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THREE GENERALIZATIONS OF FIRMLY NONEXPANSIVE MAPPINGS: THEIR RELATIONS AND CONTINUITY PROPERTIES

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Abstract. The purpose of this paper is to study three generalizations of a firmly nonexpansive mapping, which is called a mapping of type (P), (Q), and (R), respectively. Especially, we focus on relationships among them and continuity properties of them.

1. Introduction

The class of firmly nonexpansive mappings in a Hilbert space is one of the most important class of nonlinear mappings. Indeed, all metric projections onto a closed convex set and all resolvents of a monotone operator are firmly nonexpansive.

Bruck [14] introduced and discussed a firmly nonexpansive mapping in a Banach space. All norm one linear projections, all sunny nonexpansive retractions, and all resolvents of an accretive operator are firmly nonexpansive. It is also known that the class of firmly nonexpansive mappings coincides with that of resolvents of accretive operators in Banach spaces; see also Bruck and Reich [15]. Let E be a Banach space and C a nonempty subset of E. Then a mapping $V\colon C \to E$ is said to be firmly nonexpansive [14] if

$$\|t(x-y)+(1-t)(Vx-Vy)\| \geq \|Vx-Vy\|$$

for all $x, y \in C$ and $t \geq 0$. If E is smooth, it is not hard to check that a mapping $V\colon C \to E$ is firmly nonexpansive if and only if

$$\langle x - Vx - (y - Vy), J(Vx - Vy)\rangle \geq 0$$

for all $x, y \in C$, where J is the duality mapping on E. If E is a Hilbert space, then this definition is reduced to

$$\|Vx - Vy\|^2 \leq \langle x - y, Vx - Vy\rangle$$

or equivalently

$$\|Vx - Vy\|^2 + \|x - Vx - (y - Vy)\|^2 \leq \|x - y\|^2$$

for all $x, y \in C$; see, for example, Goebel and Kirk [19].

Motivated by the proximal methods for monotone operators in Banach spaces [26, 27, 29, 30, 33, 44, 47, 54, 55, 57] and the results on relatively nonexpansive mappings in Banach spaces [3, 4, 7, 36, 42, 43, 47], Kohsaka and Takahashi [37] proposed the class of mappings of firmly nonexpansive type, which contains the classes of firmly

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nonexpansive mappings in Hilbert spaces and resolvents of maximal monotone operators in Banach spaces. We know that this class is contained in the classes of D-firm operators [10] and F-firmly nonexpansive operators [12]. In [37–39], some existence theorems and convergence theorems for mappings of firmly nonexpansive type were obtained. Let E be a smooth Banach space and C a nonempty subset of E. Then a mapping $T\colon C \to E$ is said to be of firmly nonexpansive type if

$$\langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \geq 0$$

for all $x, y \in C$. In the remainder of this paper, a mapping of firmly nonexpansive type is said to be of type (Q); see §4.

The purpose of this paper is to study three generalizations of a firmly nonexpansive mapping, which is called a mapping of type (P), (Q), and (R), respectively. Especially, we focus on relationships among them and continuity properties of them.

2. Preliminaries

Throughout the present paper, E denotes a real Banach space, E^* the dual of E, $\|\cdot\|$ the norm of E or E^*, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, I the identity mapping on E, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of nonnegative integers. Strong convergence of a sequence $\{x_n\}$ in E to x is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The (normalized) duality mapping of E is denoted by J, that is,

$$Jx = \left\{x^* \in E^* : \|x\|^2 = \|x^*\|^2 = \langle x, x^*\rangle\right\}$$

for $x \in E$. It is known that if E is reflexive, then J is surjective.

Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. In this case E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, the limit (2.1) is attained uniformly for $x \in S_E$. The norm of E is said to be Fréchet differentiable if for each $x \in S_E$, the limit (2.1) is attained uniformly for $y \in S_E$. The norm of E is said to be uniformly Fréchet differentiable if the limit (2.1) is attained uniformly for $x, y \in S_E$. In this case E is said to be uniformly smooth. It is known that

- if E is smooth, then the duality mapping J is single-valued;
- if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E;
- if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous;
- if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

see [52, 53] for more details.

A Banach space E is said to be strictly convex if $\|x + y\| < 2$ whenever $x, y \in S_E$ and $x \neq y$. It is known that if E is strictly convex, then the duality mapping J is injective, that is, $x, y \in E$ and $x \neq y$ imply $Jx \cap Jy = \emptyset$. A Banach space E is said to be uniformly convex if $\|x_n - y_n\| \to 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in S_E and $\|x_n + y_n\| \to 2$. It is known that if E is uniformly convex, then

- E is strictly convex and reflexive;
- E has the Kadec–Klee property, that is, a sequence $\{x_n\}$ in E converges strongly to x whenever $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$;

see [52, 53].

Let E be a strictly convex and reflexive Banach space and C a nonempty closed convex subset of E. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Such a point z is denoted by $P_C x$ and P_C is called the metric projection of E onto C.

Lemma 2.1 ([52, Corollary 6.5.5])**.** *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, P_C the metric projection of E onto C, $x \in E$, and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle z - y, J(x - z)\rangle \geq 0$$

for all $y \in C$.

Let E be a smooth Banach space. Throughout this paper, let $\phi\colon E \times E \to \mathbb{R}$ be a function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see [1, 29]. It is easy to check that

$$(\|x\| - \|y\|)^2 \leq \frac{1}{2}(\phi(x, y) + \phi(y, x)) = \langle x - y, Jx - Jy\rangle \tag{2.2}$$

for all $x, y \in E$. Let E be a smooth, strictly convex, and reflexive Banach space. Then we also define a function $\phi_*\colon E^* \times E^* \to \mathbb{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\left\langle J^{-1}y^*, x^*\right\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$. In this case, it is clear that $\phi(x, y) = \phi_*(Jy, Jx)$ for all $x, y \in E$ and $\phi_*(x^*, y^*) = \phi\left(J^{-1}y^*, J^{-1}x^*\right)$ for all $x^*, y^* \in E^*$.

We know the following result:

Lemma 2.2 ([29, Proposition 2])**.** *Let E be a smooth and uniformly convex Banach space and $\{x_n\}$ and $\{y_n\}$ sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

We know the following lemma; see, for instance, Takahashi [53, Problem 3.2.4]. For the sake of completeness, we give another proof of it by using the functions ϕ and ϕ_*:

Lemma 2.3. *Let E be a smooth, strictly convex, and reflexive Banach space, $\{x_n\}$ a sequence in E, and $x \in E$. If $\langle x_n - x, Jx_n - Jx\rangle \to 0$, then $x_n \rightharpoonup x$, $Jx_n \rightharpoonup Jx$, and $\|x_n\| \to \|x\|$.*

Proof. It follows from (2.2) that $\phi(x_n, x) \to 0$, $\phi(x, x_n) \to 0$, and $\|x_n\| \to \|x\|$. Hence $\phi_*(Jx_n, Jx) \to 0$ and $\{x_n\}$ is bounded.

Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in E$. Since the function ϕ is weakly lower semicontinuous in its first variable, we have

$$\phi(u, x) \leq \liminf_{i\to\infty} \phi(x_{n_i}, x) = \lim_{i\to\infty} \phi(x_{n_i}, x) = 0.$$

This shows that $u = x$ because of the strict convexity of E. Consequently, $x_n \rightharpoonup x$.

Let $\{Jx_{n_i}\}$ be a subsequence of $\{Jx_n\}$ which converges weakly to $u^* \in E^*$. Since the function ϕ_* is also weakly lower semicontinuous in its first variable, we have

$$\phi_*(u^*, Jx) \le \liminf_{i\to\infty} \phi_*(Jx_{n_i}, Jx) = \lim_{i\to\infty} \phi_*(Jx_{n_i}, Jx) = 0.$$

This shows that $u^* = Jx$ and therefore $Jx_n \rightharpoonup Jx$. This completes the proof. □

Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. It is known that for each $x \in E$ there is a unique point $z \in C$ such that $\phi(z, x) \le \phi(y, x)$ for all $y \in C$. Such a point z is denoted by $\Pi_C x$ and Π_C is called the generalized projection of E onto C; see [1] and [29]. The generalized projection has the following properties:

Lemma 2.4 ([1] and [29])**.** *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and Π_C the generalized projection of E onto C. Then*

$$\langle \Pi_C x - y, Jx - J\Pi_C x\rangle \ge 0,$$

or equivalently

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x)$$

for all $x \in E$ and $y \in C$.

Let C be a subset of a Banach space E and $T\colon C \to E$ a mapping. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T [47] if C contains a sequence $\{x_n\}$ which converges weakly to p and $x_n - Tx_n \to 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. The mapping T is said to be relatively nonexpansive [42,43] if $\hat{F}(T) = F(T) \ne \emptyset$ and

$$\phi(u, Tx) \le \phi(u, x)$$

for all $u \in F(T)$ and $x \in C$. A relatively nonexpansive mapping T is also said to be strongly relatively nonexpansive [7,36,47] if $\phi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\phi(u, x_n) - \phi(u, Tx_n) \to 0$ for some $u \in F(T)$.

Let C and D be nonempty subsets of a Banach space E. Then a mapping $U\colon C \to D$ is said to be sunny if $U(Ux + t(x - Ux)) = Ux$ whenever $x \in C$, $t \ge 0$, and $Ux + t(x - Ux) \in C$. A mapping $U\colon C \to C$ is said to be a retraction if $U^2 = U$. A mapping $U\colon C \to E$ is said to be generalized nonexpansive [21,22] if $F(U) \ne \emptyset$ and $\phi(Ux, u) \le \phi(x, u)$ for all $x \in C$ and $u \in F(U)$. A subset C of E is said to be a sunny generalized nonexpansive retract (resp. generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. generalized nonexpansive retraction) of E onto C.

Lemma 2.5 ([22, Proposition 4.2])**.** *Let E be a smooth and strictly convex Banach space, C a nonempty subset of E, and R_C a retraction of E onto C. If R_C is sunny and generalized nonexpansive, then*

$$\langle x - R_C x, JR_C x - Jy\rangle \ge 0,$$

or equivalently

$$\phi(R_C x, y) + \phi(x, R_C x) \le \phi(x, y)$$

for all $x \in E$ and $y \in C$.

We also know the following result:

Lemma 2.6 ([35, Theorem 3.3]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed subset of E. Then the following are equivalent:*

(1) *C is a sunny generalized nonexpansive retract of E;*
(2) *C is a generalized nonexpansive retract of E;*
(3) *$J(C)$ is closed and convex.*

In this case, the unique sunny generalized nonexpansive retraction R_C of E onto C is given by $J^{-1}\Pi_{J(C)}J$, where $\Pi_{J(C)}$ is the generalized projection of E^* onto $J(C)$.

Let A be a set-valued mapping of E into E^*, which is often denoted by $A \subset E \times E^*$. The effective domain of A is denoted by $\operatorname{dom}(A)$ and the range of A by $\operatorname{ran}(A)$, that is, $\operatorname{dom}(A) = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran}(A) = \bigcup_{x\in\operatorname{dom}(A)} Ax$. A set-valued mapping $A \subset E \times E^*$ is said to be a monotone operator if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is a monotone operator such that $A \subset A'$. It is known that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex, where $A^{-1}0 = \{x \in E : Ax \ni 0\}$.

Let E be a smooth, strictly convex, and reflexive Banach space and $A \subset E \times E^*$ a monotone operator. Then it is known that the following three single-valued mappings are well-defined for all $r > 0$:

- $K_r = (I + rJ^{-1}A)^{-1} \colon \operatorname{ran}(I + rJ^{-1}A) \to \operatorname{dom}(A)$;
- $L_r = (J + rA)^{-1}J \colon J^{-1}(\operatorname{ran}(J + rA)) \to \operatorname{dom}(A)$;
- $M_r = (I + rA^{-1}J)^{-1} \colon \operatorname{ran}(I + rA^{-1}J) \to J^{-1}\left(\operatorname{dom}(A^{-1})\right)$,

where J^{-1} is the duality mapping on E^*. Such mappings are called the resolvents of A. These mappings play essential roles in the approximation theory for zero points of maximal monotone operators in Banach spaces. The asymptotic behavior of sequences generated by several modifications of the well-known proximal point algorithm [49] (see also [18, 28, 53, 56]) are deeply related to the properties of these operators. We refer to [8, 9, 31, 40, 45, 50, 53] for some fundamental results on the classical resolvent K_r and [2–4, 6, 7, 10–12, 16–18, 20, 26, 27, 29, 30, 32–34, 36–39, 41–44, 46, 47, 58] for some results on the resolvent L_r or its generalizations with Bregman functions. We also refer to [21–24, 35] for some results on the relatively new resolvent M_r. The papers due to Takahashi [54, 55, 57] on four types of resolvents are also useful to the reader.

For each $r > 0$, it is known that $F(K_r) = F(L_r) = A^{-1}0$ and $F(M_r) = (A^{-1}J)^{-1}(0)$,

$$\frac{J(x - K_r x)}{r} \in AK_r x \tag{2.3}$$

for all $x \in \operatorname{ran}(I + rJ^{-1}A)$,

$$\frac{Jx - JL_r x}{r} \in AL_r x \tag{2.4}$$

for all $x \in J^{-1}(\operatorname{ran}(J + rA))$, and

$$\frac{x - M_r x}{r} \in A^{-1} J M_r x \tag{2.5}$$

for all $x \in \operatorname{ran}(I + rA^{-1}J)$. It is also known that if $A \subset E \times E^*$ is a maximal monotone operator, then

$$\operatorname{ran}(I + rJ^{-1}A) = J^{-1}(\operatorname{ran}(J + rA)) = \operatorname{ran}(I + rA^{-1}J) = E$$

for all $r > 0$; see [13, 48, 53], that is, K_r, L_r, and M_r are well-defined on the whole space E.

Let C be a subset of a Banach space E. A mapping $B\colon C \to E^*$ is said to be demicontinuous if B is norm-to-weak* continuous. We also know the following results:

Lemma 2.7 ([8, Theorem 1.3] and [9, Corollary 4.2])**.** *Let E be a reflexive Banach space and $B\colon E \to E^*$ a mapping. If B is monotone and demicontinuous, then B is maximal monotone.*

Theorem 2.8 ([52, Theorem 7.1.8])**.** *Let E be a reflexive Banach space, C a nonempty bounded closed convex subset of E, and $B\colon C \to E^*$ a monotone and demicontinuous mapping. Then there exists $u \in C$ such that $\langle y - u, Bu \rangle \geq 0$ for all $y \in C$.*

3. Mappings of type (P)

In this section, we introduce and discuss a mapping of type (P) in a Banach space.

Let E be a smooth Banach space and C a nonempty subset of E. A mapping $S\colon C \to E$ is said to be of type (P) if

$$\langle Sx - Sy, J(x - Sx) - J(y - Sy) \rangle \geq 0$$

for all $x, y \in C$.

Some important examples are listed below:

Example 3.1. Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then the metric projection P_C is of type (P). Indeed: Let $x, y \in E$. Then $P_C x, P_C y \in C$. Lemma 2.1 implies that

$$\langle P_C x - P_C y, J(x - P_C x) \rangle \geq 0 \text{ and } \langle P_C y - P_C x, J(y - P_C y) \rangle \geq 0.$$

Thus we have

$$\langle P_C x - P_C y, J(x - P_C x) - J(y - P_C y) \rangle \geq 0.$$

This means that P_C is a mapping of type (P).

Example 3.2. Let E be a smooth, strictly convex, and reflexive Banach space, $A \subset E \times E^*$ a monotone operator, and $r > 0$. Then the resolvent K_r of A is of type (P). Indeed: It follows from (2.3) that

$$\frac{J(x - K_r x)}{r} \in AK_r x \text{ and } \frac{J(y - K_r y)}{r} \in AK_r y$$

for all $x, y \in \operatorname{ran}(I + rJ^{-1}A)$. The monotonicity of A implies that

$$\left\langle K_r x - K_r y, \frac{J(x - K_r x)}{r} - \frac{J(y - K_r y)}{r} \right\rangle \geq 0.$$

Therefore K_r is of type (P).

There is a close relation between a mapping of type (P) and a monotone operator.

Proposition 3.3. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, $S\colon C \to E$ a mapping, and $A_S \subset E \times E^*$ an operator defined by $A_S = J\left(S^{-1} - I\right)$. Then S is of type (P) if and only if A_S is monotone. In this case $S = \left(I + J^{-1}A_S\right)^{-1}$, that is, S is the resolvent of A_S.*

Proof. The "if" part is obvious from Example 3.2. So, it is enough to prove the "only if" part. Let $(x, x^*), (y, y^*) \in A_S$. Then we have $S(J^{-1}x^* + x) = x$ and $S(J^{-1}y^* + y) = y$. Suppose that S is of type (P). Then it follows that

$$\begin{aligned} \langle x - y, x^* - y^* \rangle &= \left\langle x - y, J(J^{-1}x^* + x - x) - J(J^{-1}y^* + y - y) \right\rangle \\ &= \left\langle Sx' - Sy', J(x' - Sx') - J(y' - Sy') \right\rangle \geq 0, \end{aligned}$$

where $x' = J^{-1}x^* + x$ and $y' = J^{-1}y^* + y$. Hence A_S is monotone. □

We know the following proposition:

Proposition 3.4 ([5])**.** *Let E be a smooth Banach space, C a nonempty subset of E, and $S\colon C \to E$ a mapping of type (P). Then the following hold:*

(1) *If C is closed and convex, then $F(S)$ is closed and convex;*
(2) *$\hat{F}(S) = F(S)$;*
(3) *a mapping $V\colon C \to E$ defined by $V = \lambda I + (1 - \lambda)S$ is of type (P), where $\lambda \in [0, 1]$ and I is the identity mapping.*

4. Mappings of type (Q)

In this section, we introduce and discuss a mapping of type (Q) in a Banach space.

Let E be a smooth Banach space and C a nonempty subset of E. A mapping $T\colon C \to E$ is said to be of type (Q) if

$$\langle Tx - Ty, Jx - JTx - (Jy - JTy) \rangle \geq 0,$$

or equivalently

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. Such a mapping was said to be of firmly nonexpansive type in [37]; see also [10, 12, 38, 39].

Some important examples are listed below:

Example 4.1. Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then the generalized projection Π_C is of type (Q). Indeed: Let $x, y \in E$. Then $\Pi_C x, \Pi_C y \in C$. Lemma 2.4 implies that

$$\langle \Pi_C x - \Pi_C y, Jx - J\Pi_C x \rangle \geq 0 \text{ and } \langle \Pi_C y - \Pi_C x, Jy - J\Pi_C y \rangle \geq 0.$$

Thus we have

$$\langle \Pi_C x - \Pi_C y, Jx - J\Pi_C x - (Jy - J\Pi_C y)\rangle \geq 0.$$

This means that Π_C is a mapping of type (Q).

Example 4.2. Let E be a smooth, strictly convex, and reflexive Banach space, $A \subset E \times E^*$ a monotone operator, and $r > 0$. Then the resolvent $L_r = (J + rA)^{-1}J$ of A is of type (Q). Indeed: It follows from (2.4) that

$$\frac{Jx - JL_r x}{r} \in AL_r x \text{ and } \frac{Jy - JL_r y}{r} \in AL_r y$$

for all $x, y \in J^{-1}(\operatorname{ran}(J + rA))$. The monotonicity of A implies that

$$\left\langle L_r x - L_r y, \frac{Jx - JL_r x}{r} - \frac{Jy - JL_r y}{r} \right\rangle \geq 0.$$

Therefore L_r is of type (Q).

There is a close relation between a mapping of type (Q) and a monotone operator.

Proposition 4.3 ([38, Proposition 3.1])**.** *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, $T\colon C \to E$ a mapping, and $A_T \subset E \times E^*$ an operator defined by $A_T = JT^{-1} - J$. Then T is of type (Q) if and only if A_T is monotone. In this case $T = (J + A_T)^{-1} J$, that is, T is the resolvent of A_T.*

We know the following result:

Proposition 4.4 ([37, Lemma 5.1])**.** *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty subset of E, and $T\colon C \to E$ a mapping of type (Q). Then $\hat{F}(T) = F(T)$. In particular, if $F(T)$ is nonempty, then T is strongly relatively nonexpansive.*

Motivated by the technique due to Takahashi [51], Kohsaka and Takahashi [37] obtained the following fixed point theorem for mappings of type (Q):

Theorem 4.5 ([37, Theorem 3.2])**.** *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $T\colon C \to C$ a mapping of type (Q). Then there exists $x \in C$ such that $\{T^n x\}$ is bounded if and only if $F(T)$ is nonempty.*

5. Mappings of type (R)

In this section, we introduce and discuss a mapping of type (R) in a Banach space.

Let E be a smooth Banach space and C a nonempty subset of E. A mapping $U\colon C \to E$ is said to be of type (R) if

$$\langle x - Ux - (y - Uy), JUx - JUy\rangle \geq 0,$$

or equivalently

$$\phi(Ux, Uy) + \phi(Uy, Ux) + \phi(x, Ux) + \phi(y, Uy) \leq \phi(x, Uy) + \phi(y, Ux)$$

for all $x, y \in C$. Such a mapping was said to be of firmly generalized nonexpansive type in [25].

Some important examples are listed below:

Example 5.1. Let E be a smooth, strictly convex, and reflexive Banach space and C a subset of E such that $J(C)$ is closed and convex in E^*. Then Lemma 2.6 implies that C is a sunny generalized nonexpansive retract of E. In this case the sunny generalized nonexpansive retraction R_C of E onto C is of type (R). Indeed: Let $x, y \in E$. Then $R_C x, R_C y \in C$. Lemma 2.5 implies that

$$\langle x - R_C x, JR_C x - JR_C y\rangle \geq 0 \text{ and } \langle y - R_C y, JR_C y - JR_C x\rangle \geq 0.$$

Thus we have

$$\langle x - R_C x - (y - R_C y), JR_C x - JR_C y\rangle \geq 0.$$

This means that R_C is a mapping of type (R).

Example 5.2. Let E be a smooth, strictly convex, and reflexive Banach space, $A \subset E \times E^*$ a monotone operator, and $r > 0$. Then the resolvent $M_r = (I + rA^{-1}J)^{-1}$ of A is of type (R). Indeed: It follows from (2.5) that

$$\frac{x - M_r x}{r} \in A^{-1}JM_r x \text{ and } \frac{y - M_r y}{r} \in A^{-1}JM_r y$$

for all $x, y \in \operatorname{ran}(I + rA^{-1}J)$. The monotonicity of A implies that

$$\left\langle \frac{x - M_r x}{r} - \frac{y - M_r y}{r}, JM_r x - JM_r y \right\rangle \geq 0.$$

Therefore M_r is of type (R).

There is a close relation between a mapping of type (R) and a monotone operator.

Proposition 5.3. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, $U\colon C \to E$ a mapping, and $A_U \subset E^* \times E$ an operator defined by $A_U = \left(U^{-1} - I\right) J^{-1}$. Then U is of type (R) if and only if A_U is monotone. In this case $U = (I + A_U J)^{-1}$, that is, U is the resolvent of A_U.*

Proof. The "if" part is obvious from Example 5.2. We show the "only if" part. Let $(x^*, x), (y^*, y) \in A_U$. Then $x^* = JU(x + J^{-1}x^*)$ and $y^* = JU(y + J^{-1}y^*)$. Suppose that U is of type (R). Then it follows that

$$\begin{aligned}\langle x - y, x^* - y^*\rangle &= \left\langle x - y, JU(x + J^{-1}x^*) - JU(y + J^{-1}y^*)\right\rangle \\ &= \left\langle x' - Ux' - (y' - Uy'), JUx' - JUy'\right\rangle \geq 0,\end{aligned}$$

where $x' = x + J^{-1}x^*$ and $y' = y + J^{-1}y^*$. Hence A_U is monotone. □

We can also show the following proposition:

Proposition 5.4. *Let E be a smooth Banach space, C a nonempty subset of E, and $U\colon C \to E$ a mapping of type (R). Then the following hold:*

1. *If C is closed and convex, then $U^{-1}0$ is closed and convex;*
2. *if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup p \in C$ and $Ux_n \to 0$, then $p \in U^{-1}0$.*

Proof. We first show the closedness of $U^{-1}0$ in (1). Let $\{u_n\}$ be a sequence in $U^{-1}0$ such that $u_n \to w$. It follows from the closedness of C that $w \in C$. Since U is of type (R), we have

$$\|Uw\|^2 \leq \langle w - u_n, JUw \rangle .$$

This gives us that $\|Uw\|^2 \leq 0$ and hence $w \in U^{-1}0$.

We next show the convexity of $U^{-1}0$ in (1). Let $u, v \in U^{-1}0$ and $\alpha \in (0,1)$ be given and put $z = \alpha u + (1-\alpha)v$. It follows from the convexity of C that $z \in C$. Since U is of type (R) and $u, v \in U^{-1}0$, we have

$$\begin{aligned} -\|Uz\|^2 &= \langle -z - Uz, JUz \rangle + \langle z, JUz \rangle \\ &= \alpha \langle -u - Uz, JUz \rangle + (1-\alpha) \langle -v - Uz, JUz \rangle + \langle z, JUz \rangle \\ &= \alpha \langle z - Uz - u, JUz \rangle + (1-\alpha) \langle z - Uz - v, JUz \rangle \geq 0. \end{aligned}$$

This implies that $z \in U^{-1}0$.

We finally show (2). Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p \in C$ and $Ux_n \to 0$. Then we have $\|JUx_n\| = \|Ux_n\| \to 0$. Since U is of type (R), we have

$$\langle Ux_n - Up, JUx_n - JUp \rangle \leq \langle x_n - p, JUx_n - JUp \rangle$$

for all $n \in \mathbb{N}$. Taking the limit in this inequality, we get $\|Up\|^2 \leq 0$ and hence $p \in U^{-1}0$. □

6. Relationships among mappings of type (P), (Q), and (R)

In this section, we study some relationships among mappings of type (P), (Q), and (R).

Proposition 6.1. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, and $S\colon C \to E$ a mapping of type (P). Let $T_*\colon J(C) \to E^*$ be a mapping defined by $T_* = J(I-S)J^{-1}$ and $U\colon C \to E$ a mapping defined by $U = I - S$. Then T_* is of type (Q) in E^* and U is of type (R). Moreover $F(S) = (T_*J)^{-1}0 = U^{-1}0$ and $S^{-1}0 = J^{-1}F(T_*) = F(U)$.*

Proof. Let $x^*, y^* \in J(C)$ be given. Then there are $x, y \in C$ such that $x^* = Jx$ and $y^* = Jy$. By the definition of T_*, it is clear that $T_*J = J(I-S)$ and $S = I - J^{-1}T_*J$. Since S is of type (P), we have

$$\begin{aligned} 0 &\leq \langle Sx - Sy, J(I-S)x - J(I-S)y \rangle \\ &= \left\langle (I - J^{-1}T_*J)x - (I - J^{-1}T_*J)y, T_*Jx - T_*Jy \right\rangle \\ &= \left\langle J^{-1}x^* - J^{-1}T_*x^* - (J^{-1}y^* - J^{-1}T_*y^*), T_*x^* - T_*y^* \right\rangle . \end{aligned}$$

Therefore T_* is of type (Q) in E^*.

By the definitions of T_* and U, it is not hard to verify that U is of type (R), $F(S) = (T_*J)^{-1}0 = U^{-1}0$, and $S^{-1}0 = J^{-1}F(T_*) = F(U)$. This completes the proof. □

Similarly, we obtain the following:

Proposition 6.2. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, and $T\colon C \to E$ a mapping of type (Q). Let $S_*\colon J(C) \to E^*$ be a mapping defined by $S_* = I_* - JTJ^{-1}$ and $U_*\colon J(C) \to E^*$ a mapping defined by $U_* = JTJ^{-1}$, where I_* is the identity mapping on E^*. Then S_* is of type (P) in E^* and U_* is of type (R) in E^*. Moreover $F(T) = (S_*J)^{-1}0 = J^{-1}F(U_*)$ and $T^{-1}0 = J^{-1}F(S_*) = (U_*J)^{-1}0$.*

Proof. Let $x^*, y^* \in J(C)$ be given. Then there are $x, y \in C$ such that $x^* = Jx$ and $y^* = Jy$. By the definition of S_*, it is clear that $T = J^{-1}(I_* - S_*)J$. Since T is of type (Q), we have

$$\begin{aligned} 0 &\le \langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \\ &= \left\langle J^{-1}(Jx - S_*Jx) - J^{-1}(Jy - S_*Jy), S_*Jx - S_*Jy\right\rangle \\ &= \left\langle J^{-1}(x^* - S_*x^*) - J^{-1}(y^* - S_*y^*), S_*x^* - S_*y^*\right\rangle . \end{aligned}$$

Therefore S_* is of type (P) in E^*. Thus Proposition 6.1 implies that $U_* = JTJ^{-1} = I_* - S_*$ is of type (R) in E^*. □

As in the proof of Propositions 6.1 and 6.2, we immediately obtain the following:

Proposition 6.3. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E, and $U\colon C \to E$ a mapping of type (R). Let $S\colon C \to E$ be a mapping defined by $S = I - U$ and $T_*\colon J(C) \to E^*$ a mapping defined by $T_* = JUJ^{-1}$. Then S is of type (P) and T_* is of type (Q) in E^*. Moreover $F(U) = S^{-1}0 = J^{-1}F(T_*)$ and $U^{-1}0 = F(S) = (T_*J)^{-1}0$.*

Using Propositions 4.4 and 6.3, we can show the following:

Proposition 6.4. *Let E be a smooth and reflexive Banach space whose dual space has a uniformly Gâteaux differentiable norm, C a nonempty subset of E, and $U\colon C \to E$ a mapping of type (R). If $\{x_n\}$ is a sequence in C such that $Jx_n \rightharpoonup u^* \in J(C)$ and $Jx_n - JUx_n \to 0$, then $J^{-1}u^* \in F(U)$.*

Proof. Let $T_*\colon J(C) \to E^*$ be a mapping defined by $T_* = JUJ^{-1}$. Since $Jx_n - JUx_n \to 0$ and $JU = T_*J$, we have $Jx_n - T_*Jx_n \to 0$. Since E is smooth and reflexive, E^* is strictly convex. It follows from Propositions 4.4 and 6.3 that T_* is of type (Q) in E^* and $\hat{F}(T_*) = F(T_*)$. Therefore $u^* \in F(T_*)$. Hence we have $u^* = JUJ^{-1}u^*$ and thus $J^{-1}u^* \in F(U)$. □

The following fixed point theorem for mappings of type (R) is deduced from Theorem 4.5 and Proposition 6.3:

Theorem 6.5. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of E such that $J(C)$ is closed and convex, and $U\colon C \to C$ a mapping of type (R). Then there exists $x \in C$ such that $\{U^n x\}$ is bounded if and only if $F(U)$ is nonempty.*

Proof. The "if" part is obvious. We show the "only if" part. Suppose that there exists $x \in C$ such that $\{U^n x\}$ is bounded and let $T_* = JUJ^{-1}$. By Proposition 6.3, $T_*\colon J(C) \to J(C)$ is a mapping of type (Q) and $F(U) = J^{-1}F(T_*)$. It is easy to see that $\{(T_*)^n Jx\}$ is bounded. Therefore, Theorem 4.5 implies that $F(T_*)$ is nonempty. Thus $F(U)$ is also nonempty. This completes the proof. □

7. Continuity of mappings of type (P), (Q), and (R)

In this section, we study some continuity properties of mappings of type (P), (Q), and (R).

From (2.2) and the definition of a mapping of type (R), we obtain the following: Let E be a smooth Banach space, C a nonempty subset of E, and $U\colon C \to E$ a mapping of type (R). Then

$$\begin{aligned}\|x-y\|\,(\|Ux\|+\|Uy\|) &\geq \langle x-y, JUx-JUy\rangle \\ &\geq \langle Ux-Uy, JUx-JUy\rangle \\ &= \frac{1}{2}(\phi(Ux,Uy)+\phi(Uy,Ux)) \\ &\geq (\|Ux\|-\|Uy\|)^2\end{aligned} \tag{7.1}$$

for all $x, y \in C$.

Theorem 7.1. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty subset of E. If a mapping $U\colon C \to E$ is of type (R), then the following hold:*

1. *U is bounded on each nonempty bounded subset of C;*
2. *if $\{x_n\}$ is a sequence in C such that $x_n \to x \in C$, then $Ux_n \rightharpoonup Ux$, $JUx_n \rightharpoonup JUx$, and $\|Ux_n\| \to \|Ux\|$;*
3. *$JU\colon C \to E^*$ is monotone and demicontinuous;*
4. *if E has the Kadec–Klee property, then U is norm-to-norm continuous;*
5. *if E is uniformly convex, then U is uniformly norm-to-norm continuous on each nonempty bounded subset of C;*
6. *if E is uniformly convex and uniformly smooth, then JU is uniformly norm-to-norm continuous on each nonempty bounded subset of C.*

Proof. We first show (1). Suppose that U is not bounded on some nonempty bounded subset of C. Then we have a bounded sequence $\{x_n\}$ in C such that $\|Ux_n\| \to \infty$. Fix $y \in C$. Since U is of type (R), it follows from (7.1) that

$$\|x_n - y\|\,(\|Ux_n\| + \|Uy\|) \geq (\|Ux_n\| - \|Uy\|)^2$$

for all $n \in \mathbb{N}$. So, we have $\|x_n\| \to \infty$. This is a contradiction.

We next show (2). Let $\{x_n\}$ be a sequence in C such that $x_n \to x \in C$. It follows from (1) that $\{Ux_n\}$ is bounded. Since U is of type (R), it follows from (7.1) that

$$\begin{aligned}0 &\leq \langle Ux_n - Ux, JUx_n - JUx\rangle \\ &\leq \|x_n - x\|\,(\|Ux_n\| + \|Ux\|) \to 0.\end{aligned}$$

Thus Lemma 2.3 implies that $Ux_n \rightharpoonup Ux$, $JUx_n \rightharpoonup JUx$, and $\|Ux_n\| \to \|Ux\|$.

The part (3) follows from (7.1) and (2).

The part (4) directly follows from (2).

We show (5). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in C such that $x_n - y_n \to 0$. Then it suffices to show that $Ux_n - Uy_n \to 0$. Since U is of type (R), it follows from (7.1) and (1) that

$$0 \leq \frac{1}{2}(\phi(Ux_n, Uy_n) + \phi(Uy_n, Ux_n))$$

$$\leq \|x_n - y_n\| (\|Ux_n\| + \|Uy_n\|) \to 0.$$

This implies that $\phi(Ux_n, Uy_n) \to 0$. Then Lemma 2.2 ensures that $Ux_n - Uy_n \to 0$.

The part (6) follows from (5) and the uniform continuity of J on each nonempty bounded subset of E. This completes the proof. □

As a direct consequence of Lemma 2.7 and Theorem 7.1, we obtain the following results.

Corollary 7.2. *Let E be a smooth, strictly convex, and reflexive Banach space. If a mapping $U\colon E \to E$ is of type (R), then JU is maximal monotone.*

Theorem 7.3. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty subset of E. If a mapping $S\colon C \to E$ is of type (P), then the following hold:*

(1) *S is bounded on each nonempty bounded subset of C;*
(2) *if $\{x_n\}$ is a sequence in C such that $x_n \to x \in C$, then $Sx_n \rightharpoonup Sx$, $J(x_n - Sx_n) \rightharpoonup J(x - Sx)$, and $\|x_n - Sx_n\| \to \|x - Sx\|$;*
(3) *$J(I - S)\colon C \to E^*$ is monotone and demicontinuous;*
(4) *if E has the Kadec–Klee property, then S is norm-to-norm continuous;*
(5) *if E is uniformly convex, then S is uniformly norm-to-norm continuous on each nonempty bounded subset of C;*
(6) *if E is uniformly convex and uniformly smooth, then $J(I - S)$ is uniformly norm-to-norm continuous on each nonempty bounded subset of C.*

Proof. Let $U\colon C \to E$ be a mapping defined by $U = I - S$. Then it follows from Proposition 6.1 that U is of type (R). Thus Theorem 7.1 implies the conclusion. □

Using Theorem 2.8 and Theorem 7.3, we can show the following fixed point theorem for mappings of type (P):

Theorem 7.4. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty bounded closed convex subset of E, P_C the metric projection of E onto C, and $S\colon C \to E$ a mapping of type (P). Then $F(P_C S)$ is nonempty. If S is a self mapping, then $F(S)$ is nonempty.*

Proof. By Theorem 2.8 and Theorem 7.3 (2), there exists $u \in C$ such that

$$\langle y - u, J(I - S)u\rangle \geq 0$$

for all $y \in C$. Lemma 2.1 implies that $P_C(Su) = u$ and hence $F(P_C S)$ is nonempty. If S is a self mapping, then $P_C S = S$ and hence $F(S)$ is nonempty. □

The following result is directly deduced from Proposition 6.1 and Corollary 7.2:

Corollary 7.5. *Let E be a smooth, strictly convex, and reflexive Banach space. If a mapping $S\colon E \to E$ is of type (P), then $J(I - S)$ is maximal monotone.*

Using Corollary 7.5, we obtain the following results:

Theorem 7.6. *Let E be a smooth, strictly convex, and reflexive Banach space, $S\colon E \to E$ a mapping of type (P), and $r > 0$. Then*

(1) *$T = (J + rJ(I - S))^{-1}J$ is a single-valued mapping of E into itself;*

(2) *T is of type (Q);*
(3) $F(S) = F(T)$.

Proof. Corollary 7.5 shows that $J(I-S)\colon E \to E^*$ is a maximal monotone operator. Thus $T = (J + rJ(I-S))^{-1}J$ is the resolvent of $J(I-S)$ for r. Therefore T is a single-valued mapping on E and is of type (Q) by Example 4.2. Moreover, it is clear that

$$F(S) = (J(I-S))^{-1}(0) = F(T).$$

This completes the proof. □

Corollary 7.7. *Let E be a smooth, strictly convex, and reflexive Banach space, $A \subset E \times E^*$ a maximal monotone operator, $r > 0$, and $S = (I + J^{-1}A)^{-1}$ the resolvent of A. Then*

(1) *$T = (J + rJ(I-S))^{-1}J$ is a single-valued mapping of E into itself;*
(2) *T is of type (Q);*
(3) $A^{-1}0 = F(T)$.

Proof. By Example 3.2, we know that S is a mapping of type (P) of E into itself. Therefore Theorem 7.6 implies the conclusion. □

Using Theorem 7.1, we also obtain the following:

Theorem 7.8. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty subset of E. If a mapping $T\colon C \to E$ is of type (Q), then the following hold:*

(1) *T is bounded on each nonempty bounded subset of C;*
(2) *if the norm of E is Fréchet differentiable and if $\{x_n\}$ is a sequence in C such that $x_n \to x \in C$, then $Tx_n \rightharpoonup Tx$, $JTx_n \rightharpoonup JTx$, and $\|Tx_n\| \to \|Tx\|$;*
(3) *if the norm of E is Fréchet differentiable and E has the Kadec–Klee property, then T is norm-to-norm continuous;*
(4) *if E is uniformly convex and uniformly smooth, then T is uniformly norm-to-norm continuous on each nonempty bounded subset of C.*

Proof. Let $U_*\colon J(C) \to E^*$ be a mapping defined by $U_* = JTJ^{-1}$. Then $T = J^{-1}U_*J$ and it follows from Proposition 6.2 that U_* is of type (R) in E^*. It is obvious that J and J^{-1} is bounded on each nonempty bounded subset of E and E^*, respectively. Thus Theorem 7.1 (1) implies (1).

We next show (2). Let $\{x_n\}$ be a sequence in C such that $x_n \to x \in C$. Then it follows from (1) that $\{Tx_n\}$ is bounded. By the Fréchet differentiability of the norm of E, J is norm-to-norm continuous. Thus we have $Jx_n \to Jx$. Since T is of type (Q), we have

$$\begin{aligned} 0 \le \langle Tx_n - Tx, JTx_n - JTx\rangle &\le \langle Tx_n - Tx, Jx_n - Jx\rangle \\ &\le \|Tx_n - Tx\|\,\|Jx_n - Jx\| \to 0. \end{aligned}$$

Hence Lemma 2.3 implies that $Tx_n \rightharpoonup Tx$, $JTx_n \rightharpoonup JTx$, and $\|Tx_n\| \to \|Tx\|$.

The part (3) directly follows from (2).

We finally show (4). Since E is uniformly convex and uniformly smooth, J and J^{-1} is uniformly norm-to-norm continuous on each nonempty bounded subset of E and E^*, respectively. On the other hand, Theorem 7.1 (5) implies that the mapping

U_* defined above is uniformly norm-to-norm continuous on each nonempty bounded subset of $J(C)$. Thus $T = J^{-1}U_*J$ is uniformly norm-to-norm continuous on each nonempty bounded subset of C. This completes the proof. $\square$

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