Journal of Nonlinear and Convex Analysis
Volume 1, Number 3, 2000, 305–311

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

LIPSCHITZ FUNCTIONS ON SUBSPACES OF ASPLUND GENERATED SPACES ARE GENERICALLY VIRTUALLY PSEUDO-REGULAR

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Abstract. Locally Lipschitz functions on separable Banach spaces are generically pseudo-regular. A slightly weaker property does hold on all Banach spaces which are closed subspaces of Asplund generated spaces. We deduce that for locally Lipschitz functions on such spaces the set of points of Gâteaux but not strict differentiability is of the first category.

A real valued function ψ on an open subset A of a Banach space X is *locally Lipschitz* if given $x \in A$ there exists $K > 0$ and $\delta > 0$ such that

$$|\psi(y) - \psi(z)| \leq K\|y - z\| \quad \text{for all } y, z \in B(x; \delta).$$

Important tools in the study of the differentiability of such a function ψ are the *Dini directional derivative* at $x \in A$ in the direction $y \in X$,

$$\psi^+(x)(y) := \limsup_{\lambda \to 0^+} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

and the *Clarke directional derivative* at $x \in A$ in the direction $y \in X$

$$\psi^\circ(x)(y) := \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda}.$$

We say that ψ is *pseudo-regular at* $x \in A$ *in the direction* $y \in X$ if

$$\psi^\circ(x)(y) = \psi^+(x)(y)$$

and is *pseudo-regular at* $x \in A$ if it is pseudo-regular at x in all directions y. In general, given $y \in X$, ψ is pseudo-regular in the direction y at the points of a residual subset of A and if X is separable then ψ is pseudo-regular at the points of a residual subset of A, [G-S, p208]. It is not known whether this property extends in general beyond separable spaces. However we can define a property slightly weaker than pseudo-regularity and show that it holds generically for all locally Lipschitz functions on spaces belonging to a large class which includes the separable spaces.

Now the Clarke directional derivative has useful continuity properties: given $x \in A$, $\psi^\circ(x)(y)$ is sublinear in y and given $y \in X$, $\psi^\circ(x)(y)$ is upper semi-continuous in x. So we are able to define the *Clarke subdifferential* at $x \in A$

$$\partial^\circ\psi(x) := \{f \in X^* : f(y) \leq \psi^\circ(x)(y) \quad \text{for all } y \in X\}$$

a non-empty weak*compact convex subset of X^* and the *Clarke subdifferential mapping* $x \mapsto \partial^\circ\psi(x)$ a locally bounded weak*upper semi-continuous set-valued

2000 *Mathematics Subject Classification.* 49J52.

mapping. The Dini directional derivative does not have comparable useful continuity properties. However we can define an associated directional derivative at $x \in A$ in the direction $y \in X$

$$\psi^{\square}(x)(y) := \psi^{\circ}(x)(y) - \psi^{+}(x)(y)$$

and aim to study the associated subdifferential

$$\partial^{\square}\psi(x) := \{f \in X^* : f(y) \leq \psi^{\square}(x)(y) \quad \text{for all } y \in X\}.$$

Now $0 \in \partial^{\square}\psi(x)$ for all $x \in A$ and if ψ is pseudo-regular at $x \in A$ then $\partial^{\square}\psi(x) = \{0\}$. But since $\psi^{\square}(x)(y)$ is not in general sublinear in y the converse does not hold. So we are led to say that ψ is *virtually pseudo-regular* at $x \in A$ if $\partial^{\square}\psi(x) = \{0\}$.

Although virtual pseudo-regularity is a generalisation of pseudo-regularity a simple example shows that it is somewhat weaker.

Example
Consider the Lipschitz function ψ on $\mathbf{R}^2$ given in polar coordinates by

$$\psi(r,\theta) = r\sin 2\theta.$$

For all directions $(r,\theta) \in \mathbf{R}^2\backslash(0,0)$, $\psi^{+}(0,0)(r,\theta) = r\sin 2\theta$; but for all $(r_1,\theta_1) \in \mathbf{R}^2\backslash(0,0)$, $\psi^{+}(r_1,\theta_1)(r,\theta) = 2\theta r_1\cos 2\theta_1 + r\sin 2\theta_1 \rightarrow r\sin 2\theta_1$ as $r_1 \rightarrow 0$. So $\psi^{\circ}(0,0)(r,\theta) = r$. Then ψ is not pseudo-regular at $(0,0)$ although $\partial^{\square}\psi(0,0) = \{0\}$.

We overcome the continuity deficiencies of the Dini directional derivative as follows. Given $p \in \mathbf{N}$ we define the *approximate Dini directional derivative* at $x \in A$ in the direction $y \in X$ as

$$\begin{cases} \psi_p^{+}(x)(y) := \sup\limits_{0<\lambda<1/p} \dfrac{\psi(x+\lambda y) - \psi(x)}{\lambda} & \text{for } \|y\| = 1 \\ \psi_p^{+}(x)(\alpha y) := \alpha\psi_p^{+}(x)(y) & \alpha \geq 0. \end{cases}$$

We observe that it has useful continuity properties: given $x \in A$, $\psi_p^{+}(x)(y)$ is continuous in y and given $y \in X$, because $\psi_p^{+}(x)(y)$ is the supremum of continuous functions, it is lower semi-continuous in x. Further,

$$\psi^{+}(x)(y) = \lim_{p\rightarrow\infty} \psi_p^{+}(x)(y).$$

For our purposes we consider the associated directional derivative at $x \in A$ in the direction $y \in X$ defined by

$$\psi_p^{\square}(x)(y) := \psi^{\circ}(x)(y) - \psi_p^{+}(x)(y)$$

which for given $y \in X$ is upper semi-continuous in x. Clearly,

$$\psi^{\square}(x)(y) = \lim_{p\rightarrow\infty} \psi_p^{\square}(x)(y).$$

Our result on virtual pseudo-regularity is derived from an analysis of the associated subdifferential of ψ at $x \in A$ defined as

$$\partial_p^{\square}\psi(x) := \{f \in X^* : f(y) \leq \max\{0, \psi_p^{\square}(x)(y)\} \quad \text{for all } y \in X\}.$$

It is not difficult to see that this is a weak*compact convex subset of X^* and that $0 \in \partial_p^{\square}\psi(x)$ for all $x \in A$. Furthermore, it follows from the continuity properties of $\psi_p^\square(x)(y)$ that the associated subdifferential mapping $x \mapsto \partial_p^\square \psi(x)$ is a locally bounded weak*upper semi-continuous set-valued mapping. Clearly

$$\partial^\square \psi(x) = \cup_{p\in\mathbf{N}} \partial_p^\square \psi(x).$$

We now proceed to define the class of spaces on which we will establish our virtual pseudo-regularity property. A real-valued function ψ on an open subset A of a Banach space X is *Gâteaux differentiable* at $x \in A$ if there exists a continuous linear functional $\psi'(x)$ on X and given $\epsilon > 0$ and $y \in X$ there exists $\delta(\epsilon, y) > 0$ such that

$$\left|\frac{\psi(x+\lambda y)-\psi(x)}{\lambda} - \psi'(x)(y)\right| < \epsilon \quad \text{for all } \lambda \in (0,\delta),$$

and is *Fréchet differentiable* at $x \in A$ if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\left|\frac{\psi(x+\lambda y)-\psi(x)}{\lambda} - \psi'(x)(y)\right| < \epsilon \quad \text{for all } \lambda \in (0,\delta) \quad \text{and } y \in X, \|y\| = 1.$$

A Banach space Z is an *Asplund (weak Asplund)* space if every continuous convex function on an open convex subset A of Z is Fréchet (Gâteaux) differentiable at the points of a residual subset of A. A Banach space Y where there exists an Asplund space Z and a continuous linear mapping $T : Z \to Y$ such that $Y = \overline{T(Z)}$ is called an *Asplund generated space*. All weakly compactly generated spaces are Asplund generated spaces, [F,p12]. We are interested in considering a Banach space X which is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$. The class of such spaces is a tractable subclass of the weak Asplund spaces, [F, p17] and we aim to show that on a space of this class locally Lipschitz functions are generically virtually pseudo-regular.

Fabian and Preiss [F-P] developed special techniques for the analysis of a Banach space X which is a closed linear subspace of an Asplund generated space $Y = \overline{T(Z)}$. Given $n \in \mathbf{N}$, they consider the set

$$A_n := (T(B(Z)) + \frac{1}{n}B(Y)) \cap X$$

whose Minkowski gauge $\|.\|_n$ is an equivalent norm for X. Our theorem is built on the following crucial properties developed for such a space X with such equivalent norms.

Given a non-empty subset E of the dual X^* of a Banach space X a *weak*slice* of E is a non-empty subset of E of the form

$$S(E) := \{f \in E : f(x) > r\}$$

where we are given $x \in X$ and $r \in \mathbf{R}$. We have a key slice property.

Lemma 1, [F-P,p738]
*Consider a Banach space X which is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$. Given $n \in \mathbf{N}$ every non-empty subset E of $B(X^*)$ has a weak*slice $S(E)$ with $\|.\|_n^*$-diam $S(E) < 3/n$.*

Accompanying this result we have an important differentiability property for locally Lipschitz functions.

Lemma 2, [F-P,p735]
*Consider a Lipschitz function ψ with Lipschitz constant 1 on an open subset A of a Banach space X which is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$. Given $n \in \mathbf{N}$ and a weak*slice $S(\partial^\circ\psi(A))$ with $\|.\|_n^*$-diam $S(\partial^\circ\psi(A)) < 3/n$ then there exists an $x_n \in A$ and $f_{x_n} \in \partial^\circ\psi(x_n) \cap S(\partial^\circ\psi(A))$ and $\delta_n > 0$ such that*

$$\left|\frac{\psi(x_n+\lambda y)-\psi(x_n)}{\lambda} - f_{x_n}(y)\right| < \frac{9}{n} \quad \text{for all } \lambda \in (0,\delta_n) \text{ and all } y \in X, \|y\|_n = 1.$$

For the proof of our theorem we need the following special property.

Lemma 3
Consider a Lipschitz function ψ with Lipschitz constant 1 on an open subset A of a Banach space X which is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$. If, given $n \in \mathbf{N}$, at $x_n \in A$ there exists $f_{x_n} \in B(X^*)$ and $\delta_n > 0$ such that

$$\left|\frac{\psi(x_n+\lambda y)-\psi(x_n)}{\lambda} - f_{x_n}(y)\right| < \frac{9}{n} \quad \textit{for all } \lambda \in (0,\delta_n) \textit{ and all } y \in X, \|y\|_n = 1$$

then given $p \in \mathbf{N}$ there exists an open neighbourhood U of x_n such that

$$f_{x_n} + \partial_p^\square\psi(U) \subset \partial^\circ\psi(U) + \frac{18}{n}B_n(X^*).$$

Proof There exists $\delta \in (0, \min(\delta_n, 9/n, 1/p))$ such that

$$\left|\frac{\psi(x_n+\lambda y)-\psi(x_n)}{\lambda} - f_{x_n}(y)\right| < \frac{9}{n} \quad \text{for all } \lambda \in (0,\delta) \text{ and all } y \in X, \|y\|_n = 1.$$

But then

$$\left|\frac{\psi(z+\lambda y)-\psi(z)}{\lambda} - f_{x_n}(y)\right| \le \frac{2\|z-x_n\|}{\lambda} + \left|\frac{\psi(x_n+\lambda y)-\psi(x_n)}{\lambda} - f_{x_n}(y)\right| < \frac{18}{n}$$

for all $z \in U := B(x_n; 9\delta/(4n))$, all $\lambda \in (\delta/2, \delta)$ and all $y \in X, \|y\|_n = 1$. Therefore,

$$f_{x_n}(y) - \psi_p^+(z)(y) < f_{x_n}(y) - \sup_{\delta/2<\lambda<\delta} \frac{\psi(z+\lambda y)-\psi(z)}{\lambda} < \frac{18}{n}$$

for all $z \in U$ and $y \in X, \|y\|_n = 1$. Then for $f \in f_{x_n} + \partial_p^\square\psi(z)$ we have

$$\begin{aligned} f(y) &\le f_{x_n}(y) + \max\{0, \psi^\circ(z)(y) - \psi_p^+(z)(y)\} \\ &\le \max\{f_{x_n}(y), \psi^\circ(z)(y) + f_{x_n}(y) - \psi_p^+(z)(y)\} \\ &\le \max\{f_{x_n}(y), \psi^\circ(z)(y) + 18/n\} \quad \text{for all } y \in X, \|y\|_n = 1 \end{aligned}$$

which implies that $f \in \partial^\circ\psi(U) + \frac{18}{n}B_n(X^*)$. □

We have seen that a Banach space X which is a closed linear subspace of an Asplund generated space $Y = \overline{T(Z)}$ has for each $n \in \mathbf{N}$ an equivalent norm $\|.\|_n$ with dual norm $\|.\|_n^*$ on X^*. We note that there is a norm $\|.\|_\rho$ on X^* related to all of these dual norms $\|.\|_n^*$ on X^* as follows.

Lemma 4 [G-S, Corol 3.4]
Consider a Banach space X which is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$. There is a norm $\|.\|_\rho$ on the dual X^* which has the property that, given a bounded subset E of X^* there exists an $\alpha > 0$ such that

$$\|f\|_\rho \le \|f\|_n^* + \alpha/n \quad \text{for all } f \in E.$$

We are now in a position to establish our theorem.

Theorem
A locally Lipschitz function ψ on an open subset A of a Banach space X which is a closed linear subspace of an Asplund generated space $Y = \overline{T(Z)}$ has the property that $\partial^\square \psi(x) = \{0\}$ at the points of a residual subset of A.

Proof We may assume without loss of generality that ψ is Lipschitz with Lipschitz constant 1 on A, [F-P, p378]. Consider a non-empty subset U of A.
By Lemma 1, given $n \in \mathbf{N}$ there exists a weak*slice generated by some $y \in X$, $\|y\| = 1$ and $r \in \mathbf{R}$

$$S(\partial^\circ \psi(U)) := \{f \in \partial^\circ \psi(U) : f(y) > r\}$$

such that $\|.\|_n^*$-diam $S(\partial^\circ \psi(U)) < 3/n$.
By Lemma 2, there exists $x_n \in U$ and $f_{x_n} \in \partial^\circ \psi(x_n) \cap S(\partial^\circ \psi(U))$ and $\delta_n > 0$ such that

$$\left| \frac{\psi(x_n + \lambda y) - \psi(x_n)}{\lambda} - f_{x_n}(y) \right| < \frac{9}{n} \quad \text{for all } \lambda \in (0, \delta_n) \text{ and all } y \in X, \|y\|_n = 1.$$

By Lemma 3, given $p \in \mathbf{N}$ there exists an open neighbourhood V of x_n such that $V \subset U$ and

$$f_{x_n} + \partial_p^\square \psi(V) \subset \partial^\circ \psi(V) + \frac{18}{n} B_n(X^*).$$

Now $f_{x_n} \in S(\partial^\circ \psi(V)) \subset S(\partial^\circ \psi(U))$ and so $\|.\|_n^*$-diam $S(\partial^\circ \psi(V)) < 3/n$. Then

$$\|.\|_n^*\text{-diam } S(f_{x_n} + \partial_p^\square \psi(V)) < \frac{39}{n}.$$

Now ψ is pseudo-regular in the direction y at the points of a residual subset P_y of A, [G-S, p208]. So for $z \in V \cap P_y$ we have

$$\psi^\circ(z)(y) - \psi_p^+(z)(y) \le 0$$

and it follows from the definition of $\partial_p^\square \psi(z)$ that $f_z(y) = 0$ for all $f_z \in \partial_p^\square \psi(z)$. Now $f_{x_n}(y) > r$ so $(f_{x_n} + f_z)(y) > r$ for all $f_z \in \partial_p^\square \psi(z)$ and then

$$f_{x_n} + \partial_p^\square \psi(z) \subset S(f_{x_n} + \partial_p^\square \psi(V)).$$

Since the mapping $x \mapsto \partial_p^\square \psi(x)$ is weak*upper semi-continuous there exists an open neighbourhood W of z where $W \subset V$ such that

$$f_{x_n} + \partial_p^\square \psi(W) \subset S(f_{x_n} + \partial_p^\square \psi(V))$$

and we conclude that

$$\|.\|_n^*\text{-diam } \partial_p^{\Box}\psi(W) < 39/n.$$

By Lemma 4, since $\partial^\circ\psi(A) \subset B(X^*)$, there exists $\alpha > 0$ such that

$$\|.\|_\rho\text{-diam } \partial_p^{\Box}\psi(W) < (39+\alpha)/n.$$

Then given $p \in \mathbf{N}$ and $\epsilon > 0$ the set

$$O_\epsilon^p := \cup\{\text{open sets } W \text{ in } A : \|.\|_\rho\text{-diam } \partial_p^{\Box}\psi(W) < \epsilon\}$$

is open and dense in A and so $D_p := \cap_{\epsilon>0} O_\epsilon^p$ is a dense G_δ subset of A at the points of which the mapping $x \mapsto \partial_p^{\Box}\psi(x)$ is single-valued and $\|.\|_\rho$-norm upper semi-continuous. We conclude that the mapping $x \mapsto \partial^{\Box}\psi(x)$ is single-valued at the points of the dense G_δ subset $D := \cap_{p\in\mathbf{N}} D_p$. Since $0 \in \partial^{\Box}\psi(x)$ for all $x \in A$ we deduce that $\partial^{\Box}\psi(x) = \{0\}$ for all $x \in D$. $\Box$

A locally Lipschitz function ψ on an open subset A of a Banach space X is said to be *strictly differentiable* at $x \in A$ if it is Gâteaux differentiable at x and

$$\lim_{\substack{z\to x\\ \lambda\to 0^+}} \frac{\psi(z+\lambda y)-\psi(z)}{\lambda} = \psi'(x)(y) \quad \text{for all } y \in X.$$

It is well known that for a continuous function on the real line the set of points where it is differentiable but not strictly differentiable is of the first category. But also, for a locally Lipschitz function on an open subset of a separable Banach space, the set of points where it is Gâteaux differentiable but not strictly differentiable is of the first category, [G-S, p210]. We now extend this result to a Banach space which is a closed subspace of an Asplund generated space.

A real valued function ψ on an open subset A of a Banach space X is *directionally differentiable* at $x \in A$ if

$$\psi'_+(x)(y) := \lim_{\lambda\to 0^+} \frac{\psi(x+\lambda y)-\psi(x)}{\lambda} \quad \text{exists for all } y \in X.$$

Corollary
For a locally Lipschitz function ψ on an open subset A of a Banach space X which is a closed linear subspace of an Asplund generated space $Y = \overline{T(Z)}$, the set of points where ψ is directionally differentiable but not strictly differentiable is of the first category.

Proof It is sufficient to consider the case where ψ is directionally differentiable at the points of a residual subset of A. Since X is a closed subspace of an Asplund generated space $Y = \overline{T(Z)}$ then ψ is Gâteaux differentiable at the points of a residual subset G of A, [Z, p185, F-P, p733]. From the Theorem, ψ has the property that $\partial^{\Box}\psi(x) = \{0\}$ at the points of a residual subset D of A. Then for $x \in D \cap G$ and $f \in \partial^\circ\psi(x)$ we have $f(y) \leq \psi^\circ(x)(y)$ for all $y \in X$ so $(f - \psi'(x))(y) \leq \psi^\circ(x)(y) - \psi'(x)(y)$ for all $y \in X$; that is, $(f - \psi'(x) \in \partial^{\Box}\psi(x)$. But then $f = \psi'(x)$ and $\partial^\circ\psi(x) = \{\psi'(x)\}$. $\Box$

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Manuscript received August 24, 2000

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