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THE BEST CONSTANT OF SOBOLEV INEQUALITY CORRESPONDING TO ANTIPERIODIC BOUNDARY VALUE PROBLEM FOR $(-1)^M(d/dx)^{2M}$

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Abstract. The variational meaning of the special values $\zeta(2M)$ $(M=1,2,3,\cdots)$ of Riemann zeta function $\zeta(s)$ is clarified. They are essentially the best constant of Sobolev inequality, which is given explicitly by investigating Green function of the "antiperiodic" boundary value problem for differential operator $(-1)^M(d/dx)^{2M}$.

1. Conclusion

Sobolev inequalities

$$||u||_{L^q(\Omega)} \leq C||u||_{W^{m,p}(\Omega)} \qquad (\Omega \subset \mathbf{R}^n)$$

play crucial roles in the development of theory of differential equations. However, it is a rare case that the best constant among such C is found explicitly. Recently, the best constant of Sobolev inequality in the case $p=2, q=\infty$ is obtained by investigating Green function for a suitable boundary value problem [2, 9]. It should be noted that conserning the above problem there is a pioneering work by Talenti [8] in 1976, who found the best constant in another special case $q=np/(n-p)$.

Let us first survey our results [4, 5, 6]. For $M=1,2,3,\cdots$, given Sobolev spaces

$$H(\mathrm{X},M)=\left\{\ u(x)\ \middle|\ u(x),\ u^{(M)}(x)\in L^2(0,1),\quad u(x)\in A(\mathrm{X})\ \right\},$$

$$\begin{aligned}
A(\mathrm{P})\ &:\ \ u^{(i)}(1)-u^{(i)}(0)=0 \quad (0\leq i\leq M-1), \quad \int_0^1 u(x)dx=0,\\
A(\mathrm{D})\ &:\ \ u^{(2i)}(0)=u^{(2i)}(1)=0 \quad (0\leq i\leq [\,(M-1)/2\,]),\\
A(\mathrm{N})\ &:\ \ u^{(2i+1)}(0)=u^{(2i+1)}(1)=0 \quad (0\leq i\leq [\,(M-2)/2\,]), \quad \int_0^1 u(x)dx=0,
\end{aligned}$$

where the boundary conditions for $u(x)$ in $A(\mathrm{N})$ are not required when $M=1$, we have found the best constants of the corresponding Sobolev inequalities, which are expressed by using Riemann zeta function as

$$\begin{aligned}
C(\mathrm{P},M) &= 2^{-(2M-1)}\pi^{-2M}\zeta(2M),\\
C(\mathrm{D},M) &= 2^{-(2M-1)}\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M),\\
C(\mathrm{N},M) &= 2\pi^{-2M}\zeta(2M).
\end{aligned}$$

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