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A RELAXED CUTTING PLANE ALGORITHM FOR SOLVING THE VASICEK-TYPE INTEREST RATE MODEL

HOMING CHEN AND CHENG-FENG HU

Abstract. This work considers the resolution of the Vasicek-type interest rate model. A deterministic process is adopted to model the random behavior of interest rate variation as a deterministic perturbation. It shows that the solution of the Vasicek-type interest rate model can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is then proposed for solving the resulting optimization problem. The numerical results illustrate that our approach essentially generates the yield functions with minimal fitting errors and small oscillation.

1. Introduction

The interest rate model plays a central role in the theory of modern economics and finance. In the past studies interest rate models described by stochastic process are widely used. It is usually assumed that the interest rates are sufficient statistics for the stochastic movement of current term structure. An enormous amount of work has been directed towards modeling and estimation of the short term interest rate dynamics. Some single-factor models [2, 3, 19] have been proposed and widely used in practice because of their tractability and their ability to fit reasonably well the dynamics of the short term interest rates. Econometric estimation of these models has also been intensively studied in the literature [2, 6, 7, 1, 14]. Recently, Kortanek and Medvedev [12] introduced a deterministic process to model the random behavior of interest rate variation as a deterministic perturbation which were later investigated by Staffa [15], and Tichatschke et al. [17]. Inspired and motivated by the recent research, this work considers a Vasicek-type interest rate model, which can be described by the following stochastic differential equation.

$$dr(t) = (\alpha + \beta r(t))dt + \sigma dB(t), \quad r(t_0) = r_0, \tag{1.1}$$

where $r(t)$ is the instance interest rate, $B(t)$ denotes the Brownian motion, σ is the instantaneous standard deviation of the interest rate, the coefficients α, β and the constant r_0 satisfy the following conditions with the pre-assigned bounds $\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, \underline{r}_0, \overline{r}_0$.

$$0 < \underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}, \underline{\beta} \leqslant \beta \leqslant \overline{\beta} < 0,\ 0 < \underline{r}_0 \leqslant r_0 \leqslant \overline{r}_0.$$

The main feature of the Vasicek-type interest rate model (1.1) is the instantaneous trend of the process to revert to its long run mean value. The mean reverting property undermines that the Vasicek-type interest rate model is an equilibrium model. The parameter $\beta(t)$ determines a speed of the adjustment and should be

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negative to ensure convergence[12]. The Vasicek-type model has also been extended in subsequent research. The work of Dothan [5], Courtadon[4], Cox et al. [3]and Stapleton and Subrahmanyam [16] can also be placed in this category.

To solve the Vasicek-type interest rate model (1.1), the concept of the deterministic perturbations is adopted to deal with the random behavior of interest rate variations. It is shown that the solution of the Vasicek-type interest rate model (1.1) can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is proposed for solving the resulting optimization problem. In each iteration, we solve a finite optimization problem and add one or some more constraints. The proposed algorithm chooses a point at which the infinite constrains are violated to a degree rather than the violation being maximized. The organization of the rest of this paper is as follows. Section 2 provides some basic definitions to formulate the Brownian motion in the Vasicek-type interest rate model in terms of the deterministic perturbation. It shows that the Vasicek-type interest rate model can be solved via a nonlinear semi-infinite programming problem. Solution algorithms are developed in section 3 for solving the resulting semi-infinite programming problem. Some numerical results of the Vasicek-type interest rate model are presented in section 4. Section 5 concludes this paper by making some remarks.

2. The Vasicek-type interest rate model with impulse perturbation

As mentioned in the previous section, in this paper a deterministic process is adopted to model the uncertainty in the interest rate behavior. It is assumed that the uncertainty is deterministic, which is depending on the time t. For convenience we denote the uncertainty as an integral function $w(t)$, and $\overline{w}(t)$, $\underline{w}(t)$ are assumed to be the pre-assigned upper and lower bounds of $w(t)$, respectively, i.e.,

$$\underline{w}(t) \leq w(t) \leq \overline{w}(t). \tag{2.1}$$

In this case, the Vasicek-type interest rate model can be formulated as the following differential equation with uncertainty.

$$dr(t) = (\alpha + \beta r(t))dt + w(t)dt. \tag{2.2}$$

To specify the perturbation function $w(t)$, here we introduce some notations and definitions. Assume that there are M observed yields, say $\bar{R}_i$, with time to maturity $\mathcal{T}$ corresponding to the $i-$th day of observation, $i = 1, 2, \ldots, M$. Let $\tilde{\aleph} \stackrel{\triangle}{=} \{t_0, t_1, \ldots, t_{M+\mathcal{T}}\}$, where $t_{i-1} < t_i$, and $\aleph_i \stackrel{\triangle}{=} [t_{i-1}, t_i], i = 1, 2, \ldots, M + \mathcal{T}$. For convenient, we denote $M + \mathcal{T} = N$.

Definition 2.1 (The Observed Treasury Yield)**.** The observed Treasury yield is defined as follows.

$$\bar{R}(t \mid \mathcal{T}) \stackrel{\triangle}{=} \bar{R}_i, \forall t \in \aleph_i, i = 1, 2, \ldots, M.$$

Definition 2.2 (The Yield Function)**.** The yield function is defined as the mean value of interest rate of integral, i.e.,

$$y(t \mid \mathcal{T}) \stackrel{\triangle}{=} \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} r(\tau)d\tau, \forall t \in \aleph_i, i = 1, 2, \ldots, M. \tag{2.3}$$

Definition 2.3 (The Function of Estimation Error)**.** The function of estimation error is defined as the difference of the yield function and the observed Treasury yield, i.e.,

$$\xi(t) \stackrel{\triangle}{=} y(t \mid \mathcal{T}) - \overline{R}(t \mid \mathcal{T}), \forall t \in \aleph_i, i = 1, 2, \ldots, M. \tag{2.4}$$

Definition 2.4 (The Impulse Perturbation)**.** Let $w(t) \stackrel{\triangle}{=} w_i(t), \forall t \in \aleph_i, i = 1, 2, \ldots, N$. The impulse perturbation is defined to be

$$w_i(t) = w_i, \forall t \in \aleph_i, i = 1, 2, \ldots, N, \tag{2.5}$$

where $w_i \in \mathbb{R}$ is a constant and $\underline{w_i} \leqslant w_i \leqslant \overline{w}_i, i = 1, 2, \ldots, N$, with $\underline{w_i}$ and $\overline{w}_i, i = 1, 2, \ldots, N$, are pre-assigned bounds for the perturbations.

The solution of the Vasicek-type interest rate model (2.2) with the impulse perturbation function defined in (2.5) can be derived in Theorem 2.5.

Theorem 2.5. *The instance interest rate function of the Vasicek-type interest rate model (2.3) is*

$$\begin{aligned} \textit{given by} \qquad r(t) &= e^{\beta t} r_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sum_{j=1}^{i-1} w_j \frac{e^{\beta t}}{\beta}(e^{-\beta t_{j-1}} - e^{-\beta t_j}) \\ &+ w_i \frac{1}{\beta}(e^{\beta(t-t_{i-1})} - 1), t \in \aleph_i, i = 1, 2, \ldots, N. \end{aligned} \tag{2.6}$$

Proof. Multiply both sides with the integrating factor $e^{-\beta(t-t_0)}$ for (2.2) we have

$$e^{-\beta(t-t_0)} \frac{dr(t)}{dt} - \beta r(t) e^{-\beta(t-t_0)} = e^{-\beta(t-t_0)}(\alpha + w(t)).$$

Integrating each side from t_0 to t

$$\begin{aligned} \int_{t_0}^t d(e^{-\beta(\tau-t_0)} r(\tau)) &= \int_{t_0}^t e^{-\beta(\tau-t_0)}(\alpha + w(\tau))d\tau \\ e^{-\beta(\tau-t_0)} r(\tau)|_{t_0}^t &= \int_{t_0}^t e^{-\beta(\tau-t_0)}(\alpha + w(\tau))d\tau \\ e^{-\beta(\tau-t_0)} r(\tau)|_{t_0}^t &= \int_{t_0}^t e^{-\beta(\tau-t_0)} \alpha d\tau + \int_{t_0}^t e^{-\beta(\tau-t_0)} w(\tau)d\tau \end{aligned}$$

$$\begin{aligned} e^{-\beta(t-t_0)} r(t) - r(t_0) &= \frac{-\alpha}{\beta} e^{-\beta(\tau-t_0)}|_{t_0}^t + \int_{t_0}^{t_1} e^{-\beta(\tau-t_0)} w_1(\tau)d\tau \\ &+ \int_{t_1}^{t_2} e^{-\beta(\tau-t_0)} w_2(\tau)d\tau + \cdots \end{aligned}$$

$$+\int_{t_{i-2}}^{t_{i-1}} e^{-\beta(\tau-t_0)} w_{i-1}(\tau)d\tau$$
$$+\int_{t_{i-1}}^{t} e^{-\beta(\tau-t_0)} w_i(\tau)d\tau.$$

Hence

$$\begin{aligned}
r(t) &= e^{\beta(t-t_0)}r(t_0) - e^{\beta(t-t_0)}\frac{\alpha}{\beta}(e^{-\beta(t-t_0)}-1) + e^{\beta(t-t_0)}\sum_{j=1}^{i-1} w_j \int_{t_{j-1}}^{t_j} e^{-\beta(\tau-t_0)}d\tau \\
&\quad + e^{\beta(t-t_0)} w_i \int_{t_{i-1}}^{t} e^{-\beta(\tau-t_0)}d\tau. \\
&= e^{\beta(t-t_0)}r(t_0) + \frac{\alpha}{\beta}(e^{\beta(t-t_0)}-1) + \sum_{j=1}^{i-1} e^{\beta(t-t_0)}\frac{w_j}{\beta}(e^{-\beta(t_{j-1}-t_0)} - e^{-\beta(t_j-t_0)}) \\
&\quad - \frac{e^{\beta(t-t_0)}w_i}{\beta}(e^{-\beta(t-t_0)} - e^{-\beta(t_{i-1}-t_0)}). \\
&= r(t_0)e^{\beta(t-t_0)} + \frac{\alpha}{\beta}(e^{\beta(t-t_0)}-1) + \sum_{j=1}^{i-1} e^{\beta(t-t_0)}\frac{w_j}{\beta}(e^{-\beta(t_{j-1}-t_0)} - e^{-\beta(t_j-t_0)}) \\
&\quad + \frac{w_i}{\beta}(e^{-\beta(t_{i-1}-t)}-1). \\
&= r_0 e^{\beta(t-t_0)} + \frac{\alpha}{\beta}(e^{\beta(t-t_0)}-1) + \sum_{j=1}^{i-1} \frac{w_j}{\beta}(e^{-\beta(t_{j-1}-t)} - e^{-\beta(t_j-t)}) \\
&\quad + \frac{w_i}{\beta}(e^{\beta(t-t_{i-1})}-1), \quad t \in \aleph_i, i = 1, 2, \ldots, N.
\end{aligned}$$

□

The solution of the Vasicek-type interest rate model (2.3) with the impulse perturbation function defined in (2.5) has the form.

$$\begin{aligned}
r(t) &= e^{\beta t} r_0 + \frac{\alpha}{\beta}(e^{\beta t}-1) + \sum_{j=1}^{i-1} w_j \frac{e^{\beta t}}{\beta}(e^{-\beta t_{j-1}} - e^{-\beta t_j}) \\
&+ \ w_i \frac{1}{\beta}(e^{\beta(t-t_{i-1})}-1), t \in \aleph_i, i = 1, 2, \ldots, N.
\end{aligned}$$

It is well known that the yield function is one of the most important financial indicators in the theory of modern economics and finance. Substituting (2.6) into (2.3) yields the following result.

Theorem 2.6. *The yield function has the form*

$$y(t|\mathcal{T}) = \frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\mathcal{T}\beta} r_0 + \left(\frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\mathcal{T}\beta^2} - \frac{1}{\beta}\right)\alpha$$

$$
\begin{aligned}
&+ \sum_{k=1}^{i-1} \frac{e^{\beta(t-t_{k-1}+\mathcal{T})} - e^{\beta(t-t_k+\mathcal{T})} + e^{\beta(t-t_k)} - e^{\beta(t-t_{k-1})}}{\mathcal{T}\beta^2} w_k \\
&+ (\frac{e^{\beta(t-t_{k-1}+\mathcal{T})} - e^{\beta(t-t_i+\mathcal{T})} - e^{\beta(t-t_{i-1})} + 1}{\mathcal{T}\beta^2} - \frac{t_i - t}{\mathcal{T}\beta}) w_i \\
&+ \sum_{k=i+1}^{i+\mathcal{T}-1} (\frac{e^{\beta(t-t_{k-1})+\mathcal{T}} - e^{\beta(t-t_{k-1}+\mathcal{T})}}{\mathcal{T}\beta^2} - \frac{t_k - t_{k-1}}{\mathcal{T}\beta}) w_k \\
&+ (\frac{e^{\beta(t-t_{i+\mathcal{T}-1}+\mathcal{T})} - 1}{\mathcal{T}\beta^2} - \frac{t - t_{i+\mathcal{T}-1} + \mathcal{T}}{\mathcal{T}\beta}) w_{i+\mathcal{T}}, \quad t \in \aleph_i, i = 1, 2, \ldots, M.
\end{aligned}
\tag{2.7}
$$

Proof.

$$
\begin{aligned}
y(t|\mathcal{T}) &= \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} r(\tau) d\tau \\
&= \frac{1}{\mathcal{T}} \Bigg\{ r_0 \int_t^{t+\mathcal{T}} e^{\beta\tau} d\tau + \frac{\alpha}{\beta} \int_t^{t+\mathcal{T}} (e^{\beta\tau} - 1) d\tau \\
&+ \int_t^{t_i} (\sum_{k=1}^{i-1} w_k \frac{e^{\beta\tau}}{\beta} (e^{-\beta t_{k-1}} - e^{-\beta t_k}) + w_i \frac{e^{\beta(\tau - t_{i-1})} - 1}{\beta}) d\tau \\
&+ \sum_{j=i+1}^{i+\mathcal{T}-1} \int_{t_{j-1}}^{t_j} (\sum_{k=1}^{j-1} w_k \frac{e^{\beta\tau}}{\beta} (e^{-\beta t_{k-1}} - e^{-\beta t_k}) + w_j \frac{e^{\beta(\tau - t_{j-1})} - 1}{\beta}) d\tau \\
&+ \int_{t_{i+\mathcal{T}-1}}^{t+\mathcal{T}} (\sum_{k=1}^{i+\mathcal{T}-1} w_k \frac{e^{\beta\tau}}{\beta} (e^{-\beta t_{k-1}} - e^{-\beta t_k}) + w_{i+\mathcal{T}} \frac{e^{\beta(\tau - t_{i+\mathcal{T}-1})} - 1}{\beta}) d\tau \Bigg\} \\
&= \frac{1}{\mathcal{T}} \Bigg\{ \frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\beta} r_0 + (\frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\beta^2} - \frac{\mathcal{T}}{\beta}) \alpha \\
&+ \sum_{k=1}^{i-1} \frac{e^{-\beta t_{k-1}} - e^{-\beta t_k}}{\beta} w_k \int_t^{t+\mathcal{T}} e^{\beta\tau} d\tau \\
&+ (\frac{1}{\beta} \int_t^{t_i} (e^{\beta(\tau - t_{i-1})} - 1) d\tau + \frac{e^{-\beta t_{i-1}} - e^{-\beta t_i}}{\beta} \int_{t_i}^{t+\mathcal{T}} e^{\beta\tau} d\tau) w_i \\
&+ \sum_{k=i+1}^{i+\mathcal{T}-1} (\frac{1}{\beta} \int_{t_{k-1}}^{t_k} (e^{\beta(\tau - t_{k-1})} - 1) d\tau + \frac{e^{-\beta t_{k-1}} - e^{-\beta t_k}}{\beta} \int_{t_k}^{t+\mathcal{T}} e^{\beta\tau} d\tau) w_k \\
&+ \frac{1}{\beta} \int_{t_{i+\mathcal{T}-1}}^{t+\mathcal{T}} (e^{\beta(\tau - t_{i+\mathcal{T}-1})} - 1) w_{i+\mathcal{T}} d\tau \Bigg\}
\end{aligned}
$$

$$
= \frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\mathcal{T}\beta} r_0 + (\frac{e^{\beta(t+\mathcal{T})} - e^{\beta t}}{\mathcal{T}\beta^2} - \frac{1}{\beta}) \alpha
$$

$$
\begin{aligned}
&+\sum_{k=1}^{i-1}\frac{e^{\beta(t-t_{k-1}+\mathcal{T})}+e^{\beta(t-t_k+\mathcal{T})}+e^{\beta(t-t_k)}-e^{\beta(t-t_{k-1})}}{\mathcal{T}\beta^2}w_k\\
&+(\frac{e^{\beta(t-t_{k-1}+\mathcal{T})}-e^{\beta(t-t_i+\mathcal{T})}-e^{\beta(t-t_{i-1})}+1}{\mathcal{T}\beta^2}-\frac{t_i-t}{\mathcal{T}\beta})w_i\\
&+\sum_{k=i+1}^{i+\mathcal{T}-1}(\frac{e^{\beta(t-t_{k-1}+\mathcal{T})}-e^{\beta(t-t_k+\mathcal{T})}}{\mathcal{T}\beta^2}-\frac{t_k-t_{k-1}}{\mathcal{T}\beta})w_k\\
&+(\frac{e^{\beta(t-t_{i+\mathcal{T}-1}+\mathcal{T})}-1}{\mathcal{T}\beta^2}-\frac{t-t_{i+\mathcal{T}-1}+\mathcal{T}}{\mathcal{T}\beta})w_{i+\mathcal{T}},\quad t\in\aleph_i, i=1,2,\ldots,M.
\end{aligned}
$$

□

To shorten the mathematical formulas in (2.7), the following notations are introduced. Let

(2.8)

$$
a_k(\beta,t|\mathcal{T})=\begin{cases}
\dfrac{e^{\beta(t-t_{k-1}+\mathcal{T})}-e^{\beta(t-t_k+\mathcal{T})}+e^{\beta(t-t_k)}-e^{\beta(t-t_{k-1})}}{\mathcal{T}\beta^2}, & \text{if } k<i,\\
\dfrac{e^{\beta(t-t_{k-1}+\mathcal{T})}-e^{\beta(t-t_k+\mathcal{T})}-e^{\beta(t-t_{k-1})}+1}{\mathcal{T}\beta^2}-\dfrac{t_k-t}{\mathcal{T}\beta}, & \text{if } k=i,\\
\dfrac{e^{\beta(t-t_{k-1}+\mathcal{T})}-e^{\beta(t-t_k+\mathcal{T})}}{\mathcal{T}\beta^2}-\dfrac{t_k-t_{k-1}}{\mathcal{T}\beta}, & \text{if } i<k<i+\mathcal{T},\\
\dfrac{e^{\beta(t-t_{k-1}+\mathcal{T})}-1}{\mathcal{T}\beta^2}-\dfrac{t-t_{k-1}+\mathcal{T}}{\mathcal{T}\beta}, & \text{if } k=i+\mathcal{T}.
\end{cases}
$$

We have

$$
\begin{aligned}
y(t|\mathcal{T}) \;&=\; \frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta}r_0+(\frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta^2}-\frac{1}{\beta})\alpha+\sum_{k=1}^{i+\mathcal{T}}a_k(\beta,t|\mathcal{T})w_k,\\
&\qquad t\in\aleph_i, i=1,2,\ldots,M. \qquad (2.9)
\end{aligned}
$$

This work considers to find the impulse perturbation $w(t)$ that minimizes the maximum absolute value of the function of estimation errors defined in (2.4). It leads to the following optimization problem.

Problem 1

$$
\begin{aligned}
&\min \epsilon\\
\textbf{subject to}\quad &\bar{R}(t\mid\mathcal{T})\leqslant y(t|\mathcal{T})+\epsilon, \forall t\in\aleph_i, i=1,2,\ldots,M,\\
&\bar{R}(t\mid\mathcal{T})\geqslant y(t|\mathcal{T})-\epsilon, \forall t\in\aleph_i, i=1,2,\ldots,M,\\
&\underline{\alpha}\leq\alpha\leq\overline{\alpha},\\
&\underline{\beta}\leq\beta\leq\overline{\beta},\\
&\underline{r}_0\leq r_0\leq\overline{r}_0,\\
&\underline{w}_i\leq w_i\leq\overline{w}_i,\quad i=1,2,\ldots,N.
\end{aligned}
$$

Substituting (2.9) into the **Problem 1** leads to the following nonlinear programming problem.

Problem 2 $\min \epsilon$

$$
\begin{aligned}
\textbf{subject to } \overline{R}_i &\leqslant \frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta} r_0 + \left(\frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta^2} - \frac{1}{\beta}\right)\alpha + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta, t|\mathcal{T}) w_j + \epsilon, \\
& \forall t \in \aleph_i, i = 1, 2, \ldots, M, \\
\overline{R}_i &\geqslant \frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta} r_0 + \left(\frac{e^{\beta(t+\mathcal{T})}-e^{\beta t}}{\mathcal{T}\beta^2} - \frac{1}{\beta}\right)\alpha + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta, t|\mathcal{T}) w_j - \epsilon, \\
& \forall t \in \aleph_i, i = 1, 2, \ldots, M, \\
& \underline{\alpha} \le \alpha \le \overline{\alpha}, \\
& \underline{\beta} \le \beta \le \overline{\beta}, \\
& \underline{r}_0 \le r_0 \le \overline{r}_0, \\
& \underline{w}_i \le w_i \le \overline{w}_i, i = 1, 2, \ldots, N.
\end{aligned}
$$

It should be noticed that the **Problem 2** is a semi-infinite programming problem with finite variables, $\alpha, \beta, r_0, \epsilon, w_i, i = 1, 2, \ldots, N$, and infinite many constraints.

3. An algorithm

In this work a cutting plane based algorithm is considered to effectively deal with the infinite number of constraints in **Problem 2** [9, 10, 11]. Following the basic concept of the cutting plane approach, we can easily design an iterative algorithm which adds one or some more constraints at a time for consideration until an optimal solution is identified. To be more specific, at the $k-th$ iteration, given subsets $N_i^k = \{\tau_1^i, \tau_2^i, \ldots, \tau_{p_i^k}^i\}$ and $\aleph_i^k = \{u_1^i, u_2^i, \ldots, u_{q_i^k}^i\}$ of $\aleph_i$, where $p_i^k, q_i^k \ge 1, i = 1, 2, \ldots, M$, we consider the following finite optimization problem.

Program SD^k

$$
\begin{aligned}
\min \quad & \phi(\alpha, \beta, r_0, w, \epsilon) = \epsilon \\
\textbf{subject to} \quad & \\
\overline{R}_i &\leqslant \frac{e^{\beta(\tau_s^i+\mathcal{T})}-e^{\beta \tau_s^i}}{\mathcal{T}\beta} r_0 + \left(\frac{e^{\beta(\tau_s^i+\mathcal{T})}-e^{\beta \tau_s^i}}{\mathcal{T}\beta^2} - \frac{1}{\beta}\right)\alpha + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta, \tau_s^i|\mathcal{T}) w_j + \epsilon, \\
& s = 1, 2, \ldots, p_i^k, i = 1, 2, \ldots, M, \\
\overline{R}_i &\geqslant \frac{e^{\beta(u_l^i+\mathcal{T})}-e^{\beta u_l^i}}{\mathcal{T}\beta} r_0 + \left(\frac{e^{\beta(u_l^i+\mathcal{T})}-e^{\beta u_l^i}}{\mathcal{T}\beta^2} - \frac{1}{\beta}\right)\alpha + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta, u_l^i|\mathcal{T}) w_j - \epsilon, \\
& l = 1, 2, \ldots, q_i^k, i = 1, \ldots, M, \\
& \underline{\alpha} \le \alpha \le \overline{\alpha}, \\
& \underline{\beta} \le \beta \le \overline{\beta}, \\
& \underline{r}_0 \le r_0 \le \overline{r}_0,
\end{aligned}
$$

$$\underline{w}i \leq w_i \leq \overline{w}_i, i = 1, 2, \ldots, N.$$

Let F^k be the feasible region of Program SD^k. Suppose that $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$ is an optimal solution of SD^k. We define the "constraint violation functions" as follows.

$$\begin{aligned} g_i^{k+1}(\tau) \quad &\triangleq \quad \overline{R}_i - \frac{e^{\beta^k(\tau+\mathcal{T})} - e^{\beta^k\tau}}{\mathcal{T}\beta^k} r_0^k - ((\frac{e^{\beta^k(\tau+\mathcal{T})} - e^{\beta^k\tau}}{\mathcal{T}(\beta^k)^2}) - \frac{1}{\beta^k})\alpha^k \\ &\quad -\sum_{j=1}^{i+\mathcal{T}} a_j(\beta^k, \tau|\mathcal{T}) w_j^k \quad - \epsilon^k, \\ &\quad \tau \in \aleph_i, i = 1, 2, \ldots, M, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} v_i^{k+1}(u) \quad &\triangleq \quad \frac{e^{\beta^k(u+\mathcal{T})} - e^{\beta^k u}}{\mathcal{T}\beta^k} r_0^k + ((\frac{e^{\beta^k(u+\mathcal{T})} - e^{\beta^k u}}{\mathcal{T}(\beta^k)^2}) - \frac{1}{\beta^k})\alpha^k \\ &\quad +\sum_{j=1}^{i+\mathcal{T}} a_j(\beta^k, u|\mathcal{T}) w_j^k - \epsilon^k - \overline{R}_i, \\ &\quad u \in \aleph_i, i = 1, 2, \ldots, M. \end{aligned} \tag{3.2}$$

Since $\overline{R}_i, a_j$ are continuous over the compact set $\aleph_i$, the function $g_i^{k+1}(\tau)$ achieves its maximum over $\aleph_i, i = 1, 2, \ldots, M$. A similar argument holds for the function $v_i^{k+1}(u), i = 1, 2, \ldots, M$. Let $\tau^i_{p_i^k+1}$ and $u^i_{q_i^k+1}$ be such maximizers, $i = 1, 2, \ldots, M$, and consider the values of $g_i^{k+1}(\tau^i_{p_i^k+1})$ and $v_i^{k+1}(u^i_{q_i^k+1}), i = 1, 2, \ldots, M$. If the values are less than or equal to zero, then $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$ becomes a feasible solution of the **Problem 2**, and hence $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$ is optimal for the **Problem 2** (because the feasible region F^k of Program SD^k is no smaller than the feasible region of the **Problem 2**). Otherwise, we know that at least $\tau^i_{p_i^k+1} \notin N_i^k$ or $u^i_{q_i^k+1} \notin \aleph_i^k, i = 1, 2, \ldots, M$. This background provides a foundation for us to outline a cutting plane algorithm for solving the **Problem 2**.

CPSD Algorithm:

Initialization

Set $k = p_i^k = q_i^k = 1, i = 1, 2, \ldots, M$; Choose any $\tau_1^i, u_1^i \in \aleph_i, i = 1, 2, \ldots, M$; Set $N_i^1 = \{\tau_1^i\}$ and $\aleph_i^1 = \{u_1^i\}, i = 1, 2, \ldots, M$.

Step 1.: Solve SD^k and obtain an optimal solution $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$.

Step 2.: Find a maximizer $\tau^i_{p_i^k+1}$ of $g_i^{k+1}(\tau)$ over $\aleph_i$ and a maximizer $u^i_{q_i^k+1}$ of $v_i^{k+1}(u)$ over $\aleph_i, i = 1, 2, \ldots, M$.

Step 3.: If $g_i^{k+1}(\tau^i_{p_i^k+1}) \leq 0$ and $v_i^{k+1}(u^i_{q_i^k+1}) \leq 0, i = 1, 2, \ldots, M$, then stop with $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$ being an optimal solution of the **Problem 2**. Otherwise, go to step 4.

Step 4.: If $g_i^{k+1}(\tau^i_{p_i^k+1}) > 0$, then set $N_i^{k+1} \leftarrow N_i^k \bigcup\{\tau^i_{p_i^k+1}\}, p_i^{k+1} \leftarrow p_i^k + 1$. Otherwise, set $N_i^{k+1} \leftarrow N_i^k, p_i^{k+1} \leftarrow p_i^k, i = 1, 2, \ldots, M$.

Step 5.: If $v_i^{k+1}(u^i_{q_i^k+1}) > 0$, then set $\aleph_i^{k+1} \leftarrow \aleph_i^k \bigcup \{u^i_{q_i^k+1}\}, q_i^{k+1} \leftarrow q_i^k + 1$. Otherwise, set $\aleph_i^{k+1} \leftarrow \aleph_i^k, q_i^{k+1} \leftarrow q_i^k, i = 1, 2, \ldots, M$.

Step 6.: Set $k \leftarrow k + 1$ go to Step 1.

When the **Problem 2** has at least one feasible solution, it can be shown without much difficulty that the CPSD algorithm either terminates in a finite number of iterations with an optimal solution or generates a sequence of points $\{(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k), k = 1, 2, \ldots\}$, which converges to an optimal solution $(\alpha^*, \beta^*, r_0^*, w^*, \epsilon^*)$, under some appropriate assumptions. However, for the above cutting plane algorithm, one major computation bottleneck lies in Step 2 of finding maximizers. Ideas of relaxing the requirement of finding global maximizers for different settings can be referred to [8, 18]. But the required computation work could still be a bottleneck. Here we propose a simple and yet very effective relaxation scheme which chooses points at which the infinite constrains are violated to a degree rather than the violation being maximized. The proposed algorithm is stated as follows.

Relaxed CPSD Algorithm:

Let $\delta > 0$ be a prescribed small number.

Initialization Set $k = p_i^k = q_i^k = 1, i = 1, 2, \ldots, M$; Choose any $\tau_1^i, u_1^i \in \aleph_i, i = 1, 2, \ldots, M$; Set $N_i^1 = \{\tau_1^i\}$ and $\aleph_i^1 = \{u_1^i\}, i = 1, 2, \ldots, M$.

Step 1.: Solve SD^k and obtain an optimal solution $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$. Define $g_i^{k+1}(\tau)$ and $v_i^{k+1}(u), i = 1, 2, \ldots, M$, according to (3.1) and (3.2),respectively.

Step 2.: Find any $\tau^i_{p_i^k+1} \in \aleph_i$ such that $g_i^{k+1}(\tau^i_{p_i^k+1}) > \delta$, and $u^i_{q_i^k+1} \in \aleph_i$ such that $v_i^{k+1}(u^i_{q_i^k+1}) > \delta, i = 1, 2, \ldots, M$.

Step 3.: If such $\tau^i_{p_i^k+1}$ and $u^i_{q_i^k+1}$ do not exist, then output $(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)$ as a solution. Otherwise, go to step 4.

Step 4.: If such $\tau^i_{p_i^k+1}$ exists, then set $N_i^{k+1} \leftarrow N_i^k \bigcup \{\tau^i_{p_i^k+1}\}, p_i^{k+1} \leftarrow p_i^k + 1$. Otherwise, set $N_i^{k+1} \leftarrow N_i^k, p_i^{k+1} \leftarrow p_i^k, i = 1, 2, \ldots, M$.

Step 5.: If such $u^i_{q_i^k+1}$ exists, then set $\aleph_i^{k+1} \leftarrow \aleph_i^k \bigcup \{u^i_{q_i^k+1}\}, q_i^{k+1} \leftarrow q_i^k + 1$. Otherwise, set $\aleph_i^{k+1} \leftarrow \aleph_i^k, q_i^{k+1} \leftarrow q_i^k, i = 1, 2, \ldots, M$.

Step 6.: Set $k \leftarrow k + 1$; go to step 1.

Note that in Step 1 of the Relaxed CPSD Algorithm, we face the challenge of solving Program SD^k, for $k \geq 1$. The presence of many non-linear inequality constrains in the problem SD^k causes difficulties in finding an optimal solution of SD^k. The method we adopted here is a so-called "aggregate constraint method" to approximate the original constraint set of SD^k by a uniform (l_p) approximation with $p \to \infty$. The detail discussion of the "aggregate constraint method" for solving the finite optimization problem SD^k can be refer to [13]. Also note that if $g_i^{k+1}(\tau^*_{i,k}) \leq 0$, where $\tau^*_{i,k}$ is a maximizer of $g_i^{k+1}(\tau)$, then $\tau^i_{p_i^k+1}$ does not exist in Step 3. A similar argument holds for checking the existence of the $u^i_{q_i^k+1}$ in Step 3. Moreover, when δ is chosen to be sufficiently small, if the relaxed algorithm terminates in a finite number of iterations at Step 3, then an optimal solution is indeed obtained, assuming that the original the **Problem 2** is feasible. We now construct a convergence proof for the relaxed CPSD algorithm.

Theorem 3.1. *Given any $\delta > 0$, assume that there is a scalar $M > 0$ such that $\| (\alpha, \beta, r_0, w, \epsilon) \| \leq M$ for each feasible solution $(\alpha, \beta, r_0, w, \epsilon)$ of SD^1(Bounded Feasible Domain Assumption), then the relaxed CPSD algorithm terminates in a finite number of iterations.*

Proof. If the relaxed CPSD algorithm does not terminate in a finite number of iterations, then the algorithm generates an infinite sequence $\{(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)\}_{k=1}^{\infty}$. We have

$$g_i^{k+1}(\tau^i_{p_i^k+1}) > \delta, i = 1, 2, \ldots, M, k = 1, 2, \ldots, \tag{3.3}$$

and

$$v_i^{k+1}(u^i_{q_i^k+1}) > \delta, i = 1, 2, \ldots, M, k = 1, 2, \ldots, \tag{3.4}$$

where $\tau^i_{p_i^k+1}$ and $u^i_{q_i^k+1}$ are generated by the relaxed CPSD algorithm.

Due to the Bounded Feasible Domain Assumption and the compactness of $\aleph_i, i = 1, 2, \ldots, M$, there exists a subsequence $\{(\alpha^{k_j}, \beta^{k_j}, r_0^{k_j}, w^{k_j}, \epsilon^{k_j})\}$ of $\{(\alpha^k, \beta^k, r_0^k, w^k, \epsilon^k)\}$ such that $\lim_{j\to\infty} (\alpha^{k_j}, \beta^{k_j}, r_0^{k_j}, w^{k_j}, \epsilon^{k_j}) = (\alpha^*, \beta^*, r_0^*, w^*, \epsilon^*)$, $\lim_{j\to\infty} \tau^i_{p_i^{k_j}+1} = \tau^*$, and $\lim_{j\to\infty} u^i_{q_i^{k_j}+1} = u^*$. Consequently, by (3.3) and (3.4), we have

$$\overline{R}_i - \tfrac{e^{\beta^*(\tau^*+\mathcal{T})} - e^{\beta^*\tau^*}}{\mathcal{T}\beta^*} r_0^* - ((\tfrac{e^{\beta^*(\tau^*+\mathcal{T})} - e^{\beta^*\tau^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* - \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, \tau^*|\mathcal{T}) w_j^* - \epsilon^* \geq \delta, i = 1, 2, \ldots, M,$$

and

$$\tfrac{e^{\beta^*(u^*+\mathcal{T})} - e^{\beta^*u^*}}{\mathcal{T}\beta^*} r_0^* + ((\tfrac{e^{\beta^*(u^*+\mathcal{T})} - e^{\beta^*u^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, u^*|\mathcal{T}) w_j^* - \epsilon^* - \overline{R}_i \geq \delta, i = 1, 2, \ldots, M.$$

However, for each $\tau^i_{p_i^k}$ and $u^i_{q_i^k}, i = 1, 2, \ldots, M, k = 1, 2, \ldots,$

$$\overline{R}_i - \tfrac{e^{\beta^l(\tau^i_{p_i^k}+\mathcal{T})} - e^{\beta^l\tau^i_{p_i^k}}}{\mathcal{T}\beta^l} r_0^l - ((\tfrac{e^{\beta^l(\tau^i_{p_i^k}+\mathcal{T})} - e^{\beta^l\tau^i_{p_i^k}}}{\mathcal{T}(\beta^l)^2}) - \tfrac{1}{\beta^l})\alpha^l - \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^l, \tau^i_{p_i^k}|\mathcal{T}) w_j^l - \epsilon^l \leq 0, i = 1, 2, \ldots, M, \ \forall\ l \geq k,$$

and

$$\tfrac{e^{\beta^l(u^i_{q_i^k}+\mathcal{T})} - e^{\beta^l u^i_{q_i^k}}}{\mathcal{T}\beta^l} r_0^l + ((\tfrac{e^{\beta^l(u^i_{q_i^k}+\mathcal{T})} - e^{\beta^l u^i_{q_i^k}}}{\mathcal{T}(\beta^l)^2}) - \tfrac{1}{\beta^l})\alpha^l + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^l, u^i_{q_i^k}|\mathcal{T}) w_j^l - \epsilon^l - \overline{R}_i \leq 0, i = 1, 2, \ldots, M, \ \forall\ l \geq k.$$

Therefore, for any fixed k, as the subsequence $\{(\alpha^{k_j}, \beta^{k_j}, r_0^{k_j}, w^{k_j}, \epsilon^{k_j})\} \rightarrow$

$(\alpha^*, \beta^*, r_0^*, w^*, \epsilon^*)$, we see that

$$\overline{R}_i - \frac{\phi e^{\beta^* \mathcal{T}} - \phi}{\mathcal{T}\beta^*} r_0^* - ((\frac{\phi e^{\beta^* \mathcal{T}} - \phi}{\mathcal{T}(\beta^*)^2}) - \frac{1}{\beta^*})\alpha^* - \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, \tau^i_{p_i^k}|\mathcal{T}) w_j^* - \epsilon^* \leq 0, i = 1, 2, \ldots, M, \text{where } \phi \stackrel{\triangle}{=} e^{\beta^*(\tau^i_{p_i^k})},$$

and

$$\frac{\varphi e^{\beta^* \mathcal{T}} - \varphi}{\mathcal{T}\beta^*} r_0^* + ((\frac{\varphi e^{\beta^* \mathcal{T}} - \varphi}{\mathcal{T}(\beta^*)^2}) - \frac{1}{\beta^*})\alpha^* + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, u^i_{q_i^k}|\mathcal{T}) w_j^* - \epsilon^* - \overline{R}_i \leq 0, i = 1, 2, \ldots, M, \text{where } \varphi \stackrel{\triangle}{=} e^{\beta^*(u^i_{q_i^k})}.$$

Since the above expression is true for all k, we have

$$\overline{R}_i - \tfrac{e^{\beta^*(\tau^* + \mathcal{T})} - e^{\beta^* \tau^*}}{\mathcal{T}\beta^*} r_0^* - ((\tfrac{e^{\beta^*(\tau^* + \mathcal{T})} - e^{\beta^* \tau^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* - \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, \tau^*|\mathcal{T}) w_j^* - \epsilon^* \leq 0, i = 1, 2, \ldots, M,$$

and

$$\tfrac{e^{\beta^*(u^* + \mathcal{T})} - e^{\beta^* u^*}}{\mathcal{T}\beta^*} r_0^* + ((\tfrac{e^{\beta^*(u^* + \mathcal{T})} - e^{\beta^* u^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, u^*|\mathcal{T}) w_j^* - \epsilon^* - \overline{R}_i \leq 0, i = 1, 2, \ldots, M,$$

which contradicts the facts that

$$\overline{R}_i - \tfrac{e^{\beta^*(\tau^* + \mathcal{T})} - e^{\beta^* \tau^*}}{\mathcal{T}\beta^*} r_0^* - ((\tfrac{e^{\beta^*(\tau^* + \mathcal{T})} - e^{\beta^* \tau^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* - \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, \tau^*|\mathcal{T}) w_j^* - \epsilon^* \geq \delta, i = 1, 2, \ldots, M,$$

and

$$\tfrac{e^{\beta^*(u^* + \mathcal{T})} - e^{\beta^* u^*}}{\mathcal{T}\beta^*} r_0^* + ((\tfrac{e^{\beta^*(u^* + \mathcal{T})} - e^{\beta^* u^*}}{\mathcal{T}(\beta^*)^2}) - \tfrac{1}{\beta^*})\alpha^* + \sum_{j=1}^{i+\mathcal{T}} a_j(\beta^*, u^*|\mathcal{T}) w_j^* - \epsilon^* - \overline{R}_i \geq \delta, i = 1, 2, \ldots, M.$$

The theorem is proved. □

4. Numerical results

The numerical examples and results of the Vasicek-type interest rate model with impulse perturbations are presented in this section. The numerical experiments are performed on the Intel Pentium 4 3.0 Ghz under the Windows XP Professional Sp2 operating system. The observed 3-MONTH TREASURY BILL RATE data of the St. Louis Federal Reserve Bank from 2007-05-15 to 2007-08-08 is employed for analysis. The initial values and bounds of the parameters of the Vasicek-type interest rate model are listed in Table 1. The numerical analysis results for different β are shown in Table 2. In Table 2, $r_0^*, \alpha^*, \beta^*, \epsilon^*$ denotes the optimal value of the **Problem 2,** and Tol is the stopping tolerance value for solving the **Problem 2**.

Figure 1(a)$\sim$(e) show the estimates of the yield curves for different initial values of β. It should be noted that when β is chosen to be large enough, the value of the mean reversion ratio will tend to a certain small value. Moreover, the numerical results of Figure 1 illustrate that our approach essentially generates the yield functions with minimal fitting errors and small oscillation.

TABLE 1. The initial values and bounds of the parameters of the Vasicek-type interest rate model

0	initial value	lower bound	upper bound
r_0	0	0	30
α	0	0	30
β	Shown in Table 2	-30	0
$w(t)$	0	-30	30
ϵ	0	0	30

TABLE 2. The numerical analysis results for different β

Case	1	2	3	4	5
β	-0.1	-1.0	-5.0	-15.0	-100.0
r_0^*	0.0165	0.0386	0.0375	0.0356	0.0329
α^*	0.8102	0.0544	0.2380	0.7066	1.3455
β^*	-3.6482	-1.0524	-4.9707	-14.9618	-28.5366
$-\frac{\alpha^*}{\beta^*}$	0.2221	0.0516	0.0479	0.0472	0.0471
ϵ^*	0.0882	0.0904	0.0906	0.0905	0.0900
Tol	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}

5. Conclusions

The Vasicek-type interest rate model with impulse perturbation is studied. The concept of deterministic perturbation is adopted to deal with the random behavior of interest rate variation. It shows that the solution of the Vasicek-type interest rate model can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is then proposed for solving the resulting optimization problem. In each iteration, we solve a finite optimization problem and add one or some more constraints. The proposed algorithm chooses a point at which the infinite constrains are violated to a degree rather than the violation being maximized. Compared to the traditional approach in the term structure literature of using stochastic processes to describe uncertainty, our method essentially generates the yield functions with minimal fitting errors and small oscillation.

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(a) β = -0.1

(b) β = -1.0

(c) β = -5

(d) β = -15

(e) β = -100

Figure 1. Yield Curves for different values of β.

Homing Chen
Department of Money and Banking, National ChengChi University, Taipei 115, Taiwan
E-mail address: 97352502@nccu.edu.tw

Cheng-Feng Hu
Department of Industrial Engineering and Management, I-Shou University, Ta-Hsu, Kaohsiung 84001, Taiwan
E-mail address: chu1@isu.edu.tw