$g(\|x - y\|) \le V(x,y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.

Let C be a nonempty closed convex subset of a smooth Banach space E and let T be a mapping from C into itself. A point p in C is said to be a generalized asymptotic fixed point [12] of T if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $\|Jx_n - JTx_n\| \to 0$. The set of all generalized asymptotic fixed points of T is denoted by $\check{F}(T)$. A mapping $T : C \to C$ is called firmly generalized nonexpansive [13] if $F(T) \neq \emptyset$ and

$$V(x,Tx) + V(Tx,p) \le V(x,p)$$

for each $x \in C$ and $p \in F(T)$. It is clear that a firmly generalized nonexpansive mapping is generalized nonexpansive in a smooth Banach space (see [13] for more details). Let D be a nonempty subset of E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for each $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a retraction if $Rx = x$ for each $x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely decided (see [8, 9]). Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A nonempty subset D of E is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto D (see [8, 9] for more details). The set of all fixed points of such a sunny generalized nonexpansive retraction of E onto D is, of course, D.

We know the following results for sunny generalized nonexpansive retractions and sunny generalized nonexpansive retracts in Banach spaces.

Lemma 2.3 ([8, 9])**.** Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R be a retraction of E onto D. Then R is sunny and generalized nonexpansive if and only if

$$\langle x - Rx, JRx - Jy \rangle \geq 0$$

for each $x \in E$ and $y \in D$.

Lemma 2.4 ([9, 11])**.** Let D be a nonempty subset of a reflexive, strictly convex and smooth Banach space E. If R is the sunny generalized nonexpansive retraction of E onto D, then R is firmly generalized nonexpansive.

Lemma 2.5 ([12])**.** Let D be a nonempty subset of a reflexive, strictly convex and smooth Banach space E and let R be a sunny generalized nonexpansive retraction of E onto D. Then $\check{F}(R) = F(R)$.

Theorem 2.6 ([16])**.** Let E be a reflexive, strictly convex and smooth Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent.

(1) D is a sunny generalized nonexpansive retract of E,
(2) D is a generalized nonexpansive retract of E,
(3) JD is closed and convex.

In this case, D is closed.

Theorem 2.7 ([12])**.** Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.

3. Two nonlinear mappings of nonexpansive type

In this section, we introduce two nonlinear mappings of nonexpansive type in a smooth Banach space. Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is of generalized nonexpansive type if

$$V(Tx, Ty) + V(Ty, Tx) \leq V(x, Ty) + V(y, Tx) \tag{3.1}$$

for all $x, y \in C$. A mapping $T : C \to C$ is of firmly generalized nonexpansive type if

$$V(x, Tx) + V(y, Ty) + V(Tx, Ty) + V(Ty, Tx) \leq V(x, Ty) + V(y, Tx) \tag{3.2}$$

for all $x, y \in C$. We obtain the following four results concerning these mappings.

Lemma 3.1. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let T be a firmly generalized nonexpansive type mapping from C into C. Then T is of generalized nonexpansive type.

Proof. From the definition of T and properties of V, we have that for each $x, y \in C$,

$$\begin{aligned} V(x,Ty) + V(y,Tx) &\geq V(x,Tx) + V(y,Ty) + V(Tx,Ty) + V(Ty,Tx) \\ &\geq V(Tx,Ty) + V(Ty,Tx). \end{aligned}$$

This implies that T is of generalized nonexpansive type. □

Lemma 3.2. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let T be a generalized nonexpansive type mapping from C into C. If $F(T)$ is nonempty, then T is generalized nonexpansive.

Proof. From the definition of T and $F(T) \neq \emptyset$, we have that for each $x \in C$ and $p \in F(T)$,

$$V(Tx,Tp) + V(Tp,Tx) \leq V(x,Tp) + V(p,Tx).$$

From $Tp = p$, we have

$$V(Tx,p) + V(p,Tx) \leq V(x,p) + V(p,Tx)$$

and hence $V(Tx,p) \leq V(x,p)$. This implies that T is generalized nonexpansive. □

Lemma 3.3. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let T be a firmly generalized nonexpansive type mapping from C into C. If $F(T)$ is nonempty, then T is firmly generalized nonexpansive.

Proof. From the definition of T and $F(T) \neq \emptyset$, we have that for each $x \in C$ and $p \in F(T)$,

$$V(x,Tx) + V(p,Tp) + V(Tx,Tp) + V(Tp,Tx) \leq V(x,Tp) + V(p,Tx).$$

From $Tp = p$, we have

$$V(x,Tx) + V(p,p) + V(Tx,p) + V(p,Tx) \leq V(x,p) + V(p,Tx)$$

and hence $V(x,Tx) + V(Tx,p) \leq V(x,p)$. This implies that T is firmly generalized nonexpansive. □

Lemma 3.4. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then, T is of firmly generalized nonexpansive type if and only if

$$\langle x - Tx - (y - Ty), JTx - JTy \rangle \geq 0 \tag{3.3}$$

for each $x, y \in C$.

Proof. Let $x, y \in C$. Then, by (2.3) we obtain that

$$V(x,Ty) = V(x,Tx) + V(Tx,Ty) + 2\langle x - Tx, JTx - JTy \rangle$$

and

$$V(y,Tx) = V(y,Ty) + V(Ty,Tx) + 2\langle y - Ty, JTy - JTx \rangle.$$

From these equalities, we have

$$V(x,Ty) + V(y,Tx)$$

$$
\begin{aligned}
&= V(x,Tx)+V(y,Ty)+V(Tx,Ty)+V(Ty,Tx)\\
&\quad +2\langle x-Tx, JTx-JTy\rangle+2\langle y-Ty, JTy-JTx\rangle\\
&= V(x,Tx)+V(y,Ty)+V(Tx,Ty)+V(Ty,Tx)\\
&\quad +2\langle (x-Tx)-(y-Ty), JTx-JTy\rangle
\end{aligned}
$$

and hence

$$
\begin{aligned}
&2\langle (x-Tx)-(y-Ty), JTx-JTy\rangle\\
&= \big\{V(x,Ty)+V(y,Tx)\big\}-\big\{V(x,Tx)+V(y,Ty)+V(Tx,Ty)+V(Ty,Tx)\big\}.
\end{aligned}
$$

Therefore, the inequality (3.3) is equivalent to the inequality (3.2). This completes the proof. □

Next, let us show an important example of these mappings: Let E be a reflexive, strictly convex and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. For each $r > 0$ and $x \in E$, consider the set

$$J_r x := \{z \in E : x \in z + rBJz\}.$$

Then $J_r x$ consists of one point. We also denote the domain and the range of J_r by $D(J_r) = R(I + rBJ)$ and $R(J_r) = D(BJ)$, respectively. Such J_r is called the generalized resolvent of B and is denoted by

$$J_r = (I + rBJ)^{-1}.$$

See [9,10] for more details. We obtain the following result for generalized resolvents of maximal monotone operators in a Banach space.

Lemma 3.5. Let E be a reflexive, smooth and strictly convex Banach space, let $B \subset E^* \times E$ be a maximal monotone operator and let J_r be the generalized resolvent of B for $r > 0$. Then J_r is of firmly generalized nonexpansive type.

Proof. Let $x, y \in E$ and $r > 0$. Put $x_r := J_r x$ and $y_r := J_r y$. Then, from the definition of the generalized resolvent, we obtain that

$$x \in x_r + rBJx_r \text{ and } y \in y_r + rBJy_r.$$

Therefore, we get

$$\frac{x - x_r}{r} \in BJx_r \text{ and } \frac{y - y_r}{r} \in BJy_r.$$

From the monotonicity of B, we have that

$$\langle \frac{x - x_r}{r} - \frac{y - y_r}{r}, Jx_r - Jy_r\rangle \geq 0.$$

Since $r > 0$, we get

$$\langle x - x_r - (y - y_r), Jx_r - Jy_r\rangle \geq 0.$$

From Lemma 3.4, we have that J_r is of firmly generalized nonexpansive type. □

From Lemmas 3.1 and 3.5, we also obtain the following result.

Lemma 3.6. Let E be a reflexive, smooth and strictly convex Banach space, let $B \subset E^* \times E$ be a maximal monotone operator and let J_r be the generalized resolvent of B for $r > 0$. Then J_r is of generalized nonexpansive type.

Proof. From Lemma 3.5, we obtain that the generalized resolvent J_r is of firmly generalized nonexpansive type. Further, by Lemma 3.1, we obtain that J_r is of generalized nonexpansive type. □

4. Fixed point theorems

In this section, using the technique developed by Takahashi [22], we first prove a fixed point theorem for generalized nonexpansive type mappings in a Banach space.

Theorem 4.1. Let E be a reflexive, smooth and strictly convex Banach space and let T be a generalized nonexpansive type mapping from E into itself. Then the following are equivalent:

(1) $F(T)$ is nonempty;
(2) $\{T^n x\}$ is bounded for some $x \in E$.

Proof. It is clear that (1) implies (2). We show that (2) implies (1). Suppose that $\{T^n x\}$ is bounded for some $x \in E$. From the definition of T, we have that

$$V(T^{k+1}x, Ty) + V(Ty, T^{k+1}x) \le V(T^k x, Ty) + V(y, T^{k+1}x) \tag{4.1}$$

for each $k = 0, 1, \dots$ and $y \in E$. It follows from (2.3) that

$$V(Ty, T^{k+1}x) = V(Ty, y) + V(y, T^{k+1}x) + 2\langle Ty - y, Jy - JT^{k+1}x\rangle$$

and hence

$$V(y, T^{k+1}x) - V(Ty, T^{k+1}x) = -V(Ty, y) + 2\langle y - Ty, Jy - JT^{k+1}x\rangle. \tag{4.2}$$

Combining this with (4.1), we obtain

$$0 \le V(T^k x, Ty) - V(T^{k+1}x, Ty) - V(Ty, y) + 2\langle y - Ty, Jy - JT^{k+1}x\rangle \tag{4.3}$$

for each $k = 0, 1, \dots$. Summing these inequalities with respect to $k = 0, 1, \dots, n-1$ and then dividing by n, we have

$$0 \le \frac{1}{n}V(x, Ty) - \frac{1}{n}V(T^n x, Ty) - V(Ty, y) + 2\langle y - Ty, Jy - S_n^* x\rangle, \tag{4.4}$$

where $S_n^* := \frac{1}{n}\sum_{k=1}^{n} JT^k$. Since $\{T^n x\}$ is bounded, $\{JT^n x\}$ is also bounded. So, we have that $\{S_n^* x\}$ is bounded. Let $\{S_{n_i}^* x\}$ be a subsequence of $\{S_n^* x\}$ such that $S_{n_i}^* x \rightharpoonup p^*$ for some $p^* \in E^*$. Letting $n_i \to \infty$ in (4.4), we get

$$0 \le -V(Ty, y) + 2\langle y - Ty, Jy - p^*\rangle. \tag{4.5}$$

Putting $p := J^{-1}p^*$ and taking $y = p$, we have

$$0 \le -V(Tp, p) + 2\langle p - Tp, Jp - Jp\rangle.$$

Therefore, we have that $V(Tp, p) \le 0$ and hence $V(Tp, p) = 0$. So, we have $Tp = p$. This implies that $F(T)$ is nonempty. □

As a direct consequence of Theorem 4.1 and Lemma 3.1, we obtain the following result.

Theorem 4.2. Let E be a reflexive, smooth and strictly convex Banach space and let T be a firmly generalized nonexpansive type mapping from E into itself. Then the following are equivalent:

(1) $F(T)$ is nonempty;

(2) $\{T^n x\}$ is bounded for some $x \in E$.

We also have the following theorem.

Theorem 4.3. Let E be a reflexive and smooth Banach space and E^* has a uniformly Gâteaux differentiable norm. Let T be a generalized nonexpansive type mapping from E into itself. If $F(T)$ is nonempty, then $\check{F}(T) = F(T)$.

Proof. It is obvious that $F(T) \subset \check{F}(T)$. We show that $\check{F}(T) \subset F(T)$. Let $p \in \check{F}(T)$, there exists a sequence $\{x_n\} \subset E$ such that $Jx_n - JTx_n \to 0$ and $Jx_n \rightharpoonup Jp$. From the definition of T, we have that

$$V(Tx_n, Tp) + V(Tp, Tx_n) \leq V(x_n, Tp) + V(p, Tx_n). \tag{4.6}$$

By (2.3), we have that

$$V(Tp, Tx_n) = V(Tp, p) + V(p, Tx_n) + 2\langle Tp - p, Jp - JTx_n\rangle$$

and hence

$$V(p, Tx_n) - V(Tp, Tx_n) = -V(Tp, p) + 2\langle p - Tp, Jp - JTx_n\rangle. \tag{4.7}$$

From (4.6) and (4.7), we obtain that

$$V(Tx_n, Tp) - V(x_n, Tp) \leq -V(Tp, p) + 2\langle p - Tp, Jp - JTx_n\rangle. \tag{4.8}$$

On the other hand, we have from the definition of V that

$$\begin{aligned} & V(Tx_n, Tp) - V(x_n, Tp) \\ = \; & \|Tx_n\|^2 - \|x_n\|^2 - 2\langle Tx_n - x_n, JTp\rangle \\ = \; & \big(\|Tx_n\| - \|x_n\|\big)\big(\|Tx_n\| + \|x_n\|\big) - 2\langle Tx_n - x_n, JTp\rangle \\ = \; & \big(\|JTx_n\| - \|Jx_n\|\big)\big(\|Tx_n\| + \|x_n\|\big) - 2\langle Tx_n - x_n, JTp\rangle \\ \geq \; & -\|Jx_n - JTx_n\|\big(\|Tx_n\| + \|x_n\|\big) - 2\langle Tx_n - x_n, JTp\rangle. \end{aligned}$$

Since E^* has a uniformly Gâteaux differentiable norm, the duality mapping J^{-1} on E^* is uniformly norm to weak continuous on each bounded set. Therefore, from $Jx_n - JTx_n \to 0$, we obtain that

$$0 \leq \liminf_{n\to\infty} \big\{V(Tx_n, Tp) - V(x_n, Tp)\big\} \tag{4.9}$$

From $Jx_n - JTx_n \to 0$ and $Jx_n \rightharpoonup Jp$, we have $JTx_n \rightharpoonup Jp$. By (4.8) and (4.9), we get

$$0 \leq -V(Tp, p) + 2\langle p - Tp, Jp - Jp\rangle.$$

Therefore, we have that $V(Tp, p) \leq 0$ and hence $V(Tp, p) = 0$. So, we have $p \in F(T)$. This implies that $\check{F}(T) \subset F(T)$. □

5. Weak convergence theorems

In this section, we prove weak convergence theorems for firmly generalized nonexpansive type mappings in a Banach space. Before proving our results, we first obtain the following result.

Lemma 5.1. Let E be a smooth and uniformly convex Banach space and let T be a generalized nonexpansive mapping from E into itself. If $F(T)$ is nonempty, then $\{RT^n x\}$ converges strongly to some element of $F(T)$ for each $x \in E$, where R is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. Let $x \in E$. Then we have from Lemma 2.4 that

$$\begin{aligned} & V(T^{n+1}x, RT^{n+1}x) \\ \leq\ & V(T^{n+1}x, RT^{n+1}x) + V(RT^{n+1}x, RT^n x) \\ \leq\ & V(T^{n+1}x, RT^n x) \\ \leq\ & V(T^n x, RT^n x) \end{aligned}$$

for each $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} V(T^n x, RT^n x)$ exists. It follows from Lemma 2.3 and (2.3) that, for each $k \in \mathbb{N}$,

$$\begin{aligned} & V(T^{n+k}x, RT^n x) \\ =\ & V(T^{n+k}x, RT^{n+k}x) + V(RT^{n+k}x, RT^n x) \\ & +2\langle T^{n+k}x - RT^{n+k}x, JRT^{n+k}x - JRT^n x\rangle \\ \geq\ & V(T^{n+k}x, RT^{n+k}x) + V(RT^{n+k}x, RT^n x) \end{aligned}$$

and hence

$$\begin{aligned} V(RT^{n+k}x, RT^n x) &\leq V(T^{n+k}x, RT^n x) - V(T^{n+k}x, RT^{n+k}x) \\ &\leq V(T^n x, RT^n x) - V(T^{n+k}x, RT^{n+k}x). \end{aligned}$$

Since $F(T) \neq \emptyset$, we also obtain

$$V(RT^n x, p) \leq V(x, p)$$

for some $p \in F(T)$ and hence $\{RT^n x\}$ is bounded. Using Lemma 2.2, we have, for $m, n \in \mathbb{N}$ with $m > n$

$$g(\|RT^n x - RT^m x\|) \leq V(T^n x, RT^n x) - V(T^m x, RT^m x),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function such that $g(0) = 0$. Then the properties of g yield that $\{RT^n x\}$ is a Cauchy sequence. Since E is complete and $F(T)$ is closed, $\{RT^n x\}$ converges strongly to some point u in $F(T)$. □

Next, we obtain the following theorem for firmly generalized nonexpansive type mappings in a Banach space.

Theorem 5.2. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let T be a firmly generalized nonexpansive type mapping from E into itself. If the duality mapping J is weakly sequentially continuous, then the following are equivalent:

(1) $F(T)$ is nonempty;
(2) $\{T^n x\}$ converges weakly for each $x \in E$.

In this case, $\{T^n x\}$ converges weakly to an element of $F(T)$.

Proof. We first show that (2) implies (1). Let $x \in E$. Since $\{T^n x\}$ converges weakly, then $\{T^n x\}$ is bounded. From Theorem 4.2, we obtain that $F(T)$ is nonempty.

Next, we show that (1) implies (2). Let $x \in E$ and $p \in F(T)$. Since T is a firmly generalized nonexpansive type mapping from E into itself, from Lemma 3.3 T is firmly generalized nonexpansive. So, we have that

$$V(T^{n+1}x, p) \leq V(T^n x, T^{n+1}x) + V(T^{n+1}x, p) \leq V(T^n x, p) \tag{5.1}$$

for each $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} V(T^n x, p)$ exists. From (5.1), we obtain that

$$V(T^n x, T^{n+1}x) \leq V(T^n x, p) - V(T^{n+1}x, p)$$

for each $n \in \mathbb{N}$. Since $\{V(T^n x, p)\}$ converges, it follows that

$$\lim_{n\to\infty} V(T^n x, T^{n+1}x) = 0. \tag{5.2}$$

Since $F(T)$ is nonempty, by Theorem 4.2, $\{T^n x\}$ is bounded. From (5.2) and Lemma 2.1, we have that

$$\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0. \tag{5.3}$$

On the other hand, from Lemma 3.4, we have that

$$\langle (T^n x - T^{n+1}x) - (y - Ty), JT^{n+1}x - JTy \rangle \geq 0 \tag{5.4}$$

for each $n \in \mathbb{N}$ and $y \in E$. Let $\{T^{n_i} x\}$ be a subsequence of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p$ for some $p \in E$. Since the duality mapping J is weakly sequentially continuous, we have that $JT^{n_i} x \rightharpoonup Jp$. Since J is norm to norm uniformly continuous on each bounded set, by (5.3) we have $\|JT^n x - JT^{n+1}x\| \to 0$ and hence $JTT^{n_i} x \rightharpoonup Jp$. Letting $n_i \to \infty$ in (5.4), we obtain that

$$\langle -(y - Ty), Jp - JTy \rangle \geq 0 \tag{5.5}$$

for each $y \in E$. Put $y = p$ in (5.5). Then we have

$$\langle Tp - p, Jp - JTp \rangle \geq 0 \tag{5.6}$$

Since J is strictly monotone, it follows that $p = Tp$. This implies that $p \in F(T)$.

Let $\{T^{n_i} x\}$ and $\{T^{n_j} x\}$ be two subsequences of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p_1$ and $T^{n_j} x \rightharpoonup p_2$. As above, we have $p_1, p_2 \in F(T)$. Put

$$a = \lim_{n\to\infty} \Big(V(T^n x, p_1) - V(T^n x, p_2) \Big).$$

Since

$$V(T^n x, p_1) - V(T^n x, p_2) = 2\langle T^n x, Jp_2 - Jp_1 \rangle + \|p_1\|^2 - \|p_2\|^2$$

for $n = 1, 2, \ldots$, from $T^{n_i} x \rightharpoonup p_1$ and $T^{n_j} x \rightharpoonup p_2$ we have

$$a = 2\langle p_1, Jp_2 - Jp_1 \rangle + \|p_1\|^2 - \|p_2\|^2 \tag{5.7}$$

and

$$a = 2\langle p_2, Jp_2 - Jp_1 \rangle + \|p_1\|^2 - \|p_2\|^2. \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\langle p_1 - p_2, Jp_1 - Jp_2 \rangle = 0.$$

Since J is strictly monotone, it follows that $p_1 = p_2$. Therefore, $\{T^n x\}$ converges weakly to an element of $F(T)$. □

Using Theorems 5.1 and 5.2, we finally have the following result.

Theorem 5.3. Let E be a uniformly smooth and uniformly convex Banach space and let T be a firmly generalized nonexpansive type mapping from E into itself. If the duality mapping J is weakly sequentially continuous, then the following are equivalent:

(1) $F(T)$ is nonempty;
(2) $\{T^n x\}$ converges weakly for each $x \in E$.

In this case, $\{T^n x\}$ converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} RT^n x$ and R is a sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. From Theorem 5.2, we know that the conditions (1) and (2) are equivalent. Moreover, in this case, we also know that, for each $x \in E$, $\{T^n x\}$ converges weakly to an element $p \in F(T)$. Since Lemma 2.3, we have that

$$\langle T^n x - RT^n x, JRT^n x - Jz\rangle \geq 0 \tag{5.9}$$

for each $z \in F(T)$. From Theorem 5.1, we have that $\{RT^n x\}$ converges strongly to some point u in $F(T)$. Since E has a Fréchet differentiable norm, the duality mapping J is norm to norm continuous. Therefore, letting $n \to \infty$ in (5.9), we obtain from $T^n x \rightharpoonup p$ and $RT^n x \to u$ that

$$\langle p - u, Ju - Jz\rangle \geq 0$$

for each $z \in F(T)$. Putting $z = p$, we get

$$\langle p - u, Ju - Jp\rangle \geq 0.$$

Since J is strictly monotone, it follows that $u = p$. Therefore, $\{T^n x\}$ converges weakly to $p = \lim_{n\to\infty} RT^n x$. This completes the proof. □

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