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A NEW INTERPRETATION OF DJOKOVIĆ'S INEQUALITY

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Dedicated to Professor Ryotaro Sato on his sixtieth birthday

Abstract. In view of the convex analysis theory, we give a new interpretation of Djoković's inequality which is an extension of Hlawka's inequality on a Hilbert space.

1. Introduction

Let H be a Hilbert space. Then the following inequality

$$\|x+y\|+\|y+z\|+\|z+x\| \le \|x\|+\|y\|+\|z\|+\|x+y+z\| \tag{1}$$

holds for all $x, y, z \in H$ (cf. [1], [3]). This is well-known as Hlawka's inequality and it has various extensions. In 1963, D. Ž. Djoković [2] showed the following extension.

Theorem D. *Let H be a Hilbert space and n, k natural numbers with $2 \le k \le n-1$. Then*

$$\sum_{1\le i_1<\cdots<i_k\le n} \|x_{i_1}+\cdots+x_{i_k}\| \le \binom{n-2}{k-1}\sum_{i=1}^{n}\|x_i\| + \binom{n-2}{k-2}\left\|\sum_{i=1}^{n} x_i\right\| \tag{2}$$

holds for all $x_1, \ldots, x_n \in H$.

In the next year, D. M. Smiley and M. F. Smiley [4] has independently shown that the same inequality (2) holds on a Banach space which satisfies Hlawka's inequality (1). It is easily to see that every Banach space which is isometric to subspace of L_1-space is such a space. However, it seems that it is difficult to determine such a space. We want to call simply "Hlawka space" such a space, but they call "quadrilateral space".

Now, we have the following natural question: What does Djoković's inequality mean, and what does the constants appearing in it represent? The purpose of this paper is to give an answer to the above problem from the standpoint of the convex analysis theory.

2. Interpretation

Let X be a (real or complex) Banach space and n a natural number. For $x_1, \ldots, x_n \in X$ and $1 \le k \le n$, set

$$\delta_k(x_1,\ldots,x_n) = \sum_{1\le i_1<\cdots<i_k\le n} \|x_{i_1}+\cdots+x_{i_k}\|.$$

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Then $\{\delta_k : 1 \leq k \leq n\}$ constitutes a system of seminorms on the linear space $X \oplus \cdots \oplus X$ such that

$$\binom{n-1}{k-1}\delta_n \leq \delta_k \leq \binom{n-1}{k-1}\delta_1 \quad (1 \leq k \leq n). \tag{3}$$

In fact, for any $(x_1, \ldots, x_n) \in X \oplus \cdots \oplus X$, we have

$$\begin{aligned}\delta_k(x_1, \ldots, x_n) &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \|x_{i_1} + \cdots + x_{i_k}\| \\ &\leq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} \|x_{i_j}\| = \binom{n-1}{k-1}\delta_1(x_1, \ldots, x_n)\end{aligned}$$

and

$$\begin{aligned}\delta_k(x_1, \ldots, x_n) &\geq \left\| \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} + \cdots + x_{i_k} \right\| \\ &= \left\| \binom{n-1}{k-1}x_1 + \cdots + \binom{n-1}{k-1}x_n \right\| = \binom{n-1}{k-1}\delta_n(x_1, \ldots, x_n).\end{aligned}$$

Hence the inequality (3) holds. Therefore it will be natural to consider the following set, say Djoković's domain:

$$D(n, k; X) = \{(\alpha, \beta) \in \boldsymbol{R}^2 : \delta_k \leq \alpha\delta_1 + \beta\delta_n \text{ on } X \oplus \cdots \oplus X\}.$$

Then Djoković's inequality (2) can be rewriten as follows: The point

$$\left(\binom{n-2}{k-1}, \binom{n-2}{k-2}\right),$$

say Djoković's point, belongs to Djoković's domain $D(n, k; H)$ for a Hlawka space H. Hence if we investigate a geometrical relation between Djoković's point and Djokovic's domain, then we will obtain a new interpretation of Djoković's inequality.

Actually, we will see in the next section that Djoković's domain for a Hlawka space H is uniquely determined independent of H and that it is the widest among Djoković's domains for all Banach spaces. Moreover, we know that Djoković's point is the only extreme point of Djoković's domain for a Hlawka space.

3. Results and Proofs

Let us state in more detail the assertion in the preceding section.

Theorem 1. *Let X be a non-trivial Banach space and $1 \leq k \leq n$. Then*

(i) *$D(n, k; X)$ is a closed convex subset of $\boldsymbol{R}^2$.*
(ii) *$D(1, 1; X) = \{(\alpha, \beta) \in \boldsymbol{R}^2 : \alpha + \beta \geq 1\}$.*
(iii) *$D(n, 1; X) = \{(\alpha, \beta) \in \boldsymbol{R}^2 : \alpha \geq 1$ and $\alpha + \beta \geq 1\}$ for $n \geq 2$.*
(iv) *$D(n, n; X) = \{(\alpha, \beta) \in \boldsymbol{R}^2 : \alpha \geq 0$ and $\alpha + \beta \geq 1\}$ for $n \geq 2$.*
(v) *$D(n, k; X) \subseteq \left\{(\alpha, \beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-2}{k-1} \text{ and } \alpha + \beta \geq \binom{n-1}{k-1}\right\}$ for $2 \leq k \leq n-1$.*

(vi) *If X is a Hlawka space, then*

$$D(n,k;X) = \left\{ (\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-2}{k-1} \text{ and } \alpha+\beta \geq \binom{n-1}{k-1} \right\}$$

for $2 \leq k \leq n-1$.

(vii) *If X is a Hlawka space, then $\left(\binom{n-2}{k-1}, \binom{n-2}{k-2}\right)$ is the only extreme point of $D(n,k;X)$ for $2 \leq k \leq n-1$.*

Remark 1. The converse of (vi) is also true. This follows immediately from Proposition 3.

Proof. (i) and (ii) These follow from an easy observation.

(iii) Let $(\alpha,\beta) \in D(n,1;X)$ and e a unit vector in X. Then

$$\delta_1(e,-e,0,\dots,0) \leq \alpha\delta_1(e,-e,0,\dots,0) + \beta\delta_n(e,-e,0,\dots,0)$$

holds and hence $1 \leq \alpha$. Also

$$\delta_1(e,0,\dots,0) \leq \alpha\delta_1(e,0,\dots,0) + \beta\delta_n(e,0,\dots,0)$$

holds and hence $1 \leq \alpha+\beta$. Therefore $D(n,1;X) \subseteq \{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq 1 \text{ and } \alpha+\beta \geq 1\}$.

Conversely, observe that all points on the semi-lines L_1 and L_2 belong to the domain $D(n,1;X)$, where

$$L_1 = \{(\alpha,\beta) : \alpha = 1,\ \beta \geq 0\}$$

and

$$L_2 = \{(\alpha,\beta) : \alpha+\beta = 1,\ \beta \leq 0\}.$$

Since $\mathrm{co}(L_1 \cup L_2) = \{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq 1 \text{ and } \alpha+\beta \geq 1\}$, it follows from (i) that the inverse inclusion holds. Here "co" denotes the convex hull.

(iv) This follows from the same observation as (iii).

(v) Suppose $2 \leq k \leq n-1$. Let $(\alpha,\beta) \in D(n,k;X)$ and e a unit vector in X. Then

$$\delta_k(e,-e,0,\dots,0) \leq \alpha\delta_1(e,-e,0,\dots,0) + \beta\delta_n(e,-e,0,\dots,0)$$

holds. Since

$$\delta_k(e,-e,0,\dots,0) = 2\binom{n-2}{k-1},\ \delta_1(e,-e,0,\dots,0) = 2 \text{ and } \delta_n(e,-e,0,\dots,0) = 0,$$

it follows that $\binom{n-2}{k-1} \leq \alpha$. Also

$$\delta_k(e,0,\dots,0) \leq \alpha\delta_1(e,0,\dots,0) + \beta\delta_n(e,0,\dots,0)$$

holds. Since $\delta_k(e,0,\dots,0) = \binom{n-1}{k-1}$ and $\delta_1(e,0,\dots,0) = \delta_n(e,0,\dots,0) = 1$, it follows that $\binom{n-1}{k-1} \leq \alpha+\beta$. Consequently, we obtain the desired result.

(vi) Suppose that X is a Hlawka space and $2 \leq k \leq n-1$. By Djoković's inequality, we see that $\left(\binom{n-2}{k-1}, \binom{n-2}{k-2}\right)$ belongs to Djoković's domain $D(n,k;X)$. This fact implies that all points on the semi-line

$$L_3 = \left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha = \binom{n-2}{k-1},\ \beta \geq \binom{n-2}{k-2}\right\}$$

also belong to $D(n,k;X)$. Moreover, all points on the semi-line

$$L_4 = \left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \beta \leq 0,\ \alpha + \beta = \binom{n-1}{k-1}\right\}$$

also belong to $D(n,k;X)$. In fact, if $\beta \leq 0$ and $x_1,\ldots,x_n \in X$, then we have

$$\begin{aligned}
\delta_k(x_1,\ldots,x_n) &- \beta\delta_n(x_1,\ldots,x_n) \\
&\leq \sum_{1\leq i_1<\cdots<i_k\leq n} (\|x_{i_1}\| + \cdots + \|x_{i_k}\|) - \beta\delta_n(x_1,\ldots,x_n) \\
&\leq \binom{n-1}{k-1}\delta_1(x_1,\ldots,x_n) - \beta\delta_1(x_1,\ldots,x_n) \\
&= \left(\binom{n-1}{k-1} - \beta\right)\delta_1(x_1,\ldots,x_n)
\end{aligned}$$

and hence $\left(\binom{n-1}{k-1} - \beta, \beta\right)$ must belong to $D(n,k;X)$. Therefore all points on L_4 belong to $D(n,k;X)$. Then we see from (i) that $\mathrm{co}(L_3 \cup L_4) \subseteq D(n,k;X)$.

On the other hand, note that $\binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$ and hence

$$\mathrm{co}(L_3 \cup L_4) = \left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-2}{k-1} \text{ and } \alpha+\beta \geq \binom{n-1}{k-1}\right\}.$$

Consequently, $\left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-2}{k-1} \text{ and } \alpha+\beta \geq \binom{n-1}{k-1}\right\} \subseteq D(n,k;X)$. The inverse inclusion follows from (v).

(vii) This follows immediately from (vi). □

The preceding theorem gives an estimate of Djoković's domain from above. The following result gives an estimate from below.

Theorem 2. *Let X be a non-trivial Banach space and $2 \leq k \leq n-1$. Then*

(i) $\left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-1}{k-1} \text{ and } \alpha+\beta \geq \binom{n-1}{k-1}\right\} \subseteq D(n,k;X)$ *for* $2 \leq k \leq \frac{n}{2}$.

(ii) $\left\{(\alpha,\beta) \in \boldsymbol{R}^2 : \alpha \geq \binom{n-1}{k},\ \alpha+\beta \geq \binom{n-1}{k-1} \text{ and } n\alpha + (2k-n)\beta \geq n\binom{n-1}{k-1}\right\} \subseteq D(n,k;X)$ *for* $\frac{n}{2} < k \leq n-1$.

Proof. Let $x_1, \dots, x_n \in X$. Then we have

$$\begin{aligned}\delta_k(x_1, \dots, x_n) &= \sum_{1 \le j_1 < \cdots < j_{n-k} \le n} \left\| x_1 + \cdots + x_n - (x_{j_1} + \cdots + x_{j_{n-k}}) \right\| \\ &\le \sum_{1 \le j_1 < \cdots < j_{n-k} \le n} \| x_1 + \cdots + x_n \| \\ &\quad + \sum_{1 \le j_1 < \cdots < j_{n-k} \le n} \left(\|x_{j_1}\| + \cdots + \|x_{j_{n-k}}\| \right) \\ &= \binom{n}{k} \delta_n(x_1, \dots, x_n) + \binom{n-1}{n-k-1} \delta_1(x_1, \dots, x_n).\end{aligned}$$

Since $\binom{n-1}{n-k-1} = \binom{n-1}{k}$, it follows that the point $\left(\binom{n-1}{k}, \binom{n}{k}\right)$ belongs to Djoković's domain $D(n, k; X)$. This implies that all points on the semi-line

$$L_5 = \left\{ (\alpha, \beta) \in \boldsymbol{R}^2 : \alpha = \binom{n-1}{k}, \ \beta \ge \binom{n}{k} \right\}$$

also belongs to $D(n, k; X)$. Also since $\delta_k \le \binom{n-1}{k-1} \delta_1$, it follows that all points on the semi-line

$$L_6 = \left\{ (\alpha, \beta) \in \boldsymbol{R}^2 : \alpha = \binom{n-1}{k-1}, \ \beta \ge 0 \right\}$$

belong to $D(n, k; X)$. Moreover, as observed in the proof of Theorem 1-(vi), all points on the semi-line $L_4 = \left\{ (\alpha, \beta) \in \boldsymbol{R}^2 : \beta \le 0, \ \alpha + \beta = \binom{n-1}{k-1} \right\}$ belong to $D(n, k; X)$. Then we have $\mathrm{co}(L_4 \cup L_5 \cup L_6) \subseteq D(n, k; X)$. Note that $\binom{n-1}{k-1} \le \binom{n-1}{k}$ if and only if $k \le \frac{n}{2}$. Hence $\mathrm{co}(L_4 \cup L_6) \subseteq D(n, k; X)$ for $2 \le k \le \frac{n}{2}$ and $\mathrm{co}(L_4 \cup L_5) \subseteq D(n, k; X)$ for $\frac{n}{2} \le k \le n - 1$. Consequently, we obtain the desired result. $\square$

Remark 2. Let X be a Banach space, and set

$$\begin{gathered} D(X) = D(3, 2; X), \\ D_\infty = \{ (\alpha, \beta) \in \boldsymbol{R}^2 : \alpha \ge 1, \ \alpha + \beta \ge 2 \text{ and } 3\alpha + \beta \ge 6 \}, \\ D_H = \{ (\alpha, \beta) \in \boldsymbol{R}^2 : \alpha \ge 1, \ \alpha + \beta \ge 2 \}. \end{gathered}$$

Then $D_\infty \subseteq D(X) \subseteq D_H$ from Theorems 1 and 2. Moreover, $D(X) = D_H$ if and only if X is a Hlawka space from Theorem 1 and the definition of Hlawka space. We also see that if $X = l_n^\infty(\boldsymbol{R})$ $(3 \le n \le \infty)$, then $D_\infty = D(X)$. In fact, consider the following three elements in $l_n^\infty(\boldsymbol{R})$:

$$x = (-1, 1, 1, 0, 0, \dots), \ y = (1, -1, 1, 0, 0, \dots) \text{ and } z = (1, 1, -1, 0, 0, \dots).$$

Then the inequality

$$\|x + y\| + \|y + z\| + \|z + x\| \le \alpha(\|x\| + \|y\| + \|z\|) + \beta \, \|x + y + z\|$$

can be rewritten by $3\alpha+\beta \geq 6$ and hence $D(l_n^\infty(\boldsymbol{R})) \subseteq \{(\alpha,\beta)\in \boldsymbol{R}^2 : 3\alpha+\beta\geq 6\}$ holds. On the other hand, since $D(l_n^\infty(\boldsymbol{R})) \subseteq D_H$ holds, it follows that $D(l_n^\infty(\boldsymbol{R})) \subseteq D_\infty$ and hence $D(l_n^\infty(\boldsymbol{R})) = D_\infty$. It seems that it is difficult to determine a class of Banach spaces X on which $D_\infty = D(X)$ holds.

The following result shows that a Banach space on which Djoković's inequality holds reduces to a Hlawka space.

Proposition 3. *A Banach space X is Hlawka if and only if there exists natural numbers n and k such that $2\leq k\leq n-1$ and*

$$\sum_{1\leq i_1<\cdots<i_k\leq n} \|x_{i_1}+\cdots+x_{i_k}\| \leq \binom{n-2}{k-1}\sum_{i=1}^n \|x_i\| + \binom{n-2}{k-2}\left\|\sum_{i=1}^n x_i\right\|$$

holds for any $x_1,\ldots,x_n\in X$.

Proof. (i) Necessity. Take $n=3$ and $k=2$.

(ii) Sufficiency. Let n and k be such that $2\leq k\leq n-1$ and suppose

$$(*)\qquad \sum_{1\leq i_1<\cdots<i_k\leq n} \|x_{i_1}+\cdots+x_{i_k}\| \leq \binom{n-2}{k-1}\sum_{i=1}^n \|x_i\| + \binom{n-2}{k-2}\left\|\sum_{i=1}^n x_i\right\|$$

holds for any $x_1,\ldots,x_n\in X$. We can assume $n\geq 4$. Let us consider the case of $x_4=\cdots=x_n=0$. Then

$$\begin{aligned}\binom{n-2}{k-1}\sum_{i=1}^n \|x_i\| &+ \binom{n-2}{k-2}\left\|\sum_{i=1}^n x_i\right\| \\ &= \binom{n-2}{k-1}(\|x_1\|+\|x_2\|+\|x_3\|) + \binom{n-2}{k-2}\|x_1+x_2+x_3\|.\end{aligned}$$

We set

$$\begin{aligned}
\rho_0 &= \#\{(i_1,\ldots,i_k) : 1\leq i_1<\cdots<i_k\leq n,\ \|x_{i_1}+\cdots+x_{i_k}\| = \|x_1+x_2+x_3\|\},\\
\rho_j &= \#\{(i_1,\ldots,i_k) : 1\leq i_1<\cdots<i_k\leq n,\ \|x_{i_1}+\cdots+x_{i_k}\| = \|x_j\|\}\ (j=1,2,3),\\
\rho_4 &= \#\{(i_1,\ldots,i_k) : 1\leq i_1<\cdots<i_k\leq n,\ \|x_{i_1}+\cdots+x_{i_k}\| = \|x_1+x_2\|\},\\
\rho_5 &= \#\{(i_1,\ldots,i_k) : 1\leq i_1<\cdots<i_k\leq n,\ \|x_{i_1}+\cdots+x_{i_k}\| = \|x_2+x_3\|\},
\end{aligned}$$

and

$$\rho_6 = \#\{(i_1,\ldots,i_k) : 1\leq i_1<\cdots<i_k\leq n,\ \|x_{i_1}+\cdots+x_{i_k}\| = \|x_3+x_1\|\},$$

where # denotes the cardinal number. Then

$$\begin{aligned}\sum_{1\leq i_1<\cdots<i_k\leq n} &\|x_{i_1}+\cdots+x_{i_k}\| \\ &= \rho_0\|x_1+x_2+x_3\| + \rho_1\|x_1\| + \rho_2\|x_2\| + \rho_3\|x_3\| \\ &\quad + \rho_4\|x_1+x_2\| + \rho_5\|x_2+x_3\| + \rho_6\|x_3+x_1\|.\end{aligned}$$

Note that

$$\rho_0 = \begin{cases} 0, & \text{if } k = 2 \\ \binom{n-3}{k-3}, & \text{if } k \geq 3 \end{cases}, \ \rho_1 = \rho_2 = \rho_3 = \binom{n-3}{k-1} \text{ and } \rho_4 = \rho_5 = \rho_6 = \binom{n-3}{k-2}.$$

Hence, if $k = 2$, then

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \|x_{i_1} + \cdots + x_{i_k}\| = (n-3)(\|x_1\| + \|x_2\| + \|x_3\|) + \|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\|$$

and

$$\binom{n-2}{k-1} \sum_{i=1}^{n} \|x_i\| + \binom{n-2}{k-1} \left\| \sum_{i=1}^{n} x_i \right\| = (n-2)(\|x_1\| + \|x_2\| + \|x_3\|) + \|x_1 + x_2 + x_3\| .$$

Then (*) implies that

$$\|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\| \leq \|x_1\| + \|x_2\| + \|x_3\| + \|x_1 + x_2 + x_3\|$$

holds for any $x_1, x_2, x_3 \in X$. Therefore X is a Hlawka space. Moreover, if $k \geq 3$, then

$$\begin{aligned} \sum_{1 \leq i_1 < \cdots < i_k \leq n} &\|x_{i_1} + \cdots + x_{i_k}\| \\ &= \binom{n-3}{k-3} \|x_1 + x_2 + x_3\| + \binom{n-3}{k-1} (\|x_1\| + \|x_2\| + \|x_3\|) \\ &\quad + \binom{n-3}{k-2} (\|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\|) \end{aligned}$$

and hence (*) implies that

$$\tag{**} \begin{aligned} \binom{n-3}{k-3} \|x_1 + x_2 + x_3\| &+ \binom{n-3}{k-1} (\|x_1\| + \|x_2\| + \|x_3\|) \\ &+ \binom{n-3}{k-2} (\|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\|) \\ \leq \binom{n-2}{k-1} (\|x_1\| + \|x_2\| + \|x_3\|) &+ \binom{n-2}{k-2} \|x_1 + x_2 + x_3\| \end{aligned}$$

holds for any $x_1, x_2, x_3 \in X$. Note that

$$\binom{n-2}{k-2} - \binom{n-3}{k-3} = \binom{n-2}{k-1} - \binom{n-3}{k-1} = \binom{n-3}{k-2}$$

and so (**) implies that

$$\|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\| \leq \|x_1\| + \|x_2\| + \|x_3\| + \|x_1 + x_2 + x_3\|$$

holds for any $x_1, x_2, x_3 \in X$. Therefore X is a Hlawka space. □

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